

INTERVAL INCIDENCE COLORING OF SUBCUBIC GRAPHS¹

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Abstract

In this paper we study the problem of interval incidence coloring of subcubic graphs. In [14] the authors proved that the interval incidence 4-coloring problem is polynomially solvable and the interval incidence 5-coloring problem is \mathcal{NP} -complete, and they asked if $\chi_{ii}(G) \leq 2\Delta(G)$ holds for an arbitrary graph G . In this paper, we prove that an interval incidence 6-coloring always exists for any subcubic graph G with $\Delta(G) = 3$.

Keywords: interval incidence coloring, incidence coloring, subcubic graph.

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1. INTRODUCTION

In the paper we consider simple nonempty graphs, and we use the standard notation of graph theory. Let $G = (V, E)$ be a simple graph, and let $X \subset V$ be a non-empty set. By $N_G(X) = \{v \in V : \exists u \in X \{v, u\} \in E\}$ we mean the *open*

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neighborhood of X , by $G[X]$ we mean the subgraph of G induced by the set X , and by $G \setminus X$ we mean the graph $G[V \setminus X]$. We say that X is a *dominating set* of G if $V = N_G(X) \cup X$, and we say that X is a *total dominating set* if $V = N_G(X)$. In what follows we use $N_G(v)$ instead of $N_G(\{v\})$. Let $\deg_G(v) = |N_G(v)|$ be the degree of a vertex $v \in V(G)$. By $n(G)$, $\Delta(G)$ and $\delta(G)$ we denote the number of vertices of G , the maximum and the minimum degree of a vertex of G , respectively. By a subcubic graph G we mean a graph with $\Delta(G) \leq 3$. By an *isolated vertex* (in a graph G) we mean a vertex $v \in V(G)$ with $\deg_G(v) = 0$, and by an *isolated edge* (in a graph G) we mean an edge $e = \{u, v\}$ such that $\deg_G(u) = \deg_G(v) = 1$. We say that $X \subset V(G)$ is an *independent set* if each vertex of $G[X]$ is isolated in $G[X]$. By a *pendant vertex* we mean a vertex of degree 1.

For a given graph $G = (V, E)$, we define an *incidence* as a pair (v, e) , where vertex $v \in V$ is one of the endpoints of edge $e \in E$, i.e., $v \in e$. The set of all incidences of G will be denoted by $I(G)$, thus $I(G) = \{(v, e) : v \in V \wedge e \in E \wedge v \in e\}$. We say that two incidences (v, e) and (w, f) are *adjacent* if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}$, $f = \{w, u\}$ and $v \neq u$.

By an *incidence coloring* of G we mean a function $c: I(G) \rightarrow \mathbb{N}$ such that $c((v, e)) \neq c((w, f))$ for any two adjacent incidences (v, e) and (w, f) . The *incidence coloring number* of G , denoted by $\chi_i(G)$, is the smallest number of colors in an incidence coloring of G . In what follows we use the simplified notation $c(v, e)$ instead of $c((v, e))$.

A finite nonempty set $A \subset \mathbb{N}$ is an *interval* if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring c of graph G and $v \in V(G)$ let $A_c(v) = \{c(v, e) : v \in e \wedge e \in E(G)\}$. By an *interval incidence coloring* of a graph G we mean an incidence coloring c of G such that for each vertex $v \in V(G)$ the set $A_c(v)$ is an interval. By an *interval incidence k -coloring* we mean an interval incidence coloring using all colors from the set $\{1, \dots, k\}$. The *interval incidence coloring number* of G , denoted by $\chi_{ii}(G)$, is the smallest number of colors in an interval incidence coloring of G .

1.1. Background and previous results

Alon *et al.* [1] defined the problem of partitioning a graph into the minimal number of star forests. Brualdi and Massey [3] formulated a model of incidence coloring of graphs with references to certain models of coloring of graphs, such as strong edge and vertex coloring of graphs. Guiduli [9] observed that the problem of incidence coloring of graphs is a special case of the problem of partitioning a symmetric digraph into directed star forests.

In [3] the authors conjectured that $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph G (*incidence coloring conjecture*, shortly ICC). This conjecture was disproved by Guiduli in [9] who observed that Paley graphs have incidence coloring number at

least $\Delta + \Omega(\log \Delta)$. In fact, he used the crucial result from [1]. For many classes of graphs it is shown that the incidence coloring number is at most $\Delta + 2$, e.g., trees and cycles [3], complete graphs [3], complete bipartite graphs [3] (proof corrected in [19]), planar graphs with girth at least 11 or with girth at least 6 and maximum degree at least 5 [5], partial 2-trees (i.e., K_4 -minor free graphs) [4], hypercubes [18], complete k -partite graphs [15].

In [17] the author proved that ICC holds for subcubic graphs. The incidence 4-colorability problem is \mathcal{NP} -complete for *semicubic* graphs (i.e., subcubic graphs with vertex degrees equal to 1 or 3) [16] and for semicubic bipartite graphs [15].

In this paper we consider a restriction of the problem of incidence coloring of graphs in which the colors of incidences at a vertex form an interval. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring (see, e.g., [2, 6, 8]), which can be applied to the open-shop scheduling problem [6, 7]. In [11] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [12] the authors studied applications in one-multicast transmission in multifiber WDM networks.

In [13] the authors proved that the problem of interval incidence k -coloring of bipartite graphs is polynomial for each $k \leq 6$ and $\Delta \leq 3$, polynomial for $k = 5$ and $\Delta = 4$, and \mathcal{NP} -complete for $k = 6$ and $\Delta = 4$. In [14] the authors proved certain lower and upper bounds on the interval incidence coloring number, e.g., $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$ for an arbitrary graph G , and they determined the exact values of χ_{ii} for some basic classes of graphs (e.g., complete k -partite graphs). In [14] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem of deciding whether $\chi_{ii} \leq 4$ is easy, and $\chi_{ii} \leq 5$ is \mathcal{NP} -complete. The problem of interval incidence 6-coloring of subcubic graphs remained unsolved.

