Postprint of: Eremeyev V.A., Lebedev L.P., Cloud M.J., Acceleration waves in the nonlinear micromorphic continuum, Mechanics Research Communications, Vol. 93 (2018), pp. 70-74, DOI: 10.1016/j.mechrescom.2017.07.004

Acceleration waves in the nonlinear micromorphic continuum☆

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ABSTRACT

Within the framework of the nonlinear elastic theory of micromorphic continua we derive the conditions for propagation of acceleration waves. An acceleration wave, also called a wave of weak discontinuity of order two, can be treated as a propagating nonmaterial surface across which the second derivatives of the placement vector and micro-distortion tensor may undergo jump discontinuities. Here we obtain the acoustic tensor for the micromorphic medium and formulate the conditions for existence of acceleration waves. As examples we consider these conditions for the linear micromorphic medium and for the relaxed micromorphic model.

Keywords:
Acceleration wave
Micromorphic continuum
Discontinuities
Ellipticity

1. Introduction

Waves of discontinuity such as shock and acceleration waves characterize material behaviour and internal structure, the internal instabilities of strain localizations, and other phenomena [1]. An acceleration wave can be represented as an isolated travelling smooth surface carrying jump discontinuities in the second derivatives of the relevant kinematic quantities with respect to the space coordinates and time. Existence conditions for the propagation of an acceleration wave can be reduced to an algebraic problem, i.e., to a spectral problem for an acoustic tensor whose eigenvalues should be positive. The condition for propagation of acceleration waves is closely related to the strong ellipticity of the equilibrium equations. Let us note that various ellipticity requirements seem to be a natural property of the elasto-statics equation, at least for small deformations. Thus the analysis of acceleration wave propagation plays an important role in the mechanics of materials.

In nonlinear elasticity, acceleration waves were studied in many works; see, e.g., [2,3] where further references can be found. For the generalized models of continua, acceleration waves are also considered in a number of papers. In particular, the propagation of acceleration waves is studied in the theory of porous media [4,5], for random materials [6], and for piezoelectric solids [7]. Acceleration waves are also studied in various theories of microstructured fluids and gases; see [8–13] and the references therein. Such analysis can also be useful in nonmechanical applications such as for social systems [14]. For the nonlinear elastic micropolar media, acceleration waves are analyzed in [15]. In [16] certain generalizations are presented for elastic and viscoelastic micropolar media. Interrelations between the existence conditions for acceleration waves and the condition of strong ellipticity for the static equations are discussed for elastic micropolar media in [17] and for thermoelastic media in [18].

Here we consider acceleration waves in the micromorphic continuum. This model was proposed in [19,20]; see also [21–24]. Within the model, the medium kinematics are described by two fields: those of the placement vector and the micro-distortion second-order tensor. The latter may be useful for modelling such microstructured materials as foams [25], heterogeneous media [26–28], metamaterials [29,30], granular media [31], and others. Let us note that as for enriched continuum models, the constitutive equations of the micromorphic continuum can be derived using homogenization technique. Wave propagation in micromorphic media is also studied, for example, in [29,32–35].

The paper is organized as follows. Following [20,21], in Section 2 we present the governing equations of the micromorphic continuum under large deformations. As a special case we also present the basic equations under small deformations. In Section 3 we

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derive the conditions for propagation of acceleration waves in the micromorphic medium and also consider the ellipticity conditions. Finally, in Section 4 we discuss the propagation of acceleration waves in a relaxed model of a micromorphic medium [36]. Direct tensor notation is used throughout (see, e.g., [37]).

2. Governing equations for the micromorphic continuum

The deformation of a micromorphic nonlinear elastic solid is described as a mapping from a reference placement into an actual one. The kinematics of the micromorphic continuum are determined by the placement vector $\mathbf{r}$ and the micro-distortion second-order tensor $\mathbf{P}$:

$$\mathbf{r} = \mathbf{r}(\mathbf{R}, t), \;\; \mathbf{P} = \mathbf{P}(\mathbf{R}, t), \tag{1}$$

where $\mathbf{R}$ is the position vector in the reference placement and t is time.

