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Comparison of anti-plane surface waves in strain-gradient materials and materials with surface stresses

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Abstract

Here we discuss similarity and difference in anti-plane surface waves propagation in an elastic half-space within the framework of the both theories of Gurtin-Murdoch surface elasticity and the Toupin–Mindlin strain gradient elasticity. The qualitative behaviour of the dispersion curves and decay of the obtained solutions are quite similar. On the other hand, we show that the solutions related to the surface elasticity model is more localized near the free surface. For the strain gradient elasticity there is a range of wavenumbers where the amplitude of displacements is decaying very slow.

keywords: strain gradient elasticity, SH waves, anti-plane waves, surface waves, surface stresses.

1 Introduction

Surface-related phenomena play an important role in the mechanics of materials at the micro- and nanoscales. Considering the mechanical models of surface phenomena we distinguish the surface elasticity models where the additional constitutive relation on the surface are introduced and the enhanced models of continua such as the strain gradient elasticity. It is worth to mention here the surface elasticity models introduced by Gurtin and Murdoch [1], by Steigmann and Ogden [2,3] and the first and second strain gradient elasticity presented in works by Toupin [4] and Mindlin [5,6], see also [7,8] and [9,10] for historical overview. Obviously, the both approaches may change the behaviour of solutions of the corresponding boundary value problems. For example, they can describe the size-effect that is the dependence of the apparent material properties on the specimen size, change the properties of solutions near crack tips and other geometrical singularities, and found many applications in micro- and nanomechanics, see e.g. [11–17] for the surface elasticity and [18–23] for the strain gradient elasticity. Among such examples there is a problem of surface antiplane waves propagation that is antiplane waves whose amplitude decays exponentially with distance from the surface. It is well-known that within the classic linear elasticity anti-plane surface waves in an elastic halfspace do not exist, see *e.g.* [24]. For extended models of continua such waves may exist, see [25] for the surface elasticity and [26–28] for the strain gradient elasticity. Dynamics within the special case of the strain gradient elasticity motivated by beam lattices was analyzed in [29,30].

In this paper we analyze the propagation of anti-plane surface waves in solids within the strain gradient elasticity and within the theory of surface stresses by Gurtin-Murdoch. The aim of the paper is the comparison of these two types of media in order to understand the similarities and difference of these models trough the problem of anti-plane surface waves. The coupling between surface elastic and kinetic energies and bulk waves has been studied in [31–33]. The analysis of propagation of shear waves in order to determine surface properties of thin films was discussed in [34].
The paper is organized as follows. In Section 2 we present the both models of the linear strain gradient elasticity using the general Mindlin's model [4,5,35] and the Gurtin-Murdoch model of surface elasticity [1,36]. Then we transform the general boundary-value problems for the case of antiplane motion. So the governing equations reduce to one scalar equations in the bulk and corresponding boundary conditions. Finally, in Section 4 we present the numerical analysis of waves propagation including the dispersion curves and the amplitude dependence on the depth-coordinate.

2 Governing equations

2.1 Strain-gradient elasticity

We consider infinitesimal deformations of an elastic solid which are described by the displacement field

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t), \tag{1}$$

where $\mathbf{u}$ is twice differentiable vector-function of displacements, $\mathbf{x}$ is the position vector and t is time. We use the strain-gradient model of elastic medium with the following constitutive equations. Strain energy density W is given by [5, 35]

$$W = \frac{1}{2}\mathbf{e} : \mathbf{C} : \mathbf{e} + \frac{1}{2}\nabla\mathbf{e} \,\vdots\, \mathbf{A} \,\vdots\, \nabla\mathbf{e}, \tag{2}$$