1.2. Main results

Our main result in the paper is Theorem 21 which states $\chi_{ii}(G) \leq 6$ for every subcubic graph G . To prove it, we state and prove Theorem 8: in any subcubic graph G with $\delta(G) \geq 2$ there is a maximal induced bipartite subgraph of G without isolated vertices, or equivalently, G has a total dominating set S such that $G[S]$ is a bipartite graph.

2. MAXIMAL INDUCED BIPARTITE SUBGRAPHS WITHOUT ISOLATED VERTICES

In this section we prove (in Theorem 8) that any subcubic graph G with $\delta(G) \geq 2$ contains a maximal induced bipartite subgraph without isolated vertices.



2.1. Introductory properties

By $H \subset G$ we mean that H is a subgraph of G . By $H \sqsubset G$ we mean that H is an induced subgraph of G , i.e., $H = G[V(H)]$.

Observation 1. *If $G_1 \sqsubset G_2$ and $G_2 \sqsubset G_3$, then $G_1 \sqsubset G_3$.*

Observation 2. *Let $G_1 \sqsubset G$ and $G_2 \sqsubset G$. If $G_1 \subset G_2$, then $G_1 \sqsubset G_2$.*

Let $\mathcal{B}(G) = \{H \sqsubset G : N_G(V(H)) = V(G) \wedge H \text{ is bipartite}\}$, i.e., the set of all induced bipartite subgraphs of a given graph G such that $V(H)$ is a total dominating set of G . If $H \in \mathcal{B}(G)$, then $V(H)$ is a total dominating set of G and, obviously, H has no isolated vertices.

In the following, let G be any graph. Let $\hat{\mathcal{B}}(G)$ be the subfamily of $\mathcal{B}(G)$ consisting of all the elements (graphs) in $\mathcal{B}(G)$ that are maximal with respect to the subgraph relation (\subset).

Observation 3. *If $H \in \mathcal{B}(G)$, then there is $H' \in \hat{\mathcal{B}}(G)$ such that $H \subset H'$.*

By Observations 2 and 3 we have

Observation 4. *Let $H \in \mathcal{B}(G)$. Then, $H \in \hat{\mathcal{B}}(G)$ if and only if for each $v \in V(G) \setminus V(H)$ the subgraph $G[V(H) \cup \{v\}]$ is not bipartite.*

Observation 5. *If $H \in \mathcal{B}(G) \setminus \hat{\mathcal{B}}(G)$, then there is a vertex $v \in V(G) \setminus V(H)$ such that $G[V(H) \cup \{v\}] \in \mathcal{B}(G)$.*

Since any dominating set $S \subset V(G)$ is a total dominating set if and only if $G[S]$ has no isolated vertices, we have

Observation 6. *Let G be an arbitrary graph and let $H \subset G$. Then, $H \in \hat{\mathcal{B}}(G)$ if and only if H is a maximal induced bipartite subgraph (of G) without isolated vertices.*

Let \mathcal{G}_3^2 be the family of subcubic graphs without isolated and pendant vertices, i.e., each vertex in a graph of this family has degree 2 or 3. Let \mathcal{M}_3^2 be the subfamily of \mathcal{G}_3^2 consisting of all the graphs for which there is no maximal induced bipartite subgraph without isolated vertices. Let us denote by \mathcal{M} the set of elements in \mathcal{M}_3^2 that are minimal with respect to the subgraph relation (\subset). By Observation 6 we have

Observation 7. *Let $G \in \mathcal{G}_3^2$. Then, $G \in \mathcal{M}_3^2 \Leftrightarrow \mathcal{B}(G) = \emptyset \Leftrightarrow \hat{\mathcal{B}}(G) = \emptyset$.*

2.2. Main Theorem

Theorem 8. *Let G be a subcubic graph with $\delta(G) \geq 2$. Then, G has a maximal induced bipartite subgraph without isolated vertices.*

By Observation 7, Theorem 8 is equivalent to $\mathcal{M} = \emptyset$. First, we prove some structural properties of graphs from \mathcal{M} .

Lemma 9. *Let $G \in \mathcal{M}$. Then, G is a connected graph and $\Delta(G) = 3$.*

Proof. Let $G \in \mathcal{M}$. Let us assume to the contrary that $G = G_1 \cup G_2$, where G_1 and G_2 are disjoint graphs (without common vertices). Since $G_i \subsetneq G \in \mathcal{M}$ and $G_i \in \mathcal{G}_3^2$, we have $G_i \notin \mathcal{M}_3^2$, for $i \in \{1, 2\}$. Hence, there exist $H_1 \in \hat{\mathcal{B}}(G_1)$ and $H_2 \in \hat{\mathcal{B}}(G_2)$. Thus, $H_1 \cup H_2 \in \hat{\mathcal{B}}(G)$, a contradiction.

Since every cycle is either a bipartite graph or it becomes a bipartite graph after deleting an arbitrary vertex, G is not a cycle, which implies $\Delta(G) = 3$. ■

Lemma 10. *Let $G \in \mathcal{M}$ and let v be a vertex of degree 2 in G . Then, every neighbor of v in G has degree 3.*

Proof. Let $G \in \mathcal{M}$. Suppose to the contrary that there are two adjacent vertices of degree 2. Since G is not a cycle (by Lemma 9), there is a subgraph P of G with vertex set $\{v_0, \dots, v_{k+1}\}$ and edges $\{v_i, v_{i+1}\}$, for $i \in \{0, \dots, k\}$, such that $\deg_G(v_0) = \deg_G(v_{k+1}) = 3$, and $\deg_G(v_i) = 2$ for $i \in \{1, \dots, k\}$, where $k \geq 2$.

Suppose $v_0 \neq v_{k+1}$. Since $G' = G \setminus \{v_1, \dots, v_k\} \sqsubset G \in \mathcal{M}$ and $G' \in \mathcal{G}_3^2$, we have $G' \notin \mathcal{M}_3^2$. Hence, there exists $H' \in \hat{\mathcal{B}}(G')$, and $H' \sqsubset G$ by Observation 1. If $v_0 \in V(H')$, then let $H = G[V(H') \cup \{v_1, \dots, v_{k-1}\}]$, otherwise, let $H = G[V(H') \cup \{v_1, \dots, v_k\}]$. In both cases, $H \sqsubset G$, H is a bipartite graph, and $V(H)$ is a total dominating set, i.e., $H \in \mathcal{B}(G)$. By Observation 7 we get a contradiction.

