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# Bernstein-type theorem for $\phi$-Laplacian

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**Abstract**

In this paper we obtain a solution to the second-order boundary value problem of the form $\frac{d}{dt}\Phi'(\dot{u}) = f(t,u,\dot{u})$, $t \in [0,1]$, $u\colon \mathbb{R} \to \mathbb{R}$ with Sturm–Liouville boundary conditions, where $\Phi\colon \mathbb{R} \to \mathbb{R}$ is a strictly convex, differentiable function and $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a suitable growth condition. Our result is based on a priori bounds for the solution and homotopical invariance of the Leray–Schauder degree.

**Keywords:** $\Phi$-Laplacian; Boundary value problem; Fixed point; A priori bounds; Leray–Schauder degree

## 1 Introduction

In this paper we study the existence of solutions to the boundary value problems (BVPs)

$$\frac{d}{dt}\Phi'(\dot{u}) = f(t,u,\dot{u}), \quad t \in [0,1] \tag{P}$$

$$-\alpha u(0) + \beta\dot{u}(0) = A, \qquad au(1) + b\dot{u}(1) = B, \tag{BC}$$

where $\Phi'$ is an increasing homeomorphism, the scalar function $f$ is continuous, $\alpha, a > 0$ and $\beta, b \geq 0$.

The solvability of various second-order two-point BVPs with p- or $\Phi$-Laplacian has been discussed extensively in the literature, see the recent works [1–9] for results, methods, and references.

In 1912, Bernstein [10] proved that the BVP

$$u'' = f\left(t,u,u'\right) \tag{1}$$

$$u(0) = A, \qquad u(1) = B, \tag{2}$$

has a unique $C^2$-solution if $f(t,u,v)$ is continuous, has continuous partial derivatives $f_u$ and $f_v$ on $[0,1] \times \mathbb{R}^2$, there is a constant $K > 0$ such that

$$f_u(t,u,v) \geq K \quad \text{on } [0,1] \times \mathbb{R}^2 \tag{3}$$

and

$$\left|f(t,u,v)\right| \leq A(t,u)v^2 + B(t,u) \quad \text{on } [0,1] \times \mathbb{R}^2,$$

where $A$, $B$ are functions bounded on each compact subset of $[0,1] \times \mathbb{R}$.

In 1978, Granas et al. [11] proved similar results for (1) with either Dirichlet, Neumann, or periodic boundary conditions. The authors have established the existence of solutions to the considered problems by replacing (3) with the following assumption: There is a constant $M > 0$ such that

$$uf(t,u,0) > 0 \quad \text{for } t \in [0,1] \text{ and } |u| > M.$$

The uniqueness of the solution to (1.4), (BC) follows from the assumption that the partial derivatives $f_u$ and $f_v$ exist, are bounded, and $f_u \geq 0$ on $[0,1] \times \mathbb{R}^2$.

In 1983, Baxley [12] proved Bernstein-type theorems for boundary value problems for (1) with nonlinear boundary conditions. In 1988, Frigon and O'Regan [13] established existence results of this type for (1), (2) and (1), (BC).

The aim of this paper is to give Bernstein-type existence theorems for BVPs with $\Phi$-Laplacian. Throughout this paper we assume that $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

($\Phi_1$) $\Phi$ is strictly convex, differentiable and $\Phi(x)/|x| \to \infty$ as $|x| \to \infty$;
($\Phi_2$) $\Phi(0) = \Phi'(0) = 0$;
($\Phi_3$) $(\Phi')^{-1}$ is continuously differentiable;
($\Phi_4$) there exists a constant $K_\Phi > 1$ such that

$$K_\Phi \Phi(x) \leq \Phi'(x)x \quad \text{for all } x \in \mathbb{R}.$$

Assumption ($\Phi_1$) guarantees that $\Phi'$ is an increasing homeomorphism and so $(\Phi')^{-1}$ exists. Note that $\Phi(x) = \frac{1}{p_1}|x|^{p_1} + \cdots \frac{1}{p_n}|x|^{p_n}$, $1 < p_i \leq 2$, is in the considered class of functions, and if $n = 1$, then the differential operator on the left-hand side of the equation is a $p$-Laplacian. A more general form of $\Phi$ is provided by an N-function satisfying the $\nabla_2$-condition (see [14]).

