

## INTERVAL INCIDENCE COLORING OF SUBCUBIC GRAPHS<sup>1</sup>

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### Abstract

In this paper we study the problem of interval incidence coloring of subcubic graphs. In [14] the authors proved that the interval incidence 4-coloring problem is polynomially solvable and the interval incidence 5-coloring problem is  $\mathcal{NP}$ -complete, and they asked if  $\chi_{ii}(G) \leq 2\Delta(G)$  holds for an arbitrary graph  $G$ . In this paper, we prove that an interval incidence 6-coloring always exists for any subcubic graph  $G$  with  $\Delta(G) = 3$ .

**Keywords:** interval incidence coloring, incidence coloring, subcubic graph.

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### 1. INTRODUCTION

In the paper we consider simple nonempty graphs, and we use the standard notation of graph theory. Let  $G = (V, E)$  be a simple graph, and let  $X \subset V$  be a non-empty set. By  $N_G(X) = \{v \in V : \exists u \in X \{v, u\} \in E\}$  we mean the *open*

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*neighborhood* of  $X$ , by  $G[X]$  we mean the subgraph of  $G$  induced by the set  $X$ , and by  $G \setminus X$  we mean the graph  $G[V \setminus X]$ . We say that  $X$  is a *dominating set* of  $G$  if  $V = N_G(X) \cup X$ , and we say that  $X$  is a *total dominating set* if  $V = N_G(X)$ . In what follows we use  $N_G(v)$  instead of  $N_G(\{v\})$ . Let  $\deg_G(v) = |N_G(v)|$  be the degree of a vertex  $v \in V(G)$ . By  $n(G)$ ,  $\Delta(G)$  and  $\delta(G)$  we denote the number of vertices of  $G$ , the maximum and the minimum degree of a vertex of  $G$ , respectively. By a subcubic graph  $G$  we mean a graph with  $\Delta(G) \leq 3$ . By an *isolated vertex* (in a graph  $G$ ) we mean a vertex  $v \in V(G)$  with  $\deg_G(v) = 0$ , and by an *isolated edge* (in a graph  $G$ ) we mean an edge  $e = \{u, v\}$  such that  $\deg_G(u) = \deg_G(v) = 1$ . We say that  $X \subset V(G)$  is an *independent set* if each vertex of  $G[X]$  is isolated in  $G[X]$ . By a *pendant vertex* we mean a vertex of degree 1.

For a given graph  $G = (V, E)$ , we define an *incidence* as a pair  $(v, e)$ , where vertex  $v \in V$  is one of the endpoints of edge  $e \in E$ , i.e.,  $v \in e$ . The set of all incidences of  $G$  will be denoted by  $I(G)$ , thus  $I(G) = \{(v, e) : v \in V \wedge e \in E \wedge v \in e\}$ . We say that two incidences  $(v, e)$  and  $(w, f)$  are *adjacent* if one of the following holds: (1)  $v = w$  and  $e \neq f$ ; (2)  $e = f$  and  $v \neq w$ ; (3)  $e = \{v, w\}$ ,  $f = \{w, u\}$  and  $v \neq u$ .

By an *incidence coloring* of  $G$  we mean a function  $c: I(G) \rightarrow \mathbb{N}$  such that  $c((v, e)) \neq c((w, f))$  for any two adjacent incidences  $(v, e)$  and  $(w, f)$ . The *incidence coloring number* of  $G$ , denoted by  $\chi_i(G)$ , is the smallest number of colors in an incidence coloring of  $G$ . In what follows we use the simplified notation  $c(v, e)$  instead of  $c((v, e))$ .

A finite nonempty set  $A \subset \mathbb{N}$  is an *interval* if it contains all integers between  $\min A$  and  $\max A$ . For a given incidence coloring  $c$  of graph  $G$  and  $v \in V(G)$  let  $A_c(v) = \{c(v, e) : v \in e \wedge e \in E(G)\}$ . By an *interval incidence coloring* of a graph  $G$  we mean an incidence coloring  $c$  of  $G$  such that for each vertex  $v \in V(G)$  the set  $A_c(v)$  is an interval. By an *interval incidence  $k$ -coloring* we mean an interval incidence coloring using all colors from the set  $\{1, \dots, k\}$ . The *interval incidence coloring number* of  $G$ , denoted by  $\chi_{ii}(G)$ , is the smallest number of colors in an interval incidence coloring of  $G$ .

### 1.1. Background and previous results

Alon *et al.* [1] defined the problem of partitioning a graph into the minimal number of star forests. Brualdi and Massey [3] formulated a model of incidence coloring of graphs with references to certain models of coloring of graphs, such as strong edge and vertex coloring of graphs. Guiduli [9] observed that the problem of incidence coloring of graphs is a special case of the problem of partitioning a symmetric digraph into directed star forests.

In [3] the authors conjectured that  $\chi_i(G) \leq \Delta(G) + 2$  holds for every graph  $G$  (*incidence coloring conjecture*, shortly ICC). This conjecture was disproved by Guiduli in [9] who observed that Paley graphs have incidence coloring number at

least  $\Delta + \Omega(\log \Delta)$ . In fact, he used the crucial result from [1]. For many classes of graphs it is shown that the incidence coloring number is at most  $\Delta + 2$ , e.g., trees and cycles [3], complete graphs [3], complete bipartite graphs [3] (proof corrected in [19]), planar graphs with girth at least 11 or with girth at least 6 and maximum degree at least 5 [5], partial 2-trees (i.e.,  $K_4$ -minor free graphs) [4], hypercubes [18], complete  $k$ -partite graphs [15].

In [17] the author proved that ICC holds for subcubic graphs. The incidence 4-colorability problem is  $\mathcal{NP}$ -complete for *semicubic* graphs (i.e., subcubic graphs with vertex degrees equal to 1 or 3) [16] and for semicubic bipartite graphs [15].

In this paper we consider a restriction of the problem of incidence coloring of graphs in which the colors of incidences at a vertex form an interval. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring (see, e.g., [2, 6, 8]), which can be applied to the open-shop scheduling problem [6, 7]. In [11] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [12] the authors studied applications in one-multicast transmission in multifiber WDM networks.

In [13] the authors proved that the problem of interval incidence  $k$ -coloring of bipartite graphs is polynomial for each  $k \leq 6$  and  $\Delta \leq 3$ , polynomial for  $k = 5$  and  $\Delta = 4$ , and  $\mathcal{NP}$ -complete for  $k = 6$  and  $\Delta = 4$ . In [14] the authors proved certain lower and upper bounds on the interval incidence coloring number, e.g.,  $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$  for an arbitrary graph  $G$ , and they determined the exact values of  $\chi_{ii}$  for some basic classes of graphs (e.g., complete  $k$ -partite graphs). In [14] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem of deciding whether  $\chi_{ii} \leq 4$  is easy, and  $\chi_{ii} \leq 5$  is  $\mathcal{NP}$ -complete. The problem of interval incidence 6-coloring of subcubic graphs remained unsolved.