Suppose $v_0 = v_{k+1}$. Since $\deg_G(v_0) = 3$, there is $c \in N_G(v_0) \setminus \{v_1, v_k\}$. If $\deg_G(c) = 3$, then let $G' = G \setminus \{v_0, \dots, v_k\}$. If $\deg_G(c) = 2$, then let $G' = G \setminus \{v_0, \dots, v_k, c\}$. In both cases, $G' \sqsubset G$ and $G \neq G' \in \mathcal{G}_3^2$. Hence, there is $H' \in \hat{\mathcal{B}}(G')$. Let $H = G[V(H') \cup \{v_0, \dots, v_{k-1}\}]$. Thus, $H \in \mathcal{B}(G)$, a contradiction. ■

Lemma 11. *If $G \in \mathcal{G}_3^2$ contains G_0 as a subgraph (see Figure 1), where vertices $v_2, v_3 \in V(G_0)$ are of degree 2 in G , then $G \notin \mathcal{M}$.*

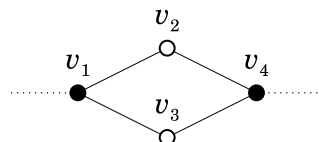


Figure 1. The subgraph G_0 of a graph G .

Proof. Suppose to the contrary that $G \in \mathcal{M}$. Suppose $G_0 \subset G$. The other possible edges in G are marked by the dotted lines (in Figure 1).

By $\deg_G(v_2) = \deg_G(v_3) = 2$, from Lemma 10 we have $\deg_G(v_1) = \deg_G(v_4) = 3$. Since $G' = G \setminus \{v_3\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. Hence, $v_1 \in V(H')$ or $v_4 \in V(H')$. Thus, $H' \in \mathcal{B}(G)$, a contradiction. ■

Lemma 12. *Let $G \in \mathcal{M}$ and let v be a vertex of degree 3 in G . Then, at most one neighbor of v has degree 2.*

Proof. Let $G \in \mathcal{M}$ and let $N_G(v) = \{x, y, z\}$. Suppose to the contrary that at least two vertices from $N_G(v)$ have degree 2. Let $\deg_G(x) = \deg_G(y) = 2$. Let $\{v_x\} = N_G(x) \setminus \{v\}$ and $\{v_y\} = N_G(y) \setminus \{v\}$. By Lemma 10, $\deg_G(v_x) = \deg_G(v_y) = 3$.

Suppose $\deg_G(z) = 2$. Let $\{v_z\} = N_G(z) \setminus \{v\}$. By Lemma 10, $\deg_G(v_z) = 3$. If any two of the vertices v_x, v_y, v_z are equal, then by Lemma 11 (i.e., because $G_0 \sqsubset G$) we get a contradiction. Hence, vertices v_x, v_y, v_z are different. Since $G' = G \setminus \{x, y, z, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. Thus, $G[V(H') \cup \{v, x\}] \in \mathcal{B}(G)$, a contradiction.

Suppose $\deg_G(z) = 3$. If $v_x = v_y$, then by Lemma 11 we get a contradiction. Hence, $v_x \neq v_y$. Suppose $z = v_x$ (the case $z = v_y$ can be treated analogously). Since $G_x = G \setminus \{x\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H_x \in \hat{\mathcal{B}}(G_x)$. Since H_x is maximal in $\mathcal{B}(G)$, we have $v \in V(H_x)$ or $z \in V(H_x)$. Thus, $H_x \in \mathcal{B}(G)$, a contradiction. Then, vertices v_x, v_y, z are different. Since $G' = G \setminus \{x, y, v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. If $z \in V(H')$, then let $A = V(H') \cup \{v\}$. If $z \notin V(H')$, then let $A = V(H') \cup \{v, x\}$. In both cases, $G[A] \in \mathcal{B}(G)$, a contradiction. ■

Let G be any subcubic graph. We say that $H \subset G$ is a Q -cycle (of G) if:

- (q₁) for each $v \in V(H)$, $\deg_G(v) = 3$, and
- (q₂) $H \sqsubset G$ and H is isomorphic to a cycle, i.e., H is an induced cycle, and
- (q₃) for each vertex $v \in V(G) \setminus V(H)$, $|N_G(v) \cap V(H)| \leq 1$.

Lemma 13. *Let $G \in \mathcal{M}$. Let $v \in V(G)$ have all neighbors of degree 3. Then, for each $x \in N_G(v)$ there is a Q -cycle C_x such that $x \in V(C_x)$, $v \notin V(C_x)$ and $N_G(v) \cap V(C_x) = \{x\}$.*

Proof. Let $G \in \mathcal{M}$ and let $v \in V(G)$ be a vertex with all neighbors of degree 3. Since $G' = G \setminus \{v\} \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$. Hence, $N_G(v) \cap V(H') = \emptyset$.

Let $x \in N_G(v)$ and let $N_G(x) = \{a, b, v\}$. Since H' is bipartite and maximal in $\mathcal{B}(G)$, we have that a and b belong to the same connected component of H' , and the length of each path in H' from a to b is odd. Let $P \subset H'$ be a path joining $x_1 = a$ and $x_{s-1} = b$ (s is odd), with vertex set $\{x_1, \dots, x_{s-1}\}$ and edges $\{x_i, x_{i+1}\}$, for $i \in \{1, \dots, s-2\}$. Let $x_0 = x$ and let C_x be the graph with $V(C_x) = V(P) \cup \{x_0\}$, and $E(C_x) = E(P) \cup \{\{x_{s-1}, x_0\}, \{x_0, x_1\}\}$. Since $P \subset H'$, we have $N_G(v) \cap V(C_x) = \{x\}$, and $v \notin V(C_x)$.

Claim 14. For each $i \in \{1, \dots, s - 1\}$, the following properties are satisfied:

- (p₁) $\deg_G(x_i) = 3$,
- (p₂) $N_G(x_i) = \{a_i, x_{(i-1) \bmod s}, x_{(i+1) \bmod s}\}$, where $a_i \in V(H') \setminus V(C_x)$,
- (p₃) $N_G(a_i) \cap V(H') = \{x_i\}$.