$$\mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right),$$

where $\mathbf{C} = C_{ijkl}\mathbf{i}_i \otimes \mathbf{i}_j \otimes \mathbf{i}_k \otimes \mathbf{i}_l$ and Einstein's summation convention is used, $\mathbf{A} = A_{ijklmn}\mathbf{i}_i \otimes \mathbf{i}_j \otimes \mathbf{i}_k \otimes \mathbf{i}_l \otimes \mathbf{i}_m \otimes \mathbf{i}_n$ are the fourth- and six-order tensors of elastic moduli, respectively, $\mathbf{i}_k$, $k = 1, 2, 3$, are vectors of Cartesian orthonormal basis, $\otimes$ stands for diadic product, $\mathbf{e}$ is the strain tensor, the superscript "T" means the transpose operation, the ":" and "$\vdots$" stand for scalar (inner) product of two second-order and two third-order tensors, and ∇ is the 3D nabla operator defined as follows: $\nabla\mathbf{u} = \frac{\partial u_i}{\partial x_j}\mathbf{i}_i \otimes \mathbf{i}_j$, where x_j are Cartesian coordinates in the basis $\mathbf{i}_k$, $k = 1, 2, 3$. The kinetic energy density is given by

$$T = \frac{1}{2}\rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{1}{2}\nabla\dot{\mathbf{u}} : \boldsymbol{\kappa} : \nabla\dot{\mathbf{u}}, \tag{3}$$

where ρ is the mass density, $\boldsymbol{\kappa} = \kappa_{ijkl}\mathbf{i}_i \otimes \mathbf{i}_j \otimes \mathbf{i}_k \otimes \mathbf{i}_l$ is the positive definite forth-order micro-inertia tensor, and overdot stands for the derivative with respect to time t.
It is worth to note that the strain gradient elasticity found many applications, see, for example, the models of beam lattices and fabrics proposed in [22, 37–40].
For derivation of governing equations, we use the variational principle for functional of action. Functional of action is defined as follows

$$\mathcal{L} = \int_0^T \int_V (T - W)\, dV\, dt \tag{4}$$

Here V is a volume occupied by the second gradient medium and ∂V is the boundary of V.
The variational equation $\delta\mathcal{L} = 0$ results in the following motion equation

$$\rho\ddot{\mathbf{u}} - \nabla \cdot (\boldsymbol{\kappa} : \nabla\ddot{\mathbf{u}}) = \nabla \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}), \tag{5}$$

where $\ddot{\mathbf{u}}$ is the acceleration, the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are defined by

$$\boldsymbol{\sigma} = \mathbf{C} : \mathbf{e}, \quad \boldsymbol{\tau} = \mathbf{A} \vdots \nabla \mathbf{e},$$

which are the second-order stress tensor and third-order hyperstress tensor, respectively.
In what follows, we use this for the constitutive law for an isotropic strain gradient solid [35]

$$\begin{aligned}
C_{ijkl} &= \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), && (6)\\
A_{ijklmn} &= a_1\left(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{in}\delta_{jk}\delta_{lm}\right) + a_2\left(\delta_{ij}\delta_{kn}\delta_{lm}\right), \\
&\quad + a_3\left(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln}\right) + a_4\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jl}\delta_{kn}\right), && (7)\\
&\quad + a_5\left(\delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}\right), \\
\kappa_{ijkl} &= \kappa_1 \delta_{ij}\delta_{kl} + \kappa_2 \delta_{ik}\delta_{jl} + k_3 \delta_{il}\delta_{jk}, && (8)
\end{aligned}$$

where δ_{ij} is the Kronecker symbol, λ, μ, a_1, a_2, a_3, a_4, and a_5 are elastic moduli, whereas κ_1, κ_2 and κ_3 are the microinertia parameters.
It should be noticed that, while the constitutive equation is equivalent, the coefficients here defined are not the same as in [5] and consequently in [28]. As already noticed in [28], the simplified form proposed in [41] does not allow for the existence of SH waves, and thus the full form is needed.