We assume also that $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following:

($f_1$) There exists a constant $M > 0$ such that

$$xf(t,x,0) > 0 \quad \text{for } |x| > M,$$

($f_2$) There exist positive functions $S$, $T$ bounded on bounded sets such that

$$\bigl|f(t,x,v)\bigr| \leq S(t,x)\bigl(\Phi'(v)\cdot v - \Phi(v)\bigr) + T(t,x).$$

Now, we can state our main result.

**Main Theorem** *Suppose that $\Phi$ and $f$ satisfy ($\Phi_1$)–($\Phi_4$) and ($f_1$), ($f_2$), respectively. Then problem (P), (BC) has at least one solution in $C^2([0,1],\mathbb{R})$.*

To establish the validity of the above result, we apply the Leray–Schauder degree theory on a suitable constructed map. To define its domain, we use a priori bounds.

To prove the existence, we use topological methods. This approach has already been used by many authors. In [11] and [13] the authors considered the case of a Laplace operator with various boundary conditions. Generalizations to the $p$-Laplacian and to the operator defined by an arbitrary increasing homeomorphism were developed in [3] and [5], respectively. The main idea in the paper [11] was to use the topological transversality theorem. This is a fixed point type theorem (see [15]). We decided to use an approach via Leray–Schauder degree theory instead, since it is essentially equivalent but the degree theory is familiar to a broader audience.

However, in [3] and [5] authors subject the equation to very specific boundary conditions, namely $u(0) = A$, $\dot{u}(1) = B$. In order to show the existence for general Sturm–Liouville conditions, more effort has to be put in as can be seen below.

## 2 Auxiliary results

**Lemma 2.1** *Let $X$ be a metric space, and let $G\colon X \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that*

(1) *for every $v \in X$, function $g_v\colon \mathbb{R} \to \mathbb{R}$, defined by $g_v(c) = G(v,c)$, is an increasing homeomorphism;*

(2) *if $\{v_n\}$ is bounded and $b_n \to \pm\infty$, then $G(v_n, b_n) \to \pm\infty$.*

*Then, for each fixed constant $C \in \mathbb{R}$, the function $c\colon X \to \mathbb{R}$ defined by $G(v, c(v)) = C$ is continuous.*

*Proof* Suppose that function $c$ is not continuous, i.e., there exist $\epsilon > 0$ and a sequence $v_n$ converging to some $v_0$ such that $|c(v_n) - c(v_0)| > \epsilon$. By the definition of $c$, $G(v_n, c(v_n)) = C$. In particular, both $v_n$ and $G(v_n, c(v_n))$ are bounded. This, together with (2), implies that $c(v_n)$ is bounded. Take a subsequence $c(v_{n_k})$ which converges to some $c'$. Note that $c' \neq c(v_0)$ because $|c(v_n) - c(v_0)| > \epsilon$. By the continuity of $G$, we have $G(v_{n_k}, c(v_{n_k})) \to G(v_0, c')$. But $G(v_{n_k}, c(v_{n_k})) = C$ and $G(v_0, c') \neq G(v_0, c(v_0)) = C$ by (1). A contradiction. □

If $g_v$ is differentiable and $g_v'$ is positive, then the conclusion follows from implicit function theorem. However, in the problem that we consider, $g_v'$ is only non-negative.

*Remark* 2.2 Note that this trivializes in [3, 5]. For boundary conditions considered therein $c_1$ and $c_2$ are constants independent of $v$. We cannot proceed in such a way here.