## 1.2. Main results

Our main result in the paper is Theorem 21 which states  $\chi_{ii}(G) \leq 6$  for every subcubic graph  $G$ . To prove it, we state and prove Theorem 8: in any subcubic graph  $G$  with  $\delta(G) \geq 2$  there is a maximal induced bipartite subgraph of  $G$  without isolated vertices, or equivalently,  $G$  has a total dominating set  $S$  such that  $G[S]$  is a bipartite graph.

## 2. MAXIMAL INDUCED BIPARTITE SUBGRAPHS WITHOUT ISOLATED VERTICES

In this section we prove (in Theorem 8) that any subcubic graph  $G$  with  $\delta(G) \geq 2$  contains a maximal induced bipartite subgraph without isolated vertices.



### 2.1. Introductory properties

By  $H \subset G$  we mean that  $H$  is a subgraph of  $G$ . By  $H \sqsubset G$  we mean that  $H$  is an induced subgraph of  $G$ , i.e.,  $H = G[V(H)]$ .

**Observation 1.** *If  $G_1 \sqsubset G_2$  and  $G_2 \sqsubset G_3$ , then  $G_1 \sqsubset G_3$ .*

**Observation 2.** *Let  $G_1 \sqsubset G$  and  $G_2 \sqsubset G$ . If  $G_1 \subset G_2$ , then  $G_1 \sqsubset G_2$ .*

Let  $\mathcal{B}(G) = \{H \sqsubset G : N_G(V(H)) = V(G) \wedge H \text{ is bipartite}\}$ , i.e., the set of all induced bipartite subgraphs of a given graph  $G$  such that  $V(H)$  is a total dominating set of  $G$ . If  $H \in \mathcal{B}(G)$ , then  $V(H)$  is a total dominating set of  $G$  and, obviously,  $H$  has no isolated vertices.

In the following, let  $G$  be any graph. Let  $\hat{\mathcal{B}}(G)$  be the subfamily of  $\mathcal{B}(G)$  consisting of all the elements (graphs) in  $\mathcal{B}(G)$  that are maximal with respect to the subgraph relation ( $\subset$ ).

**Observation 3.** *If  $H \in \mathcal{B}(G)$ , then there is  $H' \in \hat{\mathcal{B}}(G)$  such that  $H \subset H'$ .*

By Observations 2 and 3 we have

**Observation 4.** *Let  $H \in \mathcal{B}(G)$ . Then,  $H \in \hat{\mathcal{B}}(G)$  if and only if for each  $v \in V(G) \setminus V(H)$  the subgraph  $G[V(H) \cup \{v\}]$  is not bipartite.*

**Observation 5.** *If  $H \in \mathcal{B}(G) \setminus \hat{\mathcal{B}}(G)$ , then there is a vertex  $v \in V(G) \setminus V(H)$  such that  $G[V(H) \cup \{v\}] \in \mathcal{B}(G)$ .*

Since any dominating set  $S \subset V(G)$  is a total dominating set if and only if  $G[S]$  has no isolated vertices, we have

**Observation 6.** *Let  $G$  be an arbitrary graph and let  $H \subset G$ . Then,  $H \in \hat{\mathcal{B}}(G)$  if and only if  $H$  is a maximal induced bipartite subgraph (of  $G$ ) without isolated vertices.*

Let  $\mathcal{G}_3^2$  be the family of subcubic graphs without isolated and pendant vertices, i.e., each vertex in a graph of this family has degree 2 or 3. Let  $\mathcal{M}_3^2$  be the subfamily of  $\mathcal{G}_3^2$  consisting of all the graphs for which there is no maximal induced bipartite subgraph without isolated vertices. Let us denote by  $\mathcal{M}$  the set of elements in  $\mathcal{M}_3^2$  that are minimal with respect to the subgraph relation ( $\subset$ ). By Observation 6 we have

**Observation 7.** *Let  $G \in \mathcal{G}_3^2$ . Then,  $G \in \mathcal{M}_3^2 \Leftrightarrow \mathcal{B}(G) = \emptyset \Leftrightarrow \hat{\mathcal{B}}(G) = \emptyset$ .*

2.2. Main Theorem

**Theorem 8.** *Let  $G$  be a subcubic graph with  $\delta(G) \geq 2$ . Then,  $G$  has a maximal induced bipartite subgraph without isolated vertices.*

By Observation 7, Theorem 8 is equivalent to  $\mathcal{M} = \emptyset$ . First, we prove some structural properties of graphs from  $\mathcal{M}$ .

**Lemma 9.** *Let  $G \in \mathcal{M}$ . Then,  $G$  is a connected graph and  $\Delta(G) = 3$ .*

**Proof.** Let  $G \in \mathcal{M}$ . Let us assume to the contrary that  $G = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  are disjoint graphs (without common vertices). Since  $G_i \subsetneq G \in \mathcal{M}$  and  $G_i \in \mathcal{G}_3^2$ , we have  $G_i \notin \mathcal{M}_3^2$ , for  $i \in \{1, 2\}$ . Hence, there exist  $H_1 \in \hat{\mathcal{B}}(G_1)$  and  $H_2 \in \hat{\mathcal{B}}(G_2)$ . Thus,  $H_1 \cup H_2 \in \hat{\mathcal{B}}(G)$ , a contradiction.

Since every cycle is either a bipartite graph or it becomes a bipartite graph after deleting an arbitrary vertex,  $G$  is not a cycle, which implies  $\Delta(G) = 3$ . ■

**Lemma 10.** *Let  $G \in \mathcal{M}$  and let  $v$  be a vertex of degree 2 in  $G$ . Then, every neighbor of  $v$  in  $G$  has degree 3.*

**Proof.** Let  $G \in \mathcal{M}$ . Suppose to the contrary that there are two adjacent vertices of degree 2. Since  $G$  is not a cycle (by Lemma 9), there is a subgraph  $P$  of  $G$  with vertex set  $\{v_0, \dots, v_{k+1}\}$  and edges  $\{v_i, v_{i+1}\}$ , for  $i \in \{0, \dots, k\}$ , such that  $\deg_G(v_0) = \deg_G(v_{k+1}) = 3$ , and  $\deg_G(v_i) = 2$  for  $i \in \{1, \dots, k\}$ , where  $k \geq 2$ .

Suppose  $v_0 \neq v_{k+1}$ . Since  $G' = G \setminus \{v_1, \dots, v_k\} \sqsubset G \in \mathcal{M}$  and  $G' \in \mathcal{G}_3^2$ , we have  $G' \notin \mathcal{M}_3^2$ . Hence, there exists  $H' \in \hat{\mathcal{B}}(G')$ , and  $H' \sqsubset G$  by Observation 1. If  $v_0 \in V(H')$ , then let  $H = G[V(H') \cup \{v_1, \dots, v_{k-1}\}]$ , otherwise, let  $H = G[V(H') \cup \{v_1, \dots, v_k\}]$ . In both cases,  $H \sqsubset G$ ,  $H$  is a bipartite graph, and  $V(H)$  is a total dominating set, i.e.,  $H \in \mathcal{B}(G)$ . By Observation 7 we get a contradiction.