2.2 Gurtin-Murdoch model of the surface elasticity

Within the Gurtin–Murdoch approach [1, 36] in the bulk, we have classic constitutive equations of an isotropic solid

$$\mathcal{W} = \mu \mathbf{e} : \mathbf{e} + \frac{1}{2}\lambda (\operatorname{tr} \mathbf{e})^2, \tag{9}$$

$$\boldsymbol{\sigma} \equiv \frac{\partial \mathcal{W}}{\partial \mathbf{e}} = 2\mu \mathbf{e} + \lambda \mathbf{I} \operatorname{tr} \mathbf{e}, \tag{10}$$

where $\mathcal{W}$ is the strain energy density, λ and μ are Lamé moduli, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{e}$ is the strain tensor. The kinetic energy density is given by

$$\mathcal{K} = \frac{1}{2}\rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}, \tag{11}$$

where ρ is the mass volume density.
Additionally, we introduce the surface strain energy density $\mathcal{W}_s$ and surface stress tensor $\mathbf{s}$ are defined as follows [1]

$$\begin{aligned}
\mathcal{W}_s &= \mu_s \boldsymbol{\epsilon} : \boldsymbol{\epsilon} + \frac{1}{2}\lambda_s (\operatorname{tr} \boldsymbol{\epsilon})^2, \\
\mathbf{s} &\equiv \frac{\partial \mathcal{W}_s}{\partial \boldsymbol{\epsilon}} = \mu_s \boldsymbol{\epsilon} + \lambda_s \mathbf{P} (\operatorname{tr} \boldsymbol{\epsilon}), \qquad (12)\\
\boldsymbol{\epsilon} &= \frac{1}{2}\left(\mathbf{P} \cdot (\nabla_s \mathbf{u}) + (\nabla_s \mathbf{u})^T \cdot \mathbf{P}\right),
\end{aligned}$$

where λ_s and μ_s are the surface elastic moduli called also surface Lamé moduli, tr is the trace operator, ∇_s is the surface nabla operator, $\mathbf{P} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is the surface unit second-order tensor, $\mathbf{n}$ is the unit vector of outer normal to ∂V, the symbol $\otimes$ designates the diadic product of two vectors and $\boldsymbol{\epsilon}$ is the infinitesimal surface strain tensor. In addition, we take into account the mass density associated with the surface where surface stresses are defined. This assumption results in the following formula for surface kinetic energy density [36]

$$\mathcal{K}_s = \frac{1}{2} m \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\big|_{\mathbf{x} \in \partial V}, \tag{13}$$

where m is the surface mass density and ∂V is the boundary of V.
Motion and natural boundary equations can be derived using the least action principle with the functional

$$\mathcal{L} = \int_0^T \int_V (\mathcal{K} - \mathcal{W})\, dV\, dt + \int_0^T \int_{\partial V} (\mathcal{K}_s - \mathcal{W}_s)\, da\, dt. \tag{14}$$

3 Anti-plane surface waves in an elastic half-space

Let us consider stationary waves of an elastic half-space $x_1 \le 0$. In what follows we use the Cartesian coordinates x_1, x_2 and x_3 with the basis $\mathbf{i}_k$, $k = 1, 2, 3$.

3.1 Kinematics

For the anti-plane motion, the vector of displacement takes the form [24]

$$\mathbf{u} = u(x_1, x_2, t)\mathbf{i}_3. \tag{15}$$

Here x_2 is the direction of the wave propagation, whereas x_3 is the direction of displacements. From (15), it follows that

$$\nabla \mathbf{u} = u_{,\alpha} \mathbf{i}_3 \otimes \mathbf{i}_\alpha = \mathbf{i}_3 \otimes \nabla u, \quad \nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{e} = \frac{1}{2}(\nabla u \otimes \mathbf{i}_3 + \mathbf{i}_3 \otimes \nabla u), \quad \nabla \mathbf{e} = \frac{1}{2}(\mathbf{i}_3 \otimes \nabla\nabla u + \nabla u_{,\alpha} \mathbf{i}_3 \otimes \mathbf{i}_\alpha)$$

Hereafter, we used the notation $u_{,\alpha} = \frac{\partial u}{\partial x_\alpha}$, and Greek indices take values 1, or 2. $\nabla \cdot \mathbf{u}$ is the divergence of $\mathbf{u}$.
Under assumption that a steady state has been reached, we may search displacement of the form

$$u = U(x_1) \exp\left[i(kx_2 - \omega t)\right], \tag{16}$$

where k is the wavenumber, ω is the circular velocity, and $i = \sqrt{-1}$.