Proof. We proceed by induction on i . Suppose $i = 1$. Let $X = V(H') \setminus \{x_i\} \cup \{x, v\}$. Hence, $G[X]$ is bipartite. If $\deg_G(x_i) = 2$ or $N_G(a_i) \cap V(H') \neq \{x_i\}$, then $G[X] \in \mathcal{B}(G)$, a contradiction. If $a_i \notin V(H') \setminus V(C_x)$, then $a_i \notin V(H')$ or $a_i \in V(C_x)$. If $a_i \notin V(H')$, then $N_G(a_i) \cap V(H') \neq \{x_i\}$ (otherwise H' is not maximal in $\mathcal{B}(G)$), a contradiction. If $a_i \in V(C_x)$, then $G[X] \in \mathcal{B}(G)$, a contradiction.

Suppose the properties (p₁), (p₂), (p₃) hold for $1, \dots, i - 1$ ($2 \leq i \leq s - 1$). Hence, each path joining x_1 and x_{s-1} in H' contains x_1, \dots, x_i . Let $X = V(H') \setminus \{x_i\} \cup \{x, v\}$. Hence, $G[X]$ is bipartite. The rest of the proof of properties (p₁), (p₂), (p₃) for i is literally the same as in the case $i = 1$. □

We show that C_x is a Q -cycle. Since $\deg_G(x) = 3$, by (p₁) we have (q₁). Since $v \notin V(C_x)$ and $a_i \notin V(C_x)$ (by (p₂)), for $i \in \{1, \dots, s - 1\}$, we have that C_x is an induced cycle of G . Since $a_i \in V(H')$ (by (p₂)), we have $a_i \neq v$. Thus, by (p₃) we get $|N_G(a_i) \cap V(C_x)| \leq 1$, for $i \in \{1, \dots, s - 1\}$. ■

We say that H is a Q_2 -cycle (of G) if H is a Q -cycle of G , and it holds (q₄) for each $v \in N_G(V(H)) \setminus V(H)$, $\deg_G(v) = 2$.

Lemma 15. Let $G \in \mathcal{M}$ and let C be a Q -cycle of G . Then, C is a Q_2 -cycle.

Proof. Let $G \in \mathcal{M}$. Let C be a Q -cycle of G with the vertex set $\{x_0, \dots, x_{s-1}\}$, and edges $\{x_0, x_1\}, \dots, \{x_{s-2}, x_{s-1}\}, \{x_{s-1}, x_0\}$. Let $S = \{0, \dots, s - 1\}$. Let $\{a_i\} = N_G(x_i) \setminus V(C)$, for $i \in S$. If $\deg_G(a_i) = 2$, then let $\{b_i\} = N_G(a_i) \setminus \{x_i\}$. Hence, $b_i \notin V(C)$. By Lemma 10 we have $\deg_G(b_i) = 3$. Let $G' = G \setminus (V(C) \cup \{a_i : \deg_G(a_i) = 2 \wedge i \in S\})$. Since $G' \in \mathcal{G}_3^2 \setminus \mathcal{M}_3^2$, there is $H' \in \hat{\mathcal{B}}(G')$.

Suppose to the contrary that C is not a Q_2 -cycle, i.e., there exists $r \in S$ such that $\deg_G(a_r) = 3$. Let $f: V(G') \rightarrow \{0, 1\}$ be the characteristic function of $V(H')$, i.e., $f(u) = 1$ if and only if $u \in V(H')$. Let us consider two cases.

- (i) For each $i \in S$: $\deg_G(a_i) = 2 \Rightarrow f(b_i) = 0$ and $\deg_G(a_i) = 3 \Rightarrow f(a_i) = 0$.
- (ii) For some $t \in S$: $\deg_G(a_t) = 2 \wedge f(b_t) = 1$ or $\deg_G(a_t) = 3 \wedge f(a_t) = 1$.

We construct a function $\tilde{f}: V(G) \rightarrow \{0, 1\}$ such that $\tilde{f}(u) = f(u)$ for each $u \in V(G')$. Let $u \in V(G) \setminus V(G')$. We define $\tilde{f}(u)$ depending on cases (i), (ii).

- (i) Let $\tilde{f}(x_r) = 0$ and let $\tilde{f}(x_j) = 1$, for each $j \in S \setminus \{r\}$. For each $j \in S$, if $\deg_G(a_j) = 2$, then $\tilde{f}(a_j) = 1$,

- (ii) Take any $t \in S$, if exists, such that $\deg_G(a_t) = 2 \wedge f(b_t) = 1$ and let $\tilde{f}(a_t) = 1$. Then, for each $j \in S, j \neq t$, if $\deg_G(a_j) = 2$, then $\tilde{f}(a_j) = 1 - f(b_j)$. Next, for each $j \in S$, if $\deg_G(a_j) = 2 \wedge f(b_j) = 0$, then $\tilde{f}(x_j) = 1$. Finally, for each $j \in S$, if $\deg_G(a_j) = 3$ or $\deg_G(a_j) = 2 \wedge f(b_j) = 1$, then $\tilde{f}(x_j) = 1 - \tilde{f}(a_{(j+1) \bmod s})$.

Let $H = G[\{u \in V(G) : \tilde{f}(u) = 1\}]$. In the case (i), $x_r \notin V(H) \cap V(C)$. Hence, H is a bipartite graph. For each $u \in V(G) \setminus V(G'), u \neq x_r$, we have that $u \in V(H)$. Thus, $V(H)$ is a total dominating set of G and $H \in \mathcal{B}(G)$, a contradiction.

In case (ii), if there is no $t \in S$ such that $\deg_G(a_t) = 2 \wedge f(b_t) = 1$, then, by assumption, there is $t \in S$ such that $\deg_G(a_t) = 3 \wedge f(a_t) = 1$, so finally $\tilde{f}(a_t) = 1$ for some $t \in S$. Hence, there is $p \in S$ such that $\tilde{f}(x_p) = 0$. Thus, $V(C) \setminus V(H) \neq \emptyset$.