Suppose  $v_0 = v_{k+1}$ . Since  $\deg_G(v_0) = 3$ , there is  $c \in N_G(v_0) \setminus \{v_1, v_k\}$ . If  $\deg_G(c) = 3$ , then let  $G' = G \setminus \{v_0, \dots, v_k\}$ . If  $\deg_G(c) = 2$ , then let  $G' = G \setminus \{v_0, \dots, v_k, c\}$ . In both cases,  $G' \sqsubset G$  and  $G \neq G' \in \mathcal{G}_3^2$ . Hence, there is  $H' \in \hat{\mathcal{B}}(G')$ . Let  $H = G[V(H') \cup \{v_0, \dots, v_{k-1}\}]$ . Thus,  $H \in \mathcal{B}(G)$ , a contradiction. ■

**Lemma 11.** *If  $G \in \mathcal{G}_3^2$  contains  $G_0$  as a subgraph (see Figure 1), where vertices  $v_2, v_3 \in V(G_0)$  are of degree 2 in  $G$ , then  $G \notin \mathcal{M}$ .*

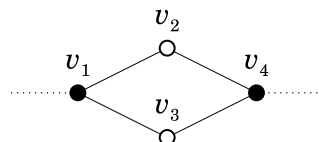


Figure 1. The subgraph  $G_0$  of a graph  $G$ .

**Proof.** Suppose to the contrary that  $G \in \mathcal{M}$ . Suppose  $G_0 \subset G$ . The other possible edges in  $G$  are marked by the dotted lines (in Figure 1).

By  $\deg_G(v_2) = \deg_G(v_3) = 2$ , from Lemma 10 we have  $\deg_G(v_1) = \deg_G(v_4) = 3$ . Since  $G' = G \setminus \{v_3\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$ , there is  $H' \in \hat{\mathcal{B}}(G')$ . Hence,  $v_1 \in V(H')$  or  $v_4 \in V(H')$ . Thus,  $H' \in \mathcal{B}(G)$ , a contradiction. ■

**Lemma 12.** *Let  $G \in \mathcal{M}$  and let  $v$  be a vertex of degree 3 in  $G$ . Then, at most one neighbor of  $v$  has degree 2.*

**Proof.** Let  $G \in \mathcal{M}$  and let  $N_G(v) = \{x, y, z\}$ . Suppose to the contrary that at least two vertices from  $N_G(v)$  have degree 2. Let  $\deg_G(x) = \deg_G(y) = 2$ . Let  $\{v_x\} = N_G(x) \setminus \{v\}$  and  $\{v_y\} = N_G(y) \setminus \{v\}$ . By Lemma 10,  $\deg_G(v_x) = \deg_G(v_y) = 3$ .

Suppose  $\deg_G(z) = 2$ . Let  $\{v_z\} = N_G(z) \setminus \{v\}$ . By Lemma 10,  $\deg_G(v_z) = 3$ . If any two of the vertices  $v_x, v_y, v_z$  are equal, then by Lemma 11 (i.e., because  $G_0 \sqsubset G$ ) we get a contradiction. Hence, vertices  $v_x, v_y, v_z$  are different. Since  $G' = G \setminus \{x, y, z, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$ , there is  $H' \in \hat{\mathcal{B}}(G')$ . Thus,  $G[V(H') \cup \{v, x\}] \in \mathcal{B}(G)$ , a contradiction.

Suppose  $\deg_G(z) = 3$ . If  $v_x = v_y$ , then by Lemma 11 we get a contradiction. Hence,  $v_x \neq v_y$ . Suppose  $z = v_x$  (the case  $z = v_y$  can be treated analogously). Since  $G_x = G \setminus \{x\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$ , there is  $H_x \in \hat{\mathcal{B}}(G_x)$ . Since  $H_x$  is maximal in  $\mathcal{B}(G)$ , we have  $v \in V(H_x)$  or  $z \in V(H_x)$ . Thus,  $H_x \in \mathcal{B}(G)$ , a contradiction. Then, vertices  $v_x, v_y, z$  are different. Since  $G' = G \setminus \{x, y, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$ , there is  $H' \in \hat{\mathcal{B}}(G')$ . If  $z \in V(H')$ , then let  $A = V(H') \cup \{v\}$ . If  $z \notin V(H')$ , then let  $A = V(H') \cup \{v, x\}$ . In both cases,  $G[A] \in \mathcal{B}(G)$ , a contradiction. ■

Let  $G$  be any subcubic graph. We say that  $H \subset G$  is a  $Q$ -cycle (of  $G$ ) if:

- (q<sub>1</sub>) for each  $v \in V(H)$ ,  $\deg_G(v) = 3$ , and
- (q<sub>2</sub>)  $H \sqsubset G$  and  $H$  is isomorphic to a cycle, i.e.,  $H$  is an induced cycle, and
- (q<sub>3</sub>) for each vertex  $v \in V(G) \setminus V(H)$ ,  $|N_G(v) \cap V(H)| \leq 1$ .

**Lemma 13.** *Let  $G \in \mathcal{M}$ . Let  $v \in V(G)$  have all neighbors of degree 3. Then, for each  $x \in N_G(v)$  there is a  $Q$ -cycle  $C_x$  such that  $x \in V(C_x)$ ,  $v \notin V(C_x)$  and  $N_G(v) \cap V(C_x) = \{x\}$ .*

**Proof.** Let  $G \in \mathcal{M}$  and let  $v \in V(G)$  be a vertex with all neighbors of degree 3. Since  $G' = G \setminus \{v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$ , there is  $H' \in \hat{\mathcal{B}}(G')$ . Hence,  $N_G(v) \cap V(H') = \emptyset$ .

Let  $x \in N_G(v)$  and let  $N_G(x) = \{a, b, v\}$ . Since  $H'$  is bipartite and maximal in  $\mathcal{B}(G)$ , we have that  $a$  and  $b$  belong to the same connected component of  $H'$ , and the length of each path in  $H'$  from  $a$  to  $b$  is odd. Let  $P \subset H'$  be a path joining  $x_1 = a$  and  $x_{s-1} = b$  ( $s$  is odd), with vertex set  $\{x_1, \dots, x_{s-1}\}$  and edges  $\{x_i, x_{i+1}\}$ , for  $i \in \{1, \dots, s-2\}$ . Let  $x_0 = x$  and let  $C_x$  be the graph with  $V(C_x) = V(P) \cup \{x_0\}$ , and  $E(C_x) = E(P) \cup \{\{x_{s-1}, x_0\}, \{x_0, x_1\}\}$ . Since  $P \subset H'$ , we have  $N_G(v) \cap V(C_x) = \{x\}$ , and  $v \notin V(C_x)$ .