3.2 Strain gradient elasticity

The motion equations (5) reduces to

$$\rho \ddot{u} - \kappa \Delta \ddot{u} = \mu(1 - \ell^2 \Delta)\Delta u, \tag{17}$$

where $\Delta u = u_{,11} + u_{,22}$ is the two-dimensional Laplace operator and where we introduced the following characteristic length

$$\ell^2 = \frac{a_3 + a_4 + a_5}{\mu}. \tag{18}$$

The boundary conditions take the form [25]

$$\mu[\ell^2 u_{,11} + \overline{a}_3 u_{,22}] = 0, \tag{19}$$

$$\mu\left[\left(\overline{a}_3 - 2\ell^2\right) u_{,221} - \ell^2 u_{,111} + u_{,1}\right] + \kappa \ddot{u}_{,1} = 0 \tag{20}$$

where $\overline{a}_3 = a_3/\mu$ and $\kappa = \kappa_2$.
Substituting (16) into (17), we obtain the ordinary differential equation with respect to U

$$-\omega^2 \left[\rho - \kappa(\partial^2 - k^2)\right] U = \mu \left[1 - \ell^2(\partial^2 - k^2)\right](\partial^2 - k^2)U, \tag{21}$$

where for brevity we use the notation $\partial = \partial/\partial x_1$. We find the decaying at $x_1 \to -\infty$ solution of (21)

$$U(x_1) = C_1 \exp(\eta_+ x_1) + C_2 \exp(\eta_- x_1), \tag{22}$$

where

$$\eta_\pm = \sqrt{k^2 - \overline{k}_\pm^2}, \quad \overline{k}_\pm^2 = \frac{\mu - \kappa\omega^2 \pm \sqrt{(\mu-\kappa\omega^2)^2 + 4\mu\ell^2\rho\omega^2}}{2\mu\ell^2} \tag{23}$$

and C_1 and C_2 are integration constants. In previous formulas, $\overline{k}_+$ and $\overline{k}_-$ correspond to the wavenumbers of an antiplane wave in the bulk. More specifically, $\overline{k}_-$ is the propagative root and $\overline{k}_+$ is the vanishing root. As a result, the displacement is given by

$$u = \left[C_1 e^{\eta_+ x_1} + C_2 e^{\eta_- x_1}\right] \exp\left[i(kx_2 - \omega t)\right]. \tag{24}$$

Substituting this form of the solution into the boundary conditions (19), we obtain the following system

$$\mathbb{A} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{25}$$

where

$$\mathbb{A} = \begin{pmatrix} \eta_+^2\ell^2\mu - \overline{a}_3k^2\mu & \eta_-^2\ell^2\mu - \overline{a}_3k^2\mu \\ \eta_+\left(\mu\left(\overline{a}_3k^2 + \ell^2\left(\eta_+^2 - 2k^2\right) - 1\right) + \kappa\omega^2\right) & \eta_-\left(\mu\left(\overline{a}_3k^2 + \ell^2\left(\eta_-^2 - 2k^2\right) - 1\right) + \kappa\omega^2\right) \end{pmatrix}. \tag{26}$$

For that the system admit a non-vanishing solution, the determinant of $\mathbb{A}$ must be zero. Thus, the dispersion equation for surface waves will be computed numerically by finding the roots of

$$\det(\mathbb{A}) = 0, \tag{27}$$

that correspond to surface waves propagating towards the direction $\mathbf{i}_2$ (with k, η_+ and η_- reals and positives).
In the numerical section, the results will be presented with respect to the phase velocity of such surface waves, *i.e.* $c = \omega/k$.