For a hyperelastic medium let us introduce the strain energy density

$$W = W(\nabla\mathbf{r}, \mathbf{P}, \nabla\mathbf{P}), \tag{2}$$

where ∇ is the gradient (nabla) operator in Lagrangian coordinates [37,38]. For example, in Cartesian coordinates $\nabla\mathbf{f} = \frac{\partial f_n}{\partial x_m}\mathbf{i}_m \otimes \mathbf{i}_n$, where $\mathbf{i}_k$, $k = 1, 2, 3$ are the basis vectors and $\otimes$ is the tensor product.

According to the principle of material frame indifference [2,3], W must satisfy the invariance property

$$W(\mathbf{F}, \mathbf{P}, \nabla\mathbf{P}) = W(\mathbf{F}\cdot\mathbf{O}, \mathbf{P}\cdot\mathbf{O}, \nabla\mathbf{P}\cdot\mathbf{O}) \tag{3}$$

for any orthogonal tensor $\mathbf{O} = \mathbf{O}^{-T}$, where $\mathbf{F} = \nabla\,\mathbf{r}$ and "$\cdot$" denotes the dot product. This invariance results in the following objective representations of W:

$$W = W(\mathbf{F}\cdot\mathbf{F}^T, \mathbf{P}\cdot\mathbf{F}^{-1}, \nabla\mathbf{P}\cdot\mathbf{F}^{-1}), \tag{4}$$

or

$$W = W(\mathbf{F}\cdot\mathbf{P}^{-1}, \mathbf{P}\cdot\mathbf{P}^T, \nabla\mathbf{P}\cdot\mathbf{P}^{-1}). \tag{5}$$

Let us note that in (4) and (5) we use different sets of strain measures (see [20,21] for details), but for simplicity we keep the notation for W. Here we assume that W is a twice continuously differentiable function. For brevity, we utilize the following notations for its derivatives:

$$W_{,\mathbf{F}} = \frac{\partial W}{\partial \mathbf{F}}, \;\; W_{,\mathbf{P}} = \frac{\partial W}{\partial \mathbf{P}},$$

$$W_{,\mathbf{FF}} = \frac{\partial^2 W}{\partial \mathbf{F}\,\partial \mathbf{F}}, \;\; W_{,\nabla\mathbf{P}\nabla\mathbf{P}} = \frac{\partial^2 W}{\partial \nabla\mathbf{P}\,\partial \nabla\mathbf{P}},$$

etc.

In addition we define the kinetic energy density

$$K = \frac{1}{2}\rho\,\mathbf{v}\cdot\mathbf{v} + j\,\dot{\mathbf{P}} : \dot{\mathbf{P}}, \tag{6}$$

where ρ is the mass density in the reference configuration, $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity (the overdot standing for the time derivative), and $j \geq 0$ is the scalar measure of microinertia.

The Lagrangian equations of motion take the form

$$\nabla\cdot\mathbf{T} + \rho\mathbf{f} = \rho\dot{\mathbf{v}}, \;\; \nabla\cdot\mathbf{M} - W_{,\mathbf{P}} + \rho\mathbf{m} = j\,\ddot{\mathbf{P}}, \tag{7}$$

where the Lagrangian stress measures of Piola–Kirchhoff type are used:

$$\mathbf{T} = W_{,\mathbf{F}}, \;\; \mathbf{M} = W_{,\nabla\mathbf{P}}. \tag{8}$$

Here $\mathbf{T}$ is the first Piola–Kirchhoff second-order stress tensor, $\mathbf{M}$ is the first Piola–Kirchhoff third-order hyper-stress tensor, and $\mathbf{f}$ and $\mathbf{m}$ denote given external loads.

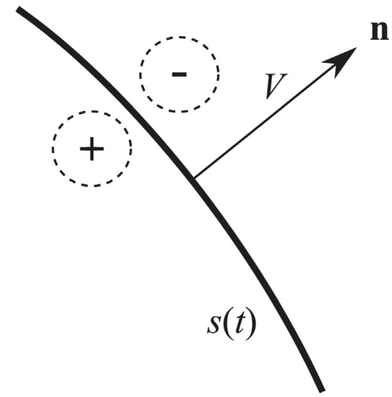

Fig. 1. Propagation of an acceleration wave.