Let us remind that for each $i \in S \setminus \{t\}$, if $\deg_G(a_i) = 2$ and $\tilde{f}(a_i) = 1$, then $f(b_i) = 0$. Let $X = \{i \in S : \deg_G(a_i) = 3 \wedge \tilde{f}(a_i) = 1\} \cup \{t\}$. Suppose that for some two $i, j \in X$, there is a path in H between a_i and a_j with successive vertices $x_i, x_{(i+1) \bmod s}, \dots, x_j$. Hence, $\tilde{f}(x_i) = \tilde{f}(x_{(i+1) \bmod s}) = \dots = \tilde{f}(x_j) = 1$, which implies that $\tilde{f}(a_{(i+1) \bmod s}) = 0, \tilde{f}(a_{(i+2) \bmod s}) = 0, \dots, \tilde{f}(a_j) = 0$, a contradiction. Thus, H is a bipartite graph.

For every $j \in S$ we have $N_G(a_j) \cap V(H) \neq \emptyset$, and $\tilde{f}(a_j) = 1$ or $\tilde{f}(a_j) = 0 \wedge \tilde{f}(x_{(j-1) \bmod s}) = 1$. Hence, we get $N_G(x_j) \cap V(H) \neq \emptyset$. Thus, $V(H)$ is a total dominating set and $H \in \mathcal{B}(G)$, a contradiction. ■

By Lemmas 10, 12, 13 and Lemma 15, and by the definition of Q_2 -cycle we have the following corollary.

Corollary 16. *Let $G \in \mathcal{M}$ and $v \in V(G)$. The following properties are satisfied:*

- (i) $\deg_G(v) = 2$ if and only if vertex v has all neighbors of degree 3,
- (ii) $\deg_G(v) = 3$ if and only if exactly one neighbor of v has degree 2,
- (iii) if $\deg_G(v) = 3$, then there is exactly one Q_2 -cycle containing v ,
- (iv) if $\deg_G(v) = 2$, then vertex v has two neighbors from disjoint Q_2 -cycles.

By Corollary 16 we have the next corollary.

Corollary 17. *Let $G \in \mathcal{M}$. The graph G satisfies the following properties:*

- (i) there is an integer $q \geq 1$ such that $V(G) = D \cup \bigcup_{i=1}^q V(C_i)$, where for each $i \in \{1, \dots, q\}$ the graph C_i is a Q_2 -cycle and D is the set of all vertices of degree 2,
- (ii) $E(G) = \{\{u, v\} : \exists i \in \{1, \dots, q\} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\}$.

Proof of Theorem 8. Suppose to the contrary that $G \in \mathcal{M}$.

By Corollary 17, there is $q \geq 1$ such that $V(G) = D \cup \bigcup_{i=1}^q V(C_i)$, where for each $i \in \{1, \dots, q\}$ the graph C_i is a Q_2 -cycle and D is the set of all vertices of degree 2, and

$$E(G) = \{\{u, v\} : \exists_{i \in \{1, \dots, q\}} (\{u, v\} \in E(C_i) \vee (u \in V(C_i) \wedge v \in D))\}.$$

Let $Q = (D \cup \bigcup_{i=1}^q \{c_i\}, E_Q)$, where for each $i \in \{1, \dots, q\}$ vertex c_i corresponds to the cycle C_i and

$$E_Q = \{\{v, c_i\} : i \in \{1, \dots, q\} \wedge v \in D \wedge \exists_{x \in V(C_i)} \{v, x\} \in E(G)\}.$$

By Corollary 16 and Corollary 17 we have that Q is a simple bipartite graph with partitions D and $C = \bigcup_{i=1}^q \{c_i\}$. Obviously, for all vertices $v \in D$ and $c \in C$ we have that $\deg_Q(v) = 2 < \deg_Q(c)$. Thus, by Hall's Marriage Theorem [10] there is a matching S in Q covering all vertices from partition C .

Let

$$S' = \{\{v, x\} \in E(G) : v \in D \wedge \exists_{i \in \{1, \dots, q\}} \{v, c_i\} \in S \wedge x \in V(C_i)\}$$

and let

$$V' = \left\{ x \in \bigcup_{i=1}^q V(C_i) : \exists_{e \in S'} x \in e \right\}.$$

Let $H = G[V(G) \setminus (D \cup V')]$. For each $i \in \{1, \dots, q\}$ there is x such that $\{x\} = V(C_i) \cap V'$ and $N_G(x) \cap V(H) \neq \emptyset$. If $y \in V(C_i)$ and $x \neq y$, then $N_G(y) \cap V(H) \neq \emptyset$. Hence, H is an induced bipartite graph without isolated vertices. Since for each $v \in D$ at most one neighbor of v belongs to V' , we have $N_G(v) \cap V(H) \neq \emptyset$. Thus, $N_G(V(H)) = V(G)$ and $H \in \mathcal{B}(G)$, a contradiction. ■

3. INTERVAL INCIDENCE 6-COLORING OF SUBCUBIC GRAPHS

In this section we prove our main result, i.e., Theorem 21, which states $\chi_{ii}(G) \leq 2\Delta(G)$ for each subcubic graph G . By Theorem 8 we have the following lemma.

Lemma 18. *Let G be a connected graph and $G \in \mathcal{G}_3^2$. Let $H \in \hat{\mathcal{B}}(G)$ and let $A, B \subset V(H)$ be any partition of $V(H)$, such that A and B are disjoint independent sets and $A \cup B = V(H)$. Then, A and B are disjoint independent dominating sets, and the graph $G[V(G) \setminus V(H)]$ has only isolated vertices and isolated edges.*

Proof. Let $v \in V(G) \setminus V(H)$. If $N_G(v) \cap V(H) \subset A$ or $N_G(v) \cap V(H) \subset B$, then $G[V(H) \cup \{v\}]$ is a bipartite graph, a contradiction. Thus, $N_G(v) \cap A \neq \emptyset$ and $N_G(v) \cap B \neq \emptyset$. Let $v \in A$ ($v \in B$). Since H is an induced graph without isolated vertices, we have $v \in N_G(B)$ ($v \in N_G(A)$). Hence, A and B are disjoint independent dominating sets.