**Claim 14.** For each  $i \in \{1, \dots, s - 1\}$ , the following properties are satisfied:

- (p<sub>1</sub>)  $\deg_G(x_i) = 3$ ,
- (p<sub>2</sub>)  $N_G(x_i) = \{a_i, x_{(i-1) \bmod s}, x_{(i+1) \bmod s}\}$ , where  $a_i \in V(H') \setminus V(C_x)$ ,
- (p<sub>3</sub>)  $N_G(a_i) \cap V(H') = \{x_i\}$ .

**Proof.** We proceed by induction on  $i$ . Suppose  $i = 1$ . Let  $X = V(H') \setminus \{x_i\} \cup \{x, v\}$ . Hence,  $G[X]$  is bipartite. If  $\deg_G(x_i) = 2$  or  $N_G(a_i) \cap V(H') \neq \{x_i\}$ , then  $G[X] \in \mathcal{B}(G)$ , a contradiction. If  $a_i \notin V(H') \setminus V(C_x)$ , then  $a_i \notin V(H')$  or  $a_i \in V(C_x)$ . If  $a_i \notin V(H')$ , then  $N_G(a_i) \cap V(H') \neq \{x_i\}$  (otherwise  $H'$  is not maximal in  $\mathcal{B}(G)$ ), a contradiction. If  $a_i \in V(C_x)$ , then  $G[X] \in \mathcal{B}(G)$ , a contradiction.

Suppose the properties (p<sub>1</sub>), (p<sub>2</sub>), (p<sub>3</sub>) hold for  $1, \dots, i - 1$  ( $2 \leq i \leq s - 1$ ). Hence, each path joining  $x_1$  and  $x_{s-1}$  in  $H'$  contains  $x_1, \dots, x_i$ . Let  $X = V(H') \setminus \{x_i\} \cup \{x, v\}$ . Hence,  $G[X]$  is bipartite. The rest of the proof of properties (p<sub>1</sub>), (p<sub>2</sub>), (p<sub>3</sub>) for  $i$  is literally the same as in the case  $i = 1$ . □

We show that  $C_x$  is a  $Q$ -cycle. Since  $\deg_G(x) = 3$ , by (p<sub>1</sub>) we have (q<sub>1</sub>). Since  $v \notin V(C_x)$  and  $a_i \notin V(C_x)$  (by (p<sub>2</sub>)), for  $i \in \{1, \dots, s - 1\}$ , we have that  $C_x$  is an induced cycle of  $G$ . Since  $a_i \in V(H')$  (by (p<sub>2</sub>)), we have  $a_i \neq v$ . Thus, by (p<sub>3</sub>) we get  $|N_G(a_i) \cap V(C_x)| \leq 1$ , for  $i \in \{1, \dots, s - 1\}$ . ■

We say that  $H$  is a  $Q_2$ -cycle (of  $G$ ) if  $H$  is a  $Q$ -cycle of  $G$ , and it holds (q<sub>4</sub>) for each  $v \in N_G(V(H)) \setminus V(H)$ ,  $\deg_G(v) = 2$ .

**Lemma 15.** Let  $G \in \mathcal{M}$  and let  $C$  be a  $Q$ -cycle of  $G$ . Then,  $C$  is a  $Q_2$ -cycle.

**Proof.** Let  $G \in \mathcal{M}$ . Let  $C$  be a  $Q$ -cycle of  $G$  with the vertex set  $\{x_0, \dots, x_{s-1}\}$ , and edges  $\{x_0, x_1\}, \dots, \{x_{s-2}, x_{s-1}\}, \{x_{s-1}, x_0\}$ . Let  $S = \{0, \dots, s - 1\}$ . Let  $\{a_i\} = N_G(x_i) \setminus V(C)$ , for  $i \in S$ . If  $\deg_G(a_i) = 2$ , then let  $\{b_i\} = N_G(a_i) \setminus \{x_i\}$ . Hence,  $b_i \notin V(C)$ . By Lemma 10 we have  $\deg_G(b_i) = 3$ . Let  $G' = G \setminus (V(C) \cup \{a_i : \deg_G(a_i) = 2 \wedge i \in S\})$ . Since  $G' \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$ , there is  $H' \in \hat{\mathcal{B}}(G')$ .

Suppose to the contrary that  $C$  is not a  $Q_2$ -cycle, i.e., there exists  $r \in S$  such that  $\deg_G(a_r) = 3$ . Let  $f: V(G') \rightarrow \{0, 1\}$  be the characteristic function of  $V(H')$ , i.e.,  $f(u) = 1$  if and only if  $u \in V(H')$ . Let us consider two cases.

- (i) For each  $i \in S$ :  $\deg_G(a_i) = 2 \Rightarrow f(b_i) = 0$  and  $\deg_G(a_i) = 3 \Rightarrow f(a_i) = 0$ .
- (ii) For some  $t \in S$ :  $\deg_G(a_t) = 2 \wedge f(b_t) = 1$  or  $\deg_G(a_t) = 3 \wedge f(a_t) = 1$ .

We construct a function  $\tilde{f}: V(G) \rightarrow \{0, 1\}$  such that  $\tilde{f}(u) = f(u)$  for each  $u \in V(G')$ . Let  $u \in V(G) \setminus V(G')$ . We define  $\tilde{f}(u)$  depending on cases (i), (ii).

- (i) Let  $\tilde{f}(x_r) = 0$  and let  $\tilde{f}(x_j) = 1$ , for each  $j \in S \setminus \{r\}$ . For each  $j \in S$ , if  $\deg_G(a_j) = 2$ , then  $\tilde{f}(a_j) = 1$ ,



- (ii) Take any  $t \in S$ , if exists, such that  $\deg_G(a_t) = 2 \wedge f(b_t) = 1$  and let  $\tilde{f}(a_t) = 1$ . Then, for each  $j \in S, j \neq t$ , if  $\deg_G(a_j) = 2$ , then  $\tilde{f}(a_j) = 1 - f(b_j)$ . Next, for each  $j \in S$ , if  $\deg_G(a_j) = 2 \wedge f(b_j) = 0$ , then  $\tilde{f}(x_j) = 1$ . Finally, for each  $j \in S$ , if  $\deg_G(a_j) = 3$  or  $\deg_G(a_j) = 2 \wedge f(b_j) = 1$ , then  $\tilde{f}(x_j) = 1 - \tilde{f}(a_{(j+1) \bmod s})$ .

Let  $H = G[\{u \in V(G) : \tilde{f}(u) = 1\}]$ . In the case (i),  $x_r \notin V(H) \cap V(C)$ . Hence,  $H$  is a bipartite graph. For each  $u \in V(G) \setminus V(G'), u \neq x_r$ , we have that  $u \in V(H)$ . Thus,  $V(H)$  is a total dominating set of  $G$  and  $H \in \mathcal{B}(G)$ , a contradiction.