Within small deformations, let us introduce the displacement vector $\mathbf{u} = \mathbf{r} - \mathbf{R}$ and small second-order tensor $\mathbf{p}$ such that

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}, \;\; \mathbf{P} = \mathbf{I} + \mathbf{p},$$

where $\mathbf{I}$ is the 3D unit tensor. In this case the two sets of strain measures used in (4) and (5) transform into the single set

$$\boldsymbol{\varepsilon} = \nabla\mathbf{u} - \mathbf{p}, \;\; \mathbf{e} = \frac{1}{2}(\mathbf{p} + \mathbf{p}^T), \;\; \varkappa = \nabla\mathbf{p}. \tag{9}$$

For infinitesimal deformations of an isotropic micromorphic solid, the following form of strain energy can be used:

$$\begin{aligned} W \; &= \frac{\lambda}{2}\mathrm{tr}^2\boldsymbol{\varepsilon} + \frac{\mu+\kappa}{2}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \frac{\mu-\kappa}{2}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}^T \\ &\quad + \frac{\alpha}{2}\mathrm{tr}^2\mathbf{e} + \frac{\beta}{2}\mathbf{e} : \mathbf{e} + \frac{\gamma}{2}\nabla\mathbf{p}\,\vdots\,\nabla\mathbf{p}, \end{aligned} \tag{10}$$

where λ, μ, κ, α, β, and γ are material parameters, and ":" and "$\vdots$" denote the full products in the spaces of second- and third-order tensors, respectively. Positive definiteness of W implies that

$$\mu > 0, \;\; 3\lambda + 2\mu > 0, \;\; \kappa > 0, \tag{11}$$

$$\beta > 0, \;\; 3\alpha + 2\beta > 0, \;\; \gamma > 0. \tag{12}$$

Under these conditions, $W = 0$ if and only if $\mathbf{u} = \mathbf{a} + \boldsymbol{\omega} \times \mathbf{R}$ and $\mathbf{p} = \mathbf{I} \times \boldsymbol{\omega}$ with constant vectors $\mathbf{a}$ and $\boldsymbol{\omega}$. In other words, W vanishes only for rigid body motions.

The constitutive equation (10) can be easily extended to the case of large deformations considering, for example, the following strain measures from (5):

$$\boldsymbol{\varepsilon} = \mathbf{F}\cdot\mathbf{P}^{-1} - \mathbf{I}, \;\; \mathbf{e} = \frac{1}{2}(\mathbf{P}\cdot\mathbf{P}^T - \mathbf{I}), \;\; \varkappa = \nabla\mathbf{P}\cdot\mathbf{P}^{-1}.$$

The corresponding micromorphic model can be called the physically linear micromorphic material. Obviously, with use of the strain measures from (5) we get another micromorphic model. Static problems for nonlinear case were analyzed in [39].

3. Acceleration waves

Let us consider discontinuous deformations of a micromorphic medium when discontinuities appear on a smooth surface $S(t)$ said to be singular (Fig. 1). We assume existence of one-sided limits at $S(t)$ for all quantities under consideration. Let us denote a jump of any quantity across $S(t)$ by doubled brackets: $[\![\mathbf{r}]\!] = \mathbf{r}_+ - \mathbf{r}_-$. Observe that S is a non-material surface propagating across material points.

By an acceleration (or weak discontinuity) wave we mean a travelling singular surface $S(t)$ on which the second spatial and time derivatives of $\mathbf{r}$ and $\mathbf{P}$ have jumps, while $\mathbf{r}$ and $\mathbf{P}$ together with all their first derivatives are continuous. So on $S(t)$ the following conditions hold:

$$[\![\nabla \mathbf{r}]\!] = \mathbf{0},\ [\![\nabla \mathbf{P}]\!] = \mathbf{0},\ [\![\mathbf{v}]\!] = \mathbf{0},\ [\![\dot{\mathbf{P}}]\!] = \mathbf{0}. \tag{13}$$

Eq. (13) imply continuity of **T** and **M** across $S(t)$:

$$[\![\mathbf{T}]\!] = \mathbf{0},\ [\![\mathbf{M}]\!] = \mathbf{0}.$$

The following theorem [2,3] will also be needed.