3.3 Surface elasticity

For the anti-plane shear deformation (15), the motion equations and the boundary conditions reduce to

$$\rho\ddot{u} = \mu\Delta u, \tag{28}$$

$$-m\ddot{u} + \mu_s u_{,22} = \mu u_{,1}. \tag{29}$$

Substituting (16) into (28), we obtain the ordinary differential equation with respect to U

$$\left[\mu(\partial^2 - k^2) + \rho\omega^2\right] U = 0. \tag{30}$$

Assuming that the displacement decays exponentially with distance from the half-space surface, we find the solution of (30)

$$U = U_0 \exp\left[\sqrt{k^2 - \omega^2/c_T^2}\, x_1\right],$$

where U_0 is an amplitude and $c_T = \sqrt{\frac{\mu}{\rho}}$ is the phase velocity of shear waves in the bulk. As a result, we obtain the expression for an anti-plane surface wave of the form

$$u = U_0 \exp\left[\sqrt{k^2 - \omega^2/c_T^2}\, x_1\right] \exp\left[i(kx_2 - \omega t)\right]. \tag{31}$$

Substituting (31) into (29), we obtain the dispersion equation [25]

$$D_S(\omega, k) \equiv m\omega^2 - \mu_s k^2 - \mu\sqrt{k^2 - \frac{\omega^2}{c_T^2}} = 0. \tag{32}$$

μ (Pa)	ρ (kg/m^3)	ℓ (m)	κ (kg/m)	$\bar{a}_{33}$ (m^2)
34.6×10^6	195	170×10^{-6}	136×10^{-6}	5.8×10^{-9}

Table 1: Parameters of the strain gradient model

μ (Pa)	ρ (kg/m^3)	μ_s (Pa m)	m (kg/m^2)
34.6×10^6	195	52.6	0.2

Table 2: Parameters of the surface energy model

The latter equation transforms to

$$c^2 = \frac{\mu_s}{m} + \frac{\mu}{m}\frac{1}{|k|}\sqrt{1 - \frac{c^2}{c_T^2}} \tag{33}$$

with solution of the form

$$|k| = \frac{\mu\sqrt{1 - \frac{c^2}{c_T{}^2}}}{m(c^2 - c_S^2)}, \tag{34}$$

where $c_S = \sqrt{\mu_s/m}$ is the shear wave velocity in the thin film associated with the Gurtin–Murdoch model. Obviously, the wavenumber k is real if and only if

$$c \leq c_T, \quad c > c_S. \tag{35}$$

The last equation means that the surface antiplane waves exist if the surface film is softer than material in the bulk as for the Love waves [24].

4 Numerical results and comparison

In this section some numerical results are presented. The parameters used in the computation are listed in Tables 1 and 2. The parameters have been chosen so that the limit values of the phase velocity for $k = 0$ and $k \to \infty$ are the same for both models. Concerning the strain gradient model, this choice of parameters corresponds to normal dispersion (*i.e.* the phase velocity decreases when increasing the frequency).

4.1 Dispersion relations

We start the analysis by the description of the dispersion relations of surface waves, that are depicted in Fig. 1. As expected, the two models share the same phase velocities c_T and c_S, respectively for $k = 0$ and $k \to +\infty$. These limit values are represented by a black dotted line for c_S and a black dashed line for c_T. From the same figure, it can be observed that the strain gradient model (black solid line) is decreasing faster towards the limit value c_S than the surface energy model (gray solid line). Using a different combination of material properties produces different results, but the overall behaviour is qualitatively the same. The choice of the parameters have been done by fitting the curve for low values of k (*i.e.* very long wavelengths).