In case (ii), if there is no  $t \in S$  such that  $\deg_G(a_t) = 2 \wedge f(b_t) = 1$ , then, by assumption, there is  $t \in S$  such that  $\deg_G(a_t) = 3 \wedge f(a_t) = 1$ , so finally  $\tilde{f}(a_t) = 1$  for some  $t \in S$ . Hence, there is  $p \in S$  such that  $\tilde{f}(x_p) = 0$ . Thus,  $V(C) \setminus V(H) \neq \emptyset$ .

Let us remind that for each  $i \in S \setminus \{t\}$ , if  $\deg_G(a_i) = 2$  and  $\tilde{f}(a_i) = 1$ , then  $f(b_i) = 0$ . Let  $X = \{i \in S : \deg_G(a_i) = 3 \wedge \tilde{f}(a_i) = 1\} \cup \{t\}$ . Suppose that for some two  $i, j \in X$ , there is a path in  $H$  between  $a_i$  and  $a_j$  with successive vertices  $x_i, x_{(i+1) \bmod s}, \dots, x_j$ . Hence,  $\tilde{f}(x_i) = \tilde{f}(x_{(i+1) \bmod s}) = \dots = \tilde{f}(x_j) = 1$ , which implies that  $\tilde{f}(a_{(i+1) \bmod s}) = 0, \tilde{f}(a_{(i+2) \bmod s}) = 0, \dots, \tilde{f}(a_j) = 0$ , a contradiction. Thus,  $H$  is a bipartite graph.

For every  $j \in S$  we have  $N_G(a_j) \cap V(H) \neq \emptyset$ , and  $\tilde{f}(a_j) = 1$  or  $\tilde{f}(a_j) = 0 \wedge \tilde{f}(x_{(j-1) \bmod s}) = 1$ . Hence, we get  $N_G(x_j) \cap V(H) \neq \emptyset$ . Thus,  $V(H)$  is a total dominating set and  $H \in \mathcal{B}(G)$ , a contradiction. ■

By Lemmas 10, 12, 13 and Lemma 15, and by the definition of  $Q_2$ -cycle we have the following corollary.

**Corollary 16.** *Let  $G \in \mathcal{M}$  and  $v \in V(G)$ . The following properties are satisfied:*

- (i)  $\deg_G(v) = 2$  if and only if vertex  $v$  has all neighbors of degree 3,
- (ii)  $\deg_G(v) = 3$  if and only if exactly one neighbor of  $v$  has degree 2,
- (iii) if  $\deg_G(v) = 3$ , then there is exactly one  $Q_2$ -cycle containing  $v$ ,
- (iv) if  $\deg_G(v) = 2$ , then vertex  $v$  has two neighbors from disjoint  $Q_2$ -cycles.

By Corollary 16 we have the next corollary.

**Corollary 17.** *Let  $G \in \mathcal{M}$ . The graph  $G$  satisfies the following properties:*

- (i) there is an integer  $q \geq 1$  such that  $V(G) = D \cup \bigcup_{i=1}^q V(C_i)$ , where for each  $i \in \{1, \dots, q\}$  the graph  $C_i$  is a  $Q_2$ -cycle and  $D$  is the set of all vertices of degree 2,
- (ii)  $E(G) = \{\{u, v\} : \exists i \in \{1, \dots, q\} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\}$ .

**Proof of Theorem 8.** Suppose to the contrary that  $G \in \mathcal{M}$ .



By Corollary 17, there is  $q \geq 1$  such that  $V(G) = D \cup \bigcup_{i=1}^q V(C_i)$ , where for each  $i \in \{1, \dots, q\}$  the graph  $C_i$  is a  $Q_2$ -cycle and  $D$  is the set of all vertices of degree 2, and

$$E(G) = \{\{u, v\} : \exists_{i \in \{1, \dots, q\}} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\}.$$

Let  $Q = (D \cup \bigcup_{i=1}^q \{c_i\}, E_Q)$ , where for each  $i \in \{1, \dots, q\}$  vertex  $c_i$  corresponds to the cycle  $C_i$  and

$$E_Q = \{\{v, c_i\} : i \in \{1, \dots, q\} \wedge v \in D \wedge \exists_{x \in V(C_i)} \{v, x\} \in E(G)\}.$$

By Corollary 16 and Corollary 17 we have that  $Q$  is a simple bipartite graph with partitions  $D$  and  $C = \bigcup_{i=1}^q \{c_i\}$ . Obviously, for all vertices  $v \in D$  and  $c \in C$  we have that  $\deg_Q(v) = 2 < \deg_Q(c)$ . Thus, by Hall's Marriage Theorem [10] there is a matching  $S$  in  $Q$  covering all vertices from partition  $C$ .

Let

$$S' = \{\{v, x\} \in E(G) : v \in D \wedge \exists_{i \in \{1, \dots, q\}} \{v, c_i\} \in S \wedge x \in V(C_i)\}$$

and let

$$V' = \left\{ x \in \bigcup_{i=1}^q V(C_i) : \exists_{e \in S'} x \in e \right\}.$$

Let  $H = G[V(G) \setminus (D \cup V')]$ . For each  $i \in \{1, \dots, q\}$  there is  $x$  such that  $\{x\} = V(C_i) \cap V'$  and  $N_G(x) \cap V(H) \neq \emptyset$ . If  $y \in V(C_i)$  and  $x \neq y$ , then  $N_G(y) \cap V(H) \neq \emptyset$ . Hence,  $H$  is an induced bipartite graph without isolated vertices. Since for each  $v \in D$  at most one neighbor of  $v$  belongs to  $V'$ , we have  $N_G(v) \cap V(H) \neq \emptyset$ . Thus,  $N_G(V(H)) = V(G)$  and  $H \in \mathcal{B}(G)$ , a contradiction. ■

### 3. INTERVAL INCIDENCE 6-COLORING OF SUBCUBIC GRAPHS

In this section we prove our main result, i.e., Theorem 21, which states  $\chi_{ii}(G) \leq 2\Delta(G)$  for each subcubic graph  $G$ . By Theorem 8 we have the following lemma.

**Lemma 18.** *Let  $G$  be a connected graph and  $G \in \mathcal{G}_3^2$ . Let  $H \in \hat{\mathcal{B}}(G)$  and let  $A, B \subset V(H)$  be any partition of  $V(H)$ , such that  $A$  and  $B$  are disjoint independent sets and  $A \cup B = V(H)$ . Then,  $A$  and  $B$  are disjoint independent dominating sets, and the graph  $G[V(G) \setminus V(H)]$  has only isolated vertices and isolated edges.*

**Proof.** Let  $v \in V(G) \setminus V(H)$ . If  $N_G(v) \cap V(H) \subset A$  or  $N_G(v) \cap V(H) \subset B$ , then  $G[V(H) \cup \{v\}]$  is a bipartite graph, a contradiction. Thus,  $N_G(v) \cap A \neq \emptyset$  and  $N_G(v) \cap B \neq \emptyset$ . Let  $v \in A$  ( $v \in B$ ). Since  $H$  is an induced graph without isolated vertices, we have  $v \in N_G(B)$  ( $v \in N_G(A)$ ). Hence,  $A$  and  $B$  are disjoint independent dominating sets.