Maxwell's theorem. For a continuously differentiable field **L** such that $[\![\mathbf{L}]\!]=\mathbf{0}$, the following relations hold:

$$[\![\dot{\mathbf{L}}]\!] = -V\mathbf{l},\ [\![\nabla \mathbf{L}]\!] = \mathbf{n}\otimes\mathbf{l}, \tag{14}$$

where **l** is the tensor amplitude of the jump of the first gradient of **L**, the tensor amplitude is a tensor of the same order as **L**. A straightforward application of Maxwell's theorem to the continuous fields **v**, $\dot{\mathbf{P}}$, **T**, and **M** results in the relations

$$[\![\dot{\mathbf{v}}]\!] = -V\mathbf{a},\ [\![\nabla \mathbf{v}]\!] = \mathbf{n}\otimes\mathbf{a},$$
$$[\![\ddot{\mathbf{P}}]\!] = -V\mathbf{A},\ [\![\nabla \dot{\mathbf{P}}]\!] = \mathbf{n}\otimes\mathbf{A},$$
$$V[\![\nabla\cdot\mathbf{T}]\!] = -\mathbf{n}\cdot[\![\dot{\mathbf{T}}]\!],$$
$$V[\![\nabla\cdot\mathbf{M}]\!] = -\mathbf{n}\cdot[\![\dot{\mathbf{M}}]\!],$$

where **a** and **A** are the vectorial and tensorial amplitudes of the jumps. With these, the equations of motion become

$$\mathbf{n}\cdot[\![\dot{\mathbf{T}}]\!] = \rho V^2\mathbf{a},\ \mathbf{n}\cdot[\![\dot{\mathbf{M}}]\!] = jV^2\mathbf{A}. \tag{15}$$

Representing $\dot{\mathbf{T}}$ and $\dot{\mathbf{M}}$ through W we obtain

$$\mathbf{n}\cdot(W_{,\mathbf{FF}} : (\mathbf{n}\otimes\mathbf{a}) + W_{,\mathbf{F}\nabla\mathbf{P}} \vdots (\mathbf{n}\otimes\mathbf{A})) = \rho V^2\mathbf{a},$$

$$\mathbf{n}\cdot(W_{,\nabla\mathbf{PF}} : (\mathbf{n}\otimes\mathbf{a}) + W_{,\nabla\mathbf{P}\nabla\mathbf{P}} \vdots (\mathbf{n}\otimes\mathbf{A})) = jV^2\mathbf{A}.$$

These equations can be written in matrix form:

$$\mathcal{Q}(\mathbf{n})\cdot\boldsymbol{\xi} = V^2\mathcal{B}\cdot\boldsymbol{\xi}, \tag{16}$$

where $\boldsymbol{\xi} = (\mathbf{a}, \mathbf{A})$ and

$$\mathcal{Q}(\mathbf{n}) \equiv \begin{bmatrix} W_{,\mathbf{FF}}\{\mathbf{n}\} & W_{,\mathbf{F}\nabla\mathbf{P}}\{\mathbf{n}\} \\ W_{,\nabla\mathbf{PF}}\{\mathbf{n}\} & W_{,\nabla\mathbf{P}\nabla\mathbf{P}}\{\mathbf{n}\} \end{bmatrix},$$

$$\mathcal{B} \equiv \begin{bmatrix} \rho\mathbf{I} & \mathbf{0} \\ \mathbf{0} & j\mathbf{1} \end{bmatrix}.$$