Since G is subcubic and $|N_G(v) \cap V(H)| \geq 2$ for any $v \in V(G) \setminus V(H)$, graph $G[V(G) \setminus V(H)]$ has only isolated vertices and isolated edges. ■

Lemma 19. *Let G be a subcubic non-bipartite graph with $\Delta(G) = 3$. Then, there is a vertex coloring $c: V(G) \rightarrow \{1, 2, 3, 4\}$ such that for each $v \in V(G)$ the following properties hold:*

- (i) if $\deg_G(v) = 1$, then $c(v) \in \{1, 4\}$,
 - (ii) if $\deg_G(v) \geq 2$ and $c(v) \neq p$, then $a_p(v) \geq 1$, for $p \in \{1, 4\}$,
 - (iii) $a_i(v) \leq |c(v) - i|$, for $i \in \{1, 2, 3, 4\}$,
- where $a_i(v) = |\{w \in N_G(v) : c(w) = i\}|$, for $i \in \{1, 2, 3, 4\}$.

Proof. If $\delta(G) = 1$, then we successively remove pendant vertices from graph G , until there is no pendant vertex. Let us denote the resulting graph by G' . Obviously, $\delta(G') \geq 2$. Let us observe that we cut off all trees attached to G .

By Theorem 8 we have $\hat{B}(G') \neq \emptyset$. Let H be any element of $\hat{B}(G')$ with the largest possible number of vertices.

Let $A, B \subset V(H)$ be any two partite sets of $V(H)$, i.e., A and B are disjoint independent sets and $A \cup B = V(H)$. By Lemma 18, A and B are disjoint independent dominating sets of G' , and the graph $G[V(G') \setminus V(H)]$ has only isolated vertices and isolated edges. Let $I_i \subset V(G') \setminus V(H)$ be the set of all vertices of degree i in G' , for $i \in \{2, 3\}$. Let us define the partition $I_3 = I_3^A \cup I_3^B \cup I_3^2$:

- $I_3^A = \{v \in I_3 : |N_{G'}(v) \cap A| = 2 \wedge |N_{G'}(v) \cap B| = 1\}$,
- $I_3^B = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \wedge |N_{G'}(v) \cap B| = 2\}$,
- $I_3^2 = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \wedge |N_{G'}(v) \cap B| = 1\}$.

Note that I_2, I_3^A, I_3^B are independent sets in G' , each vertex $v \in I_3^2$ belongs to an isolated edge in $G'[I_3^2]$, and each vertex from I_2 has neighbors from A and B .

Let us define a coloring $c: V(G) \rightarrow \{1, 2, 3, 4\}$ in the following steps.

- (C₁) If $v \in A$, then $c(v) = 1$, and if $v \in B$, then $c(v) = 4$.
- (C₂) If $v \in I_3^B$, then $c(v) = 2$, and if $v \in I_3^A$, then $c(v) = 3$.
- (C₃) For each successive $v \in I_2$ we assign a color following the algorithm: if $c(v)$ is not determined, then let $\{u\} = N_{G'}(v) \cap A$. If there is $x \in N_{G'}(u)$ such that $c(x) = 2$, then let $c(v) = 3$. Otherwise, for each vertex $x \in N_{G'}(u)$ either $c(x) \in \{3, 4\}$ or $c(x)$ is not determined, and then let $c(v) = 2$.
- (C₄) For each successive $\{v, w\} \in E(G'[I_3^2])$ we assign colors to both v and w following the algorithm: if $c(v)$ and $c(w)$ are not determined, then let $\{u\} = N_{G'}(v) \cap A$. If there is $x \in N_{G'}(u)$ such that $c(x) = 2$, then let $c(v) = 3$ and $c(w) = 2$. Otherwise, for each vertex $x \in N_{G'}(u)$ either $c(x) \in \{3, 4\}$ or $c(x)$ is not determined, and then let $c(v) = 2$ and $c(w) = 3$.
- (C₅) For each $v \in V(G')$ such that $\deg_{G'}(v) < \deg_G(v)$, there is a tree T_v such that $V(T_v) \subset V(G) \setminus V(G')$ and let $\{w\} = V(T_v) \cap N_G(v)$. Let $d: V(T_v) \rightarrow$

$\{a, b\}$ be a 2-coloring of T_v such that $d(w) = a$. Suppose $c(v) \leq 2$. For each $u \in V(T_v)$, if $d(u) = a$, then let $c(u) = 4$, and if $d(u) = b$, then let $c(u) = 1$. Suppose $c(v) \geq 3$. For each $u \in V(T_v)$, if $d(u) = a$, then let $c(u) = 1$, and if $d(u) = b$, then let $c(u) = 4$.

In step (C_1) we colored $V(H) = A \cup B$ with colors 1 and 4, in steps (C_2) – (C_4) we colored vertices from $I_2 \cup I_3$ with colors 2 or 3, and in step (C_5) we colored vertices from $V(G) \setminus V(G')$ with colors 1 or 4. Since vertices colored with an arbitrary color form an independent set, c is a vertex 4-coloring of G .

Let $v \in V(G)$ and let $\deg_G(v) = 1$. Then, $v \in V(G) \setminus V(G')$ and, by (C_5) , $c(v) \in \{1, 4\}$. Thus, we get the property (i). Let $\deg_G(v) \geq 2$. If $v \in V(G) \setminus V(G')$, then, by (C_5) , the property (ii) holds. Let $v \in V(G')$. Since A and B are disjoint independent dominating sets of G' , the property (ii) holds.

Since c is a proper coloring of G , there is $a_{c(v)}(v) = 0$ for each $v \in V(G)$.

Let $v \in V(G) \setminus V(G')$. By step (C_5) , $c(v) \in \{1, 4\}$. If $c(v) = 1$, then $a_2(v) = 0$, $a_3(v) \leq 1$ and $a_4(v) \leq 3$. If $c(v) = 4$, then $a_3(v) = 0$, $a_2(v) \leq 1$ and $a_1(v) \leq 3$.