Since  $G$  is subcubic and  $|N_G(v) \cap V(H)| \geq 2$  for any  $v \in V(G) \setminus V(H)$ , graph  $G[V(G) \setminus V(H)]$  has only isolated vertices and isolated edges. ■

**Lemma 19.** *Let  $G$  be a subcubic non-bipartite graph with  $\Delta(G) = 3$ . Then, there is a vertex coloring  $c: V(G) \rightarrow \{1, 2, 3, 4\}$  such that for each  $v \in V(G)$  the following properties hold:*

- (i) if  $\deg_G(v) = 1$ , then  $c(v) \in \{1, 4\}$ ,
  - (ii) if  $\deg_G(v) \geq 2$  and  $c(v) \neq p$ , then  $a_p(v) \geq 1$ , for  $p \in \{1, 4\}$ ,
  - (iii)  $a_i(v) \leq |c(v) - i|$ , for  $i \in \{1, 2, 3, 4\}$ ,
- where  $a_i(v) = |\{w \in N_G(v) : c(w) = i\}|$ , for  $i \in \{1, 2, 3, 4\}$ .

**Proof.** If  $\delta(G) = 1$ , then we successively remove pendant vertices from graph  $G$ , until there is no pendant vertex. Let us denote the resulting graph by  $G'$ . Obviously,  $\delta(G') \geq 2$ . Let us observe that we cut off all trees attached to  $G$ .

By Theorem 8 we have  $\hat{B}(G') \neq \emptyset$ . Let  $H$  be any element of  $\hat{B}(G')$  with the largest possible number of vertices.

Let  $A, B \subset V(H)$  be any two partite sets of  $V(H)$ , i.e.,  $A$  and  $B$  are disjoint independent sets and  $A \cup B = V(H)$ . By Lemma 18,  $A$  and  $B$  are disjoint independent dominating sets of  $G'$ , and the graph  $G[V(G') \setminus V(H)]$  has only isolated vertices and isolated edges. Let  $I_i \subset V(G') \setminus V(H)$  be the set of all vertices of degree  $i$  in  $G'$ , for  $i \in \{2, 3\}$ . Let us define the partition  $I_3 = I_3^A \cup I_3^B \cup I_3^2$ :

- $I_3^A = \{v \in I_3 : |N_{G'}(v) \cap A| = 2 \wedge |N_{G'}(v) \cap B| = 1\}$ ,
- $I_3^B = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \wedge |N_{G'}(v) \cap B| = 2\}$ ,
- $I_3^2 = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \wedge |N_{G'}(v) \cap B| = 1\}$ .

Note that  $I_2, I_3^A, I_3^B$  are independent sets in  $G'$ , each vertex  $v \in I_3^2$  belongs to an isolated edge in  $G'[I_3^2]$ , and each vertex from  $I_2$  has neighbors from  $A$  and  $B$ .

Let us define a coloring  $c: V(G) \rightarrow \{1, 2, 3, 4\}$  in the following steps.

- (C<sub>1</sub>) If  $v \in A$ , then  $c(v) = 1$ , and if  $v \in B$ , then  $c(v) = 4$ .
- (C<sub>2</sub>) If  $v \in I_3^B$ , then  $c(v) = 2$ , and if  $v \in I_3^A$ , then  $c(v) = 3$ .
- (C<sub>3</sub>) For each successive  $v \in I_2$  we assign a color following the algorithm: if  $c(v)$  is not determined, then let  $\{u\} = N_{G'}(v) \cap A$ . If there is  $x \in N_{G'}(u)$  such that  $c(x) = 2$ , then let  $c(v) = 3$ . Otherwise, for each vertex  $x \in N_{G'}(u)$  either  $c(x) \in \{3, 4\}$  or  $c(x)$  is not determined, and then let  $c(v) = 2$ .
- (C<sub>4</sub>) For each successive  $\{v, w\} \in E(G'[I_3^2])$  we assign colors to both  $v$  and  $w$  following the algorithm: if  $c(v)$  and  $c(w)$  are not determined, then let  $\{u\} = N_{G'}(v) \cap A$ . If there is  $x \in N_{G'}(u)$  such that  $c(x) = 2$ , then let  $c(v) = 3$  and  $c(w) = 2$ . Otherwise, for each vertex  $x \in N_{G'}(u)$  either  $c(x) \in \{3, 4\}$  or  $c(x)$  is not determined, and then let  $c(v) = 2$  and  $c(w) = 3$ .
- (C<sub>5</sub>) For each  $v \in V(G')$  such that  $\deg_{G'}(v) < \deg_G(v)$ , there is a tree  $T_v$  such that  $V(T_v) \subset V(G) \setminus V(G')$  and let  $\{w\} = V(T_v) \cap N_G(v)$ . Let  $d: V(T_v) \rightarrow$

$\{a, b\}$  be a 2-coloring of  $T_v$  such that  $d(w) = a$ . Suppose  $c(v) \leq 2$ . For each  $u \in V(T_v)$ , if  $d(u) = a$ , then let  $c(u) = 4$ , and if  $d(u) = b$ , then let  $c(u) = 1$ . Suppose  $c(v) \geq 3$ . For each  $u \in V(T_v)$ , if  $d(u) = a$ , then let  $c(u) = 1$ , and if  $d(u) = b$ , then let  $c(u) = 4$ .

In step  $(C_1)$  we colored  $V(H) = A \cup B$  with colors 1 and 4, in steps  $(C_2)$ – $(C_4)$  we colored vertices from  $I_2 \cup I_3$  with colors 2 or 3, and in step  $(C_5)$  we colored vertices from  $V(G) \setminus V(G')$  with colors 1 or 4. Since vertices colored with an arbitrary color form an independent set,  $c$  is a vertex 4-coloring of  $G$ .

Let  $v \in V(G)$  and let  $\deg_G(v) = 1$ . Then,  $v \in V(G) \setminus V(G')$  and, by  $(C_5)$ ,  $c(v) \in \{1, 4\}$ . Thus, we get the property (i). Let  $\deg_G(v) \geq 2$ . If  $v \in V(G) \setminus V(G')$ , then, by  $(C_5)$ , the property (ii) holds. Let  $v \in V(G')$ . Since  $A$  and  $B$  are disjoint independent dominating sets of  $G'$ , the property (ii) holds.

Since  $c$  is a proper coloring of  $G$ , there is  $a_{c(v)}(v) = 0$  for each  $v \in V(G)$ .

Let  $v \in V(G) \setminus V(G')$ . By step  $(C_5)$ ,  $c(v) \in \{1, 4\}$ . If  $c(v) = 1$ , then  $a_2(v) = 0$ ,  $a_3(v) \leq 1$  and  $a_4(v) \leq 3$ . If  $c(v) = 4$ , then  $a_3(v) = 0$ ,  $a_2(v) \leq 1$  and  $a_1(v) \leq 3$ .