Here **1** is the four-dimensional unit tensor [37]. Let us introduce the following saturation operation [18] for an arbitrary Mth-order tensor **H** and vector **n**. For **H** and **n** represented in a Cartesian basis $\mathbf{i}_k$ $(k = 1, 2, 3)$, that is

$$\mathbf{H} = H_{i_1 i_2 \cdots i_M} \underbrace{\mathbf{i}_{i_1}\otimes\mathbf{i}_{i_2}\otimes\cdots\otimes\mathbf{i}_{i_M}}_{M\text{ times}},\ \ \mathbf{n} = n_k\mathbf{i}_k,$$

$\mathbf{H}\{\mathbf{n}\}$ denotes the following $(M-2)$th-order tensor:

$$\mathbf{H}\{\mathbf{n}\} \equiv H_{i_1 i_2 \cdots i_M} n_{i_1} n_{i_{M-N}} \underbrace{\mathbf{i}_{i_2}\otimes\cdots\otimes\mathbf{i}_M}_{M-2\text{ times}}. \tag{17}$$

$\mathcal{Q}(\mathbf{n})$ is called the *acoustic tensor* for the medium with microstructure. Due to the symmetry of the mixed derivatives of W, the tensor $\mathcal{Q}(\mathbf{n})$ is symmetric. Hence its eigenvalues, which are the squared velocities, are real. But for positivity of V one also needs the positive definiteness of $\mathcal{Q}(\mathbf{n})$:

$$\boldsymbol{\xi}\cdot\mathcal{Q}(\mathbf{n})\cdot\boldsymbol{\xi} > 0,\ \forall\,\boldsymbol{\xi}\neq\mathbf{0},\ \ \forall\,|\mathbf{n}| = 1. \tag{18}$$

Inequality (18) coincides with the strong ellipticity condition for the equilibrium equations for a micromorphic continuum.

Condition (18) also has the form

$$\mathbf{n}\cdot[W_{,\mathbf{FF}} : (\mathbf{n}\otimes\mathbf{a}) + W_{,\mathbf{F}\nabla\mathbf{P}} \vdots (\mathbf{n}\otimes\mathbf{P})]\cdot\mathbf{a}$$
$$+\mathbf{n}\cdot[W_{,\nabla\mathbf{PF}} : (\mathbf{n}\otimes\mathbf{a})$$
$$+W_{,\nabla\mathbf{P}\nabla\mathbf{P}} \vdots (\mathbf{n}\otimes\mathbf{A})]\cdot\mathbf{A} > 0,$$
$$\forall\,\mathbf{a}\neq\mathbf{0},\ \ \forall\,\mathbf{A}\neq\mathbf{0},\ \ \forall\,|\mathbf{n}| = 1,$$

and, moreover, is equivalent to

$$\left.\frac{d^2}{d\varepsilon^2}W(\nabla\mathbf{r}+\varepsilon\mathbf{n}\otimes\mathbf{a},\ \mathbf{P},\ \nabla\mathbf{P}+\varepsilon\mathbf{n}\otimes\mathbf{A})\right|_{\varepsilon=0} > 0,\quad \forall\,|\mathbf{n}| = 1,\ \forall\,\mathbf{a}\neq\mathbf{0},\ \forall\,\mathbf{A}\neq\mathbf{0}. \tag{19}$$

The latter form is the condition of the strict rank-one convexity of W.

As an example, let us consider a micromorphic material with the energy (10). Here (18) reduces to two inequalities

$$\lambda(\mathbf{n}\cdot\mathbf{a})^2 + (\mu+\kappa)(\mathbf{n}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{a}) + (\mu-\kappa)(\mathbf{n}\cdot\mathbf{a})^2 > 0,$$

$$\gamma(\mathbf{n}\cdot\mathbf{n})(\mathbf{A}:\mathbf{A}) > 0,$$

which result in

$$\mu+\kappa > 0,\ \lambda+2\mu > 0,\ \gamma > 0. \tag{20}$$

The same conditions can be derived for the physically linear micromorphic material undergoing large deformations.

Obviously, conditions (20) are weaker than (11) and (12). Moreover, there are no constraints for α and β. In particular this means that unlike nonlinear elasticity, the strong ellipticity conditions are too weak for real wave propagations [35]. A similar situation can be observed for micropolar materials [17,18]. Nevertheless, (20) provides for the propagation of waves of weak discontinuity in micromorphic solids.