4.2 Decay rate

With the aim of comparing the surface wave solutions, it is also interesting to see how rapidly the perturbation is decaying with the depth. To this end, in Fig. 2, the amplitude of the solution $U(x_1)$ is plotted in function of the depth and of the wavenumber for the strain gradient model (left) and the surface energy model (right). As it can be observed, for a given wavenumber, the solution for the

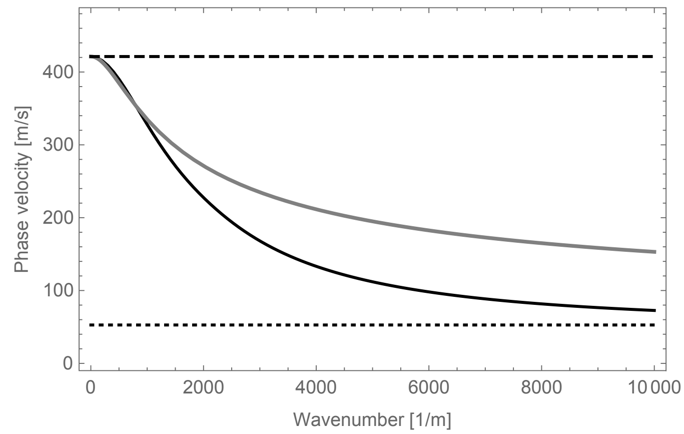

Figure 1: Dispersion relations for strain gradient model (black) and surface energy model (grey). The dashed line represents the phase velocity c_T, the dotted line the velocity c_S

surface energy model is always decaying more rapidly. In order to better compare these results, in Figure 4 we plotted the depth for which the amplitude is divided by two, in function of the wavenumber.

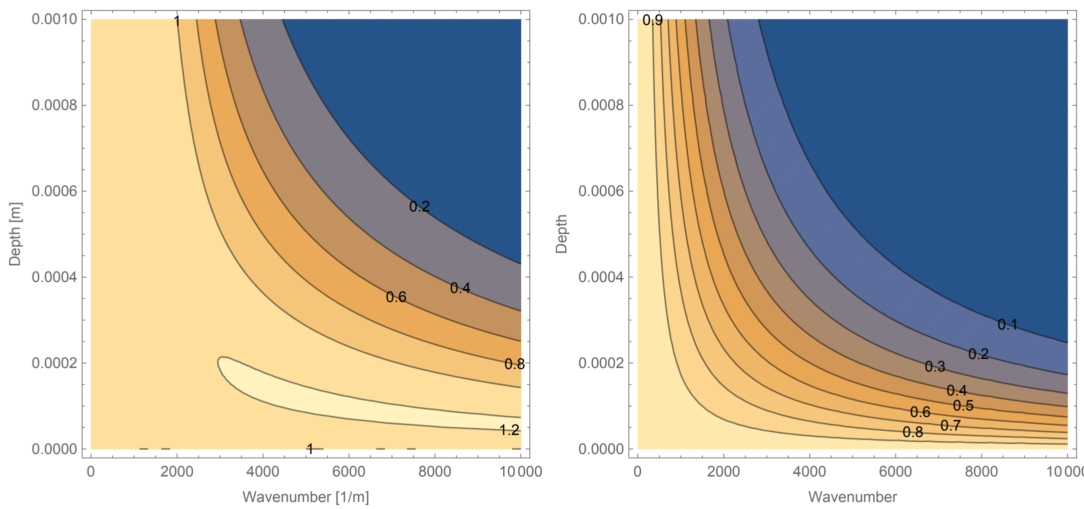

Figure 2: Decaying of the solutions with respect to depth for the strain gradient (left) and the surface energy (right) models.

It can be also remarked that for very low wavenumbers, *i.e.* for very long wavelengths, the amplitude is almost constant with depth. This means that, even if the surface wave exists for every value of k (*i.e.* the amplitude of the perturbation is going to zero for $x_1 \to -\infty$), there is a range of wavenumbers for which the amplitude is decaying at infinity very slowly. Analyzing solution (23) and (31), it can be observed that this happens when the velocity of the surface wave approaches the phase velocity in the bulk and the root corresponding to the vanishing part of the perturbation is almost zero. For the surface energy model, given that the phase velocity of the bulk does not depend on the frequency, this happens only in the vicinity $k = 0$, where the two phase velocities match. However, this is not the case for the strain gradient model. Indeed, the plot of dispersion relations of both antiplane bulk and surface waves in Fig. 3, shows that the two curves almost superpose up to a given value of the wave number k, in this numerical case 2000 [1/m], and then they start to diverge. As shown in equation

(23), this causes the root η^- to be almost vanishing. In formulas, we have

$$k \simeq \overline{k}_- \Rightarrow \eta^- \simeq 0. \tag{36}$$

This means that the dispersion in the bulk has an impact on the decaying rate of the surface waves, for the strain gradient model.