Let  $v \in V(G') \setminus V(H)$ . If  $v \in I_3^A$ , then  $c(v) = 3$ ,  $a_1(v) = 2$ ,  $a_2(v) = 0$ ,  $a_4(v) = 1$ . If  $v \in I_3^B$ , then  $c(v) = 2$ ,  $a_1(v) = 1$ ,  $a_3(v) = 0$ ,  $a_4(v) = 2$ . If  $v \in I_2$ , then  $c(v) \in \{2, 3\}$ . If  $\deg_{G'}(v) = \deg_G(v)$ , then  $a_1(v) = a_4(v) = 1$ , and  $a_2(v) = a_3(v) = 0$ . If  $\deg_{G'}(v) < \deg_G(v)$ , then if  $c(v) = 2$ , then  $a_1(v) = 1$ ,  $a_2(v) = a_3(v) = 0$ ,  $a_4(v) = 2$ , and if  $c(v) = 3$ , then  $a_1(v) = 2$ ,  $a_2(v) = a_3(v) = 0$ ,  $a_4(v) = 1$ . If  $v \in I_3^2$ , then  $c(v) \in \{2, 3\}$ . If  $c(v) = 2$ , then  $a_1(v) = a_3(v) = a_4(v) = 1$ . If  $c(v) = 3$ , then  $a_1(v) = a_2(v) = a_4(v) = 1$ .

Let  $v \in A \cup B$ . Since  $A$  and  $B$  are disjoint dominating sets of  $G'$  and  $H \in \hat{\mathcal{B}}(G')$ , it suffices to prove that if  $c(v) = 1$ , then  $a_2(v) \leq 1$ , and if  $c(v) = 4$ , then  $a_3(v) \leq 1$ .

Suppose to the contrary that  $c(v) = 1$  and  $a_2(v) = 2$  for some  $v \in A$ . The case  $c(v) = 4$  and  $a_3(v) = 2$ , for some  $v \in B$ , is analogous. Let  $x, y \in N_{G'}(v)$  such that  $c(x) = c(y) = 2$ . Since  $B$  is a dominating set of  $G'$ , there is  $w \in N_{G'}(v) \cap B$  with  $c(w) = 4$ . By the definition of coloring  $c$ , we have  $v, x, y, w \in V(G')$  and  $v, w \in V(H)$ .

Since  $c(x) = c(y) = 2$ , we have  $a_1(x) = a_1(y) = 1$ ,  $a_3(x) \leq 1$ ,  $a_3(y) \leq 1$ ,  $1 \leq a_4(x) \leq 2$  and  $1 \leq a_4(y) \leq 2$ . Let us consider the following cases:

- $x \notin N_{G'}(w)$  and  $y \notin N_{G'}(w)$ . If edge  $\{v, w\}$  is isolated in  $H$ , then let  $W = V(H) \cup \{x, y\}$ . Otherwise, let  $W = V(H) \cup \{x, y\} \setminus \{v\}$ .
- $x \in N_{G'}(w)$  or  $y \in N_{G'}(w)$ . Let  $W = V(H) \cup \{x, y\} \setminus \{v\}$ .

In both cases, the graph  $G'[W] \in \mathcal{B}(G')$  and  $|V(G'[W])| > |V(H)|$ , a contradiction. Thus, the coloring  $c$  satisfies the property (iii). ■

**Proposition 20.** [14] *For any graph  $G$ ,  $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$ .*

We prove that an interval incidence 6-coloring always exists for any subcubic graph  $G$  with  $\Delta(G) = 3$ .

**Theorem 21.** *Let  $G$  be a subcubic graph. Then,  $\chi_{ii}(G) \leq 2\Delta(G)$ .*

**Proof.** If  $G$  is a subcubic bipartite graph, then by Proposition 20 we have  $\chi_{ii}(G) \leq 2\Delta(G)$ . If  $\Delta(G) = 2$ , then one can easily construct an interval incidence 4-coloring. Thus,  $\chi_{ii}(G) \leq 2\Delta(G)$ . Let  $G$  be a subcubic non-bipartite graph with  $\Delta(G) = 3$ . By Lemma 19, there is a vertex coloring  $c: V(G) \rightarrow \{1, 2, 3, 4\}$  satisfying the properties (i), (ii), (iii) from Lemma 19.

We construct an incidence coloring  $f: I(G) \rightarrow \{1, 2, 3, 4, 5, 6\}$  in three steps.

In the first step, using the coloring  $c$ , we define the interval  $A_f(v)$  for each vertex  $v \in V(G)$ , as follows. If  $\deg_G(v) = 2$  and  $c(v) \in \{2, 3\}$ , then let  $A_f(v) = \{3, 4\}$ . If  $c(v) = 4$  and  $\deg_G(v) = 1$ , then  $A_f(v) = \{6\}$ . If  $c(v) = 4$  and  $\deg_G(v) = 2$ , then  $A_f(v) = \{5, 6\}$ . In the other cases, let  $A_f(v) = \{c(v), \dots, c(v) + \deg_G(v) - 1\}$ . Thus, by Lemma 19 (i)–(iii) we get

- (a<sub>1</sub>) if  $\deg_G(v) = 1$ , then  $c(v) \in \{1, 4\}$  and  $A_f(v) = \{c(v)\}$ ,
- (a<sub>2</sub>) if  $\deg_G(v) = 2$ , then if  $c(v) \in \{1, 3\}$ , then  $A_f(v) = \{c(v), c(v) + 1\}$  and if  $c(v) \in \{2, 4\}$ , then  $A_f(v) = \{c(v) + 1, c(v) + 2\}$ ,
- (a<sub>3</sub>) if  $\deg_G(v) = 3$ , then  $A_f(v) = \{c(v), c(v) + 1, c(v) + 2\}$ .

In the second step, for each  $v \in V(G)$ , we construct a sequence  $L_f(v)$  (i.e., a linear ordered set) from elements of  $N_G(v)$ , as follows (see Figure 2).

- (l<sub>1</sub>) Suppose  $\deg_G(v) = 1$ . If  $N_G(v) = \{x\}$ , then let  $L_f(v) = (x)$ .
- (l<sub>2</sub>) Suppose  $\deg_G(v) = 2$ . Let  $N_G(v) = \{x, y\}$ , where  $c(x) \leq c(y)$ . Then,
  - if  $c(v) \in \{1, 4\}$ , then let  $L_f(v) = (x, y)$ ,
  - if  $c(v) \in \{2, 3\}$ , then let  $L_f(v) = (y, x)$ .
- (l<sub>3</sub>) Suppose  $\deg_G(v) = 3$ . Let  $N_G(v) = \{x, y, z\}$ , where  $c(x) \leq c(y) \leq c(z)$ . Then,
  - if  $c(v) \in \{1, 4\}$ , then let  $L_f(v) = (x, y, z)$ ,
  - if  $c(v) = 2$ , then let  $L_f(v) = (y, z, x)$ ,
  - if  $c(v) = 3$ , then let  $L_f(v) = (z, x, y)$ .