Let $v \in V(G') \setminus V(H)$. If $v \in I_3^A$, then $c(v) = 3$, $a_1(v) = 2$, $a_2(v) = 0$, $a_4(v) = 1$. If $v \in I_3^B$, then $c(v) = 2$, $a_1(v) = 1$, $a_3(v) = 0$, $a_4(v) = 2$. If $v \in I_2$, then $c(v) \in \{2, 3\}$. If $\deg_{G'}(v) = \deg_G(v)$, then $a_1(v) = a_4(v) = 1$, and $a_2(v) = a_3(v) = 0$. If $\deg_{G'}(v) < \deg_G(v)$, then if $c(v) = 2$, then $a_1(v) = 1$, $a_2(v) = a_3(v) = 0$, $a_4(v) = 2$, and if $c(v) = 3$, then $a_1(v) = 2$, $a_2(v) = a_3(v) = 0$, $a_4(v) = 1$. If $v \in I_3^2$, then $c(v) \in \{2, 3\}$. If $c(v) = 2$, then $a_1(v) = a_3(v) = a_4(v) = 1$. If $c(v) = 3$, then $a_1(v) = a_2(v) = a_4(v) = 1$.

Let $v \in A \cup B$. Since A and B are disjoint dominating sets of G' and $H \in \hat{\mathcal{B}}(G')$, it suffices to prove that if $c(v) = 1$, then $a_2(v) \leq 1$, and if $c(v) = 4$, then $a_3(v) \leq 1$.

Suppose to the contrary that $c(v) = 1$ and $a_2(v) = 2$ for some $v \in A$. The case $c(v) = 4$ and $a_3(v) = 2$, for some $v \in B$, is analogous. Let $x, y \in N_{G'}(v)$ such that $c(x) = c(y) = 2$. Since B is a dominating set of G' , there is $w \in N_{G'}(v) \cap B$ with $c(w) = 4$. By the definition of coloring c , we have $v, x, y, w \in V(G')$ and $v, w \in V(H)$.

Since $c(x) = c(y) = 2$, we have $a_1(x) = a_1(y) = 1$, $a_3(x) \leq 1$, $a_3(y) \leq 1$, $1 \leq a_4(x) \leq 2$ and $1 \leq a_4(y) \leq 2$. Let us consider the following cases:

- $x \notin N_{G'}(w)$ and $y \notin N_{G'}(w)$. If edge $\{v, w\}$ is isolated in H , then let $W = V(H) \cup \{x, y\}$. Otherwise, let $W = V(H) \cup \{x, y\} \setminus \{v\}$.
- $x \in N_{G'}(w)$ or $y \in N_{G'}(w)$. Let $W = V(H) \cup \{x, y\} \setminus \{v\}$.

In both cases, the graph $G'[W] \in \mathcal{B}(G')$ and $|V(G'[W])| > |V(H)|$, a contradiction. Thus, the coloring c satisfies the property (iii). ■

Proposition 20. [14] *For any graph G , $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$.*

We prove that an interval incidence 6-coloring always exists for any subcubic graph G with $\Delta(G) = 3$.

Theorem 21. *Let G be a subcubic graph. Then, $\chi_{ii}(G) \leq 2\Delta(G)$.*

Proof. If G is a subcubic bipartite graph, then by Proposition 20 we have $\chi_{ii}(G) \leq 2\Delta(G)$. If $\Delta(G) = 2$, then one can easily construct an interval incidence 4-coloring. Thus, $\chi_{ii}(G) \leq 2\Delta(G)$. Let G be a subcubic non-bipartite graph with $\Delta(G) = 3$. By Lemma 19, there is a vertex coloring $c: V(G) \rightarrow \{1, 2, 3, 4\}$ satisfying the properties (i), (ii), (iii) from Lemma 19.

We construct an incidence coloring $f: I(G) \rightarrow \{1, 2, 3, 4, 5, 6\}$ in three steps.

In the first step, using the coloring c , we define the interval $A_f(v)$ for each vertex $v \in V(G)$, as follows. If $\deg_G(v) = 2$ and $c(v) \in \{2, 3\}$, then let $A_f(v) = \{3, 4\}$. If $c(v) = 4$ and $\deg_G(v) = 1$, then $A_f(v) = \{6\}$. If $c(v) = 4$ and $\deg_G(v) = 2$, then $A_f(v) = \{5, 6\}$. In the other cases, let $A_f(v) = \{c(v), \dots, c(v) + \deg_G(v) - 1\}$. Thus, by Lemma 19 (i)–(iii) we get

- (a₁) if $\deg_G(v) = 1$, then $c(v) \in \{1, 4\}$ and $A_f(v) = \{c(v)\}$,
- (a₂) if $\deg_G(v) = 2$, then if $c(v) \in \{1, 3\}$, then $A_f(v) = \{c(v), c(v) + 1\}$ and if $c(v) \in \{2, 4\}$, then $A_f(v) = \{c(v) + 1, c(v) + 2\}$,
- (a₃) if $\deg_G(v) = 3$, then $A_f(v) = \{c(v), c(v) + 1, c(v) + 2\}$.

In the second step, for each $v \in V(G)$, we construct a sequence $L_f(v)$ (i.e., a linear ordered set) from elements of $N_G(v)$, as follows (see Figure 2).

- (l₁) Suppose $\deg_G(v) = 1$. If $N_G(v) = \{x\}$, then let $L_f(v) = (x)$.
- (l₂) Suppose $\deg_G(v) = 2$. Let $N_G(v) = \{x, y\}$, where $c(x) \leq c(y)$. Then,
 - if $c(v) \in \{1, 4\}$, then let $L_f(v) = (x, y)$,
 - if $c(v) \in \{2, 3\}$, then let $L_f(v) = (y, x)$.
- (l₃) Suppose $\deg_G(v) = 3$. Let $N_G(v) = \{x, y, z\}$, where $c(x) \leq c(y) \leq c(z)$. Then,
 - if $c(v) \in \{1, 4\}$, then let $L_f(v) = (x, y, z)$,
 - if $c(v) = 2$, then let $L_f(v) = (y, z, x)$,
 - if $c(v) = 3$, then let $L_f(v) = (z, x, y)$.

By v_i we mean the i -th element of the sequence $L_f(v)$, i.e., $L_f(v) = (v_1, \dots)$.

In the final step, for each vertex v , we define the incidence coloring f as follows: $f(v, \{v, v_i\}) = \min A_f(v) + i - 1$, for $i \in \{1, \dots, \deg_G(v)\}$.