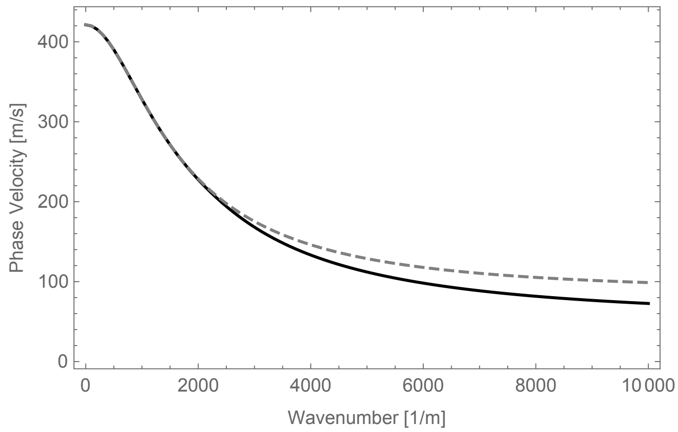

Figure 3: Plot of the phase velocity in function of the wave number for antiplane bulk waves (gray dashed) and surface waves (black solid).

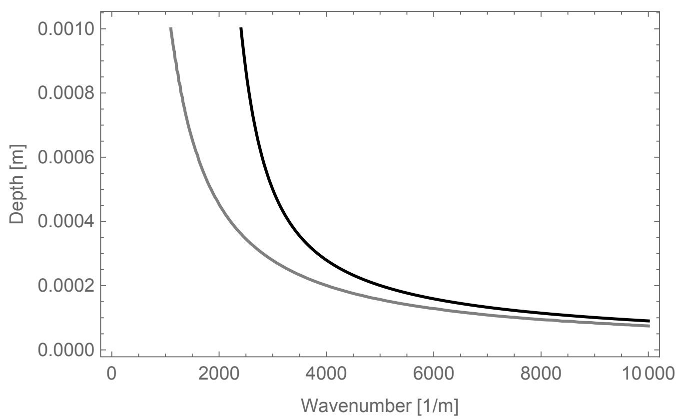

Figure 4: Plot of the depth for which the displacement is divided by two for the strain gradient (black solid) and the surface energy (gray solid) models.

5 Conclusion

We consider the propagation of surface waves in an elastic halfspace regarding the both theory of surface elasticity by Gurtin-Murdoch and the Mindlin's strain gradient elasticity. We demonstrate that the both theories give similar qualitative results for the phase velocities and for the displacement amplitudes. Nevertheless, there are some quantitative differences in the dispersion curves, the both theories are in a good coincidence for the limiting cases of long waves ($k \to 0$) and short waves ($k \to \infty$). There is a range of wavenumbers where the difference in the phase velocities is more significant. Moreover, analyzing the decaying of the displacements with the depth, we can see, that the displacements obtained within the surface elasticity decaying much faster than one derived using the strain gradient elasticity. So, the surface elasticity model produces antiplane waves which are more localized near the surface whereas within the strain gradient elasticity we have more penetrating solutions. This difference seems to be natural. Indeed, from the physical point of view the surface elasticity introduces the changes in material properties just on the surface whereas the strain gradient elasticity describes the microstructure of material in the bulk. Considering short waves limit ($k \to \infty$) one should be aware of influence of grains and subgrains, dislocations and other defects presented in the real material microstructure. So, in this case discrete models may be more efficient, such as used in the lattice dynamics [42].

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