By  $v_i$  we mean the  $i$ -th element of the sequence  $L_f(v)$ , i.e.,  $L_f(v) = (v_1, \dots)$ .

In the final step, for each vertex  $v$ , we define the incidence coloring  $f$  as follows:  $f(v, \{v, v_i\}) = \min A_f(v) + i - 1$ , for  $i \in \{1, \dots, \deg_G(v)\}$ .

In Figure 2 the *white* vertex is the vertex  $v$ , and the list above is  $L_f(v)$ . By Lemma 19 (i)–(iii), the set of all possible values of  $c$  of a vertex is as given in the curly brackets below the vertex. The colors of incidences at the white vertex (i.e.,  $v$ ) are given at the edges adjacent to  $v$ .

Obviously, all the incidences at vertex  $v$  are colored with different colors from  $A_f(v)$ . Observe that the set of colors  $A_f(v)$  is an interval of integers.

We prove that the coloring  $f$  is an incidence coloring. It is enough to prove that for each vertex  $v \in V(G)$  and each vertex  $w \in N_G(v)$  we have  $f(v, \{v, w\}) \notin A_f(w)$ , or, equivalently,  $f(v, \{v, w\}) < \min A_f(w)$  or  $f(v, \{v, w\}) > \max A_f(w)$ .

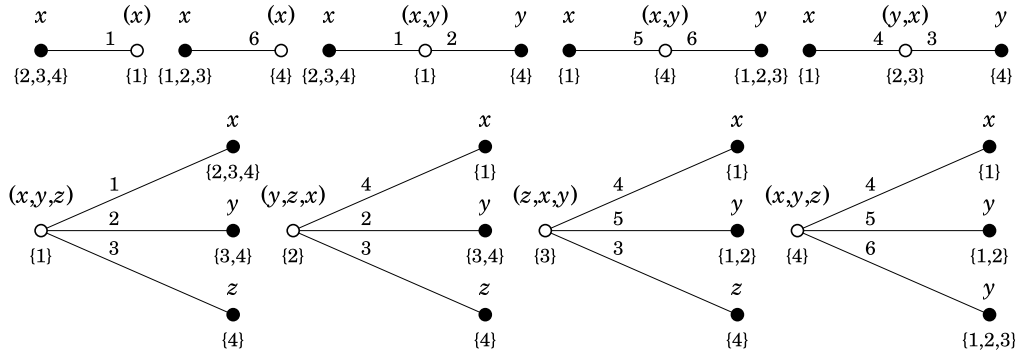


Figure 2. Interval coloring of incidences at the white vertex  $v$ , according to its degree and the values of  $c$  at the neighbors  $x, y, z$  of  $v$ . The set of possible values of  $c$  of a vertex is given in the curly brackets below the vertex. The list  $L_f(v)$  is given above the white vertex  $v$ .

Suppose that  $c(v) = 1$ . Then,  $A_f(v) \subset \{1, 2, 3\}$  and  $\min A_f(v) = 1$ . By the construction of  $L_f(v)$  we have: if  $\deg_G(v) \geq 1$ , then  $c(v_1) \in \{2, 3, 4\}$ , and if  $\deg_G(v) = 2$ , then  $c(v_2) = 4$ , and if  $\deg_G(v) = 3$ , then  $c(v_2) \in \{3, 4\}$  and  $c(v_3) = 4$  (see Figure 2). Hence, for each  $i \in \{1, \dots, \deg_G(v)\}$  we have  $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 < i + 1 \leq \min A_f(v_i)$ .

Suppose that  $c(v) = 2$ . Then,  $A_f(v) \subset \{2, 3, 4\}$ . Let  $\deg_G(v) = 3$ . Hence,  $\min A_f(v) = 2$ , and  $c(v_1) \in \{3, 4\}$  and  $c(v_2) = 4 \wedge c(v_3) = 1$ . Thus,  $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 = i + 1 < i + 2 \leq \min A_f(v_i)$ , for  $i \in \{1, 2\}$ , and  $f(v, \{v, v_3\}) = \min A_f(v) + 2 = 4 > 3 \geq \max A_f(v_3)$ . Let  $\deg_G(v) = 2$ . Hence,  $\min A_f(v) = 3$ , and  $c(v_1) = 4$  and  $c(v_2) = 1$ . Thus,  $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$  and  $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$ .

Suppose that  $c(v) = 3$ . Then,  $A_f(v) \subset \{3, 4, 5\}$  and  $\min A_f(v) = 3$ . Let  $\deg_G(v) = 3$ . Hence,  $c(v_1) = 4$  and  $c(v_2) = 1$  and  $c(v_3) \in \{1, 2\}$ . Thus,  $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$ , and  $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 > i + 1 \geq \max A_f(v_i)$ , for  $i \in \{2, 3\}$ . Let  $\deg_G(v) = 2$ . Hence,  $c(v_1) = 4$  and  $c(v_2) = 1$ . Thus,  $f(v, \{v, v_1\}) = 3 < 4 \leq \min A_f(v_1)$  and  $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$ .

Suppose that  $c(v) = 4$ . Then,  $A_f(v) \subset \{4, 5, 6\}$ . Let  $\deg_G(v) = 3$ . Hence,  $c(v_1) = 1$  and  $c(v_2) \in \{1, 2\}$  and  $c(v_3) \in \{1, 2, 3\}$  and  $c(v_2) \leq c(v_3)$ . Thus,  $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 \geq i + 3 > i + 2 \geq \max A_f(v_i)$ , for each  $i \in \{1, 2, 3\}$ . Let  $\deg_G(v) = 2$ . Hence,  $c(v_1) = 1$  and  $c(v_2) \in \{1, 2, 3\}$ , and  $A_f(v) = \{5, 6\}$ . Thus,  $f(v, \{v, v_1\}) = 5 > \max A_f(v_1)$  and  $f(v, \{v, v_2\}) = 6 > \max A_f(v_2)$ . Let  $\deg_G(v) = 1$ . Hence,  $c(v_1) \in \{1, 2, 3\}$ . Thus,  $f(v, \{v, v_1\}) = 6 > 5 \geq \max A_f(v_1)$ .

In all the cases we proved that  $f(v, \{v, v_i\}) \notin A_f(v_i)$  for each  $v_i \in N_G(v)$ . Thus,  $f$  is an interval incidence 6-coloring of  $G$ . ■

## 4. SUMMARY

In this paper we proved that for any subcubic graph  $G$ ,  $\chi_{ii}(G) \leq 2\Delta(G)$ . In [14] we proved that the upper bound of  $2\Delta(G)$  on  $\chi_{ii}(G)$  holds for each complete  $k$ -partite graph  $G$  and this bound is valid for other classes of graphs. Thus, we state the following

**Conjecture 22** [Interval Incidence Coloring Conjecture (IICC)]. *For any graph  $G$ ,  $\chi_{ii}(G) \leq 2\Delta(G)$ .*

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