In Figure 2 the *white* vertex is the vertex v , and the list above is $L_f(v)$. By Lemma 19 (i)–(iii), the set of all possible values of c of a vertex is as given in the curly brackets below the vertex. The colors of incidences at the white vertex (i.e., v) are given at the edges adjacent to v .

Obviously, all the incidences at vertex v are colored with different colors from $A_f(v)$. Observe that the set of colors $A_f(v)$ is an interval of integers.

We prove that the coloring f is an incidence coloring. It is enough to prove that for each vertex $v \in V(G)$ and each vertex $w \in N_G(v)$ we have $f(v, \{v, w\}) \notin A_f(w)$, or, equivalently, $f(v, \{v, w\}) < \min A_f(w)$ or $f(v, \{v, w\}) > \max A_f(w)$.

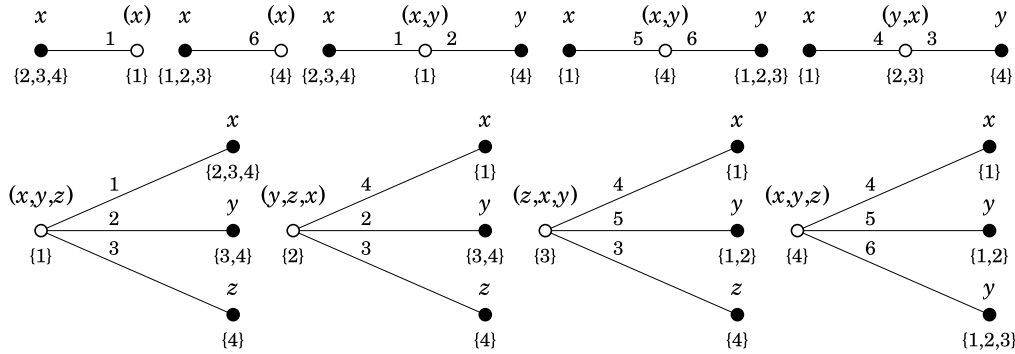


Figure 2. Interval coloring of incidences at the white vertex v , according to its degree and the values of c at the neighbors x, y, z of v . The set of possible values of c of a vertex is given in the curly brackets below the vertex. The list $L_f(v)$ is given above the white vertex v .

Suppose that $c(v) = 1$. Then, $A_f(v) \subset \{1, 2, 3\}$ and $\min A_f(v) = 1$. By the construction of $L_f(v)$ we have: if $\deg_G(v) \geq 1$, then $c(v_1) \in \{2, 3, 4\}$, and if $\deg_G(v) = 2$, then $c(v_2) = 4$, and if $\deg_G(v) = 3$, then $c(v_2) \in \{3, 4\}$ and $c(v_3) = 4$ (see Figure 2). Hence, for each $i \in \{1, \dots, \deg_G(v)\}$ we have $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 < i + 1 \leq \min A_f(v_i)$.

Suppose that $c(v) = 2$. Then, $A_f(v) \subset \{2, 3, 4\}$. Let $\deg_G(v) = 3$. Hence, $\min A_f(v) = 2$, and $c(v_1) \in \{3, 4\}$ and $c(v_2) = 4 \wedge c(v_3) = 1$. Thus, $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 = i + 1 < i + 2 \leq \min A_f(v_i)$, for $i \in \{1, 2\}$, and $f(v, \{v, v_3\}) = \min A_f(v) + 2 = 4 > 3 \geq \max A_f(v_3)$. Let $\deg_G(v) = 2$. Hence, $\min A_f(v) = 3$, and $c(v_1) = 4$ and $c(v_2) = 1$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$ and $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$.

Suppose that $c(v) = 3$. Then, $A_f(v) \subset \{3, 4, 5\}$ and $\min A_f(v) = 3$. Let $\deg_G(v) = 3$. Hence, $c(v_1) = 4$ and $c(v_2) = 1$ and $c(v_3) \in \{1, 2\}$. Thus, $f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1)$, and $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 > i + 1 \geq \max A_f(v_i)$, for $i \in \{2, 3\}$. Let $\deg_G(v) = 2$. Hence, $c(v_1) = 4$ and $c(v_2) = 1$. Thus, $f(v, \{v, v_1\}) = 3 < 4 \leq \min A_f(v_1)$ and $f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2)$.

Suppose that $c(v) = 4$. Then, $A_f(v) \subset \{4, 5, 6\}$. Let $\deg_G(v) = 3$. Hence, $c(v_1) = 1$ and $c(v_2) \in \{1, 2\}$ and $c(v_3) \in \{1, 2, 3\}$ and $c(v_2) \leq c(v_3)$. Thus, $f(v, \{v, v_i\}) = \min A_f(v) + i - 1 \geq i + 3 > i + 2 \geq \max A_f(v_i)$, for each $i \in \{1, 2, 3\}$. Let $\deg_G(v) = 2$. Hence, $c(v_1) = 1$ and $c(v_2) \in \{1, 2, 3\}$, and $A_f(v) = \{5, 6\}$. Thus, $f(v, \{v, v_1\}) = 5 > \max A_f(v_1)$ and $f(v, \{v, v_2\}) = 6 > \max A_f(v_2)$. Let $\deg_G(v) = 1$. Hence, $c(v_1) \in \{1, 2, 3\}$. Thus, $f(v, \{v, v_1\}) = 6 > 5 \geq \max A_f(v_1)$.

In all the cases we proved that $f(v, \{v, v_i\}) \notin A_f(v_i)$ for each $v_i \in N_G(v)$. Thus, f is an interval incidence 6-coloring of G . ■

4. SUMMARY

In this paper we proved that for any subcubic graph G , $\chi_{ii}(G) \leq 2\Delta(G)$. In [14] we proved that the upper bound of $2\Delta(G)$ on $\chi_{ii}(G)$ holds for each complete k -partite graph G and this bound is valid for other classes of graphs. Thus, we state the following

Conjecture 22 [Interval Incidence Coloring Conjecture (IICC)]. *For any graph G , $\chi_{ii}(G) \leq 2\Delta(G)$.*

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