If (20) holds, the velocities of acceleration waves calculated from the generalized spectral problem (16) are given by

$$V_1 = \sqrt{\frac{2\mu+\lambda}{\rho}},\ \ V_2 = \sqrt{\frac{\mu+\kappa}{\rho}},\ \ V_3 = \sqrt{\frac{\gamma}{j}}. \tag{21}$$

Here V_1 and V_2 correspond to the longitudinal and transverse acceleration waves, respectively. The corresponding eigenvalues are given by

$$\boldsymbol{\xi}_1 = (\mathbf{n}, \mathbf{0}),\ \ \boldsymbol{\xi}_{2\alpha} = (\mathbf{e}_\alpha, \mathbf{0}), \tag{22}$$

where $\mathbf{e}_\alpha$ $(\alpha = 1, 2)$ are unit vectors tangent to S and $\mathbf{n}\cdot\mathbf{e}_\alpha = 0$. Thus V_2 is a multiple eigenvalue with two corresponding eigenvectors $\boldsymbol{\xi}_{21}$ and $\boldsymbol{\xi}_{22}$. V_3 is the velocity of the acceleration wave of microdistortion; it is also a multiple eigenvalue of (16) with multiplicity 9. The corresponding set of eigenvectors is given by

$$\boldsymbol{\xi}_{333} = (\mathbf{0}, \mathbf{n}\otimes\mathbf{n}),\ \ \boldsymbol{\xi}_{33\alpha} = (\mathbf{0}, \mathbf{n}\otimes\mathbf{e}_\alpha), \tag{23}$$

$$\boldsymbol{\xi}_{3\alpha 3} = (\mathbf{0}, \mathbf{e}_\alpha\otimes\mathbf{n}),\ \ \boldsymbol{\xi}_{3\alpha\beta} = (\mathbf{0}, \mathbf{e}_\alpha\otimes\mathbf{e}_\beta). \tag{24}$$

Obviously, the acceleration waves and acceleration waves of microdistortion are decoupled. Finally, we conclude that the acceleration waves in the micromorphic continuum are carriers of various jumps of weak discontinuities.

4. Acceleration waves in relaxed micromorphic continuum

For small deformations, the relaxed micromorphic medium model was introduced by Neff et al. in a series of papers [29,35,36]. Unlike the classical micromorphic model, the relaxed micromorphic model allows us to describe complete band gaps in acoustic wave transmission problems. Within the framework of this model, the strain energy density is

$$\begin{aligned} W &= \frac{\lambda}{2}\mathrm{tr}^2\boldsymbol{\varepsilon} + \frac{\mu+\kappa}{2}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} + \frac{\mu-\kappa}{2}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}^T \\ &\quad + \frac{\alpha}{2}\mathrm{tr}^2\mathbf{e} + \frac{\beta}{2}\mathbf{e}:\mathbf{e} \\ &\quad + \frac{\gamma}{2}(\nabla\times\mathbf{p}):(\nabla\times\mathbf{p}). \end{aligned} \tag{25}$$

Here $\mu_c = 2\kappa \geq 0$ is called the Cosserat couple modulus. If $\kappa = 0$ the strain energy density depends only on symmetric part of ε. The nonlinear counterpart of (25) is

$$W = W(\mathbf{F}\cdot\mathbf{P}^{-1}, \mathbf{P}\cdot\mathbf{P}^T, (\nabla\times\mathbf{P})\cdot\mathbf{P}^{-1}) \tag{26}$$

and the Lagrangian equations of motion transform into

$$\nabla\cdot\mathbf{T} + \rho\mathbf{f} = \rho\dot{\mathbf{v}}, \;\; \nabla\times\mathbf{M} - W_{,\mathbf{P}} + \rho\mathbf{m} = j\,\ddot{\mathbf{P}}, \tag{27}$$

where

$$\mathbf{T} = W_{,\mathbf{F}}, \;\; \mathbf{M} = W_{,\nabla\times\mathbf{P}}. \tag{28}$$

Note that unlike the general micromorphic model, here $\mathbf{M}$ is a second-order tensor.

Applying the same techniques, we derive the formula for the acoustic tensor

$$\mathcal{Q}(\mathbf{n}) \equiv \begin{bmatrix} W_{,\mathbf{FF}}\{\mathbf{n}\} & W_{,\mathbf{F}\nabla\times\mathbf{P}}\{\mathbf{n}\} \\ W_{,\nabla\times\mathbf{PF}}\{\mathbf{n}\} & W_{,\nabla\times\mathbf{P}\nabla\times\mathbf{P}}\{\mathbf{n}\} \end{bmatrix}.$$

The strong ellipticity conditions now become

$$\begin{aligned} &\frac{d^2}{d\varepsilon^2}W(\nabla\mathbf{r}+\varepsilon\mathbf{n}\otimes\mathbf{a}, \mathbf{P}, \nabla\mathbf{P}+\varepsilon\mathbf{n}\times\mathbf{A})\Big|_{\varepsilon=0} > 0, \\ &\qquad \forall\, |\mathbf{n}| = 1, \;\forall\, \mathbf{a}\neq\mathbf{0}, \;\forall\, \mathbf{A}\neq\mathbf{0}. \end{aligned} \tag{29}$$

For (25), inequality (29) takes the form

$$\lambda(\mathbf{n}\cdot\mathbf{a})^2 + (\mu+\kappa)(\mathbf{n}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{a}) + (\mu-\kappa)(\mathbf{n}\cdot\mathbf{a})^2 > 0,$$

$$\gamma(\mathbf{n}\times\mathbf{A}):(\mathbf{n}\times\mathbf{A}) > 0,$$

which also imply (20). However in this case (29) is violated as $\boldsymbol{\xi}\cdot\mathcal{Q}(\mathbf{n})\cdot\boldsymbol{\xi} = 0$ if $\boldsymbol{\xi} = (\mathbf{0}, \mathbf{n}\otimes\mathbf{e})$ for any vector $\mathbf{e}$. So the general strong ellipticity condition for the equilibrium equations for the relaxed micromorphic continuum is violated. Here only the weak form of the strong ellipticity condition holds:

$$\boldsymbol{\xi}\cdot\mathcal{Q}(\mathbf{n})\cdot\boldsymbol{\xi} \geq 0, \;\; \forall\, \boldsymbol{\xi}\neq\mathbf{0},$$

which can be treated as a counterpart of the Hadamard inequality in nonlinear elasticity [2,3]. Nevertheless, note that the corresponding boundary-value problems are still well-posed [40], including the case $\kappa = 0$ ($\mu_c = 0$). Results of this type are based on Korn's inequality for incompatible tensor fields [41]. For $\kappa = 0$, the positive definiteness of the strain energy density is violated but the conditions (20) are still fulfilled. The velocities of acceleration waves are given by (21), whereas the corresponding eigenvectors are (22) and (24). So, (23) are excluded. This means that in the relaxed micromorphic continuum we have fewer possible propagating weak discontinuities.

5. Conclusion

We have formulated conditions for propagation of acceleration waves in the nonlinear micromorphic continuum including the relaxed micromorphic model. The formulas for the acoustic tensors were derived. We have shown that for the relaxed micromorphic continuum, the general strong ellipticity condition is violated. Nevertheless, for both models there exists a set of weak discontinuity surfaces propagating with certain velocities. Under the derived conditions, there are three acceleration waves: one longitudinal with velocity V_1, and two transverse with velocity V_2. Within the general model, there are nine acceleration waves of micro-distortion travelling with velocity V_3, whereas for the relaxed micromorphic continuum there are only six acceleration waves of micro-distortion propagating with velocity V_3.

Acknowledgement

V.A.E. acknowledges financial support from the Russian Science Foundation under the grant "Methods of microstructural nonlinear analysis, wave dynamics and mechanics of composites for research and design of modern metamaterials and elements of structures made on its base" (No. 15-19-10008).

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