Now introduce the map $\hat{K}\colon C^0([0,1]) \times \mathbb{R} \times \mathbb{R} \to C^1([0,1])$ defined by

$$\hat{K}(v, c_1, c_2)(t) = c_1 + \int_0^t (\Phi')^{-1} \left( \int_0^\tau v(s)\, ds + c_2 \right) d\tau,$$

and by $C^1_{\mathrm{BC}}([0,1])$ denote the set of the functions in $C^1([0,1])$ which satisfies (BC).

For every $v$, we would like to choose $c_1$ and $c_2$ in such a way that $u = \hat{K}(v, c_1, c_2)$ is an element of $C^1_{\mathrm{BC}}$. Moreover, we need that $c_1$ and $c_2$ depend continuously on $v$.

**Lemma 2.3** *Let ($\Phi_1$) and ($\Phi_3$) hold. Then, for every fixed $v \in C^0([0,1])$, there exists a unique pair of constants $c_1(v)$, $c_2(v)$ such that $\hat{K}(v, c_1(v), c_2(v)) \in C^1_{\mathrm{BC}}([0,1])$. Moreover, the functions $c_1, c_2\colon C^0([0,1]) \to \mathbb{R}$ are continuous.*

*Proof* Put $u = K(v, c_1, c_2)$. Then

$$u(0) = c_1, \qquad u(1) = c_1 + \int_0^1 (\Phi')^{-1} \left( \int_0^\tau v(s)\, ds + c_2 \right) d\tau$$

and

$$\dot{u}(0) = \left(\Phi'\right)^{-1}(c_2), \qquad \dot{u}(1) = \left(\Phi'\right)^{-1}\left(\int_0^1 v(s)\,ds + c_2\right).$$

Clearly, $u$ will satisfy the boundary conditions (BC) if $c_1$ and $c_2$ are such that $-\alpha c_1 + \beta(\Phi')^{-1}(c_2) = A$ and

$$a\left[c_1 + \int_0^t \left(\Phi'\right)^{-1}\left(\int_0^\tau v(s)\,ds + c_2\right)d\tau\right] + b\left(\Phi'\right)^{-1}\left(\int_0^1 v(s)\,ds + c_2\right) = B,$$

from where we get

$$\begin{aligned} &a\left[-\frac{A}{\alpha} + \frac{\beta}{\alpha}\left(\Phi'\right)^{-1}(c_2) + \int_0^1 \left(\Phi'\right)^{-1}\left(\int_0^\tau v(s)\,ds + c_2\right)d\tau\right] \\ &\quad + b\left(\Phi'\right)^{-1}\left(\int_0^1 v(s)\,ds + c_2\right) = B. \end{aligned} \tag{4}$$

Since $(\Phi')^{-1}$ is increasing, the function

$$\begin{aligned} G(v,c) = a&\left[-\frac{A}{\alpha} + \frac{\beta}{\alpha}\left(\Phi'\right)^{-1}(c) + \int_0^1 \left(\Phi'\right)^{-1}\left(\int_0^\tau v(s)\,ds + c\right)d\tau\right] + \\ &+ b\left(\Phi'\right)^{-1}\left(\int_0^1 v(s)\,ds + c\right) \end{aligned}$$

is increasing with respect to $c$. We can apply Lemma 2.1 for $C = B$ to conclude that (4) defines a unique constant $c_2$ depending continuously on $v$, and so $c_1$ is also unique and depends continuously on $v$. □

Now, for $\lambda \in [0,1]$, consider the family of differential equations

$$\frac{d}{dt}\Phi'(\dot{u}) = \lambda f(t,u,\dot{u}), \quad t \in [0,1]. \tag{$P_\lambda$}$$

Note that if $u$ is a $C^1$ solution to problem ($P_\lambda$), then $u \in C^2$. Indeed, $\dot{u}$ reads

$$\dot{u}(t) = \left(\Phi'\right)^{-1}\left(\int_0^t \lambda f(\tau,u,\dot{u})\,d\tau + c\right),$$

and by assumption ($\Phi_3$) and the continuity of $f$, it is continuously differentiable.

The next lemma is a variant of [13, Theorem 3.3].

**Lemma 2.4** *Assume that ($\Phi_1$)–($\Phi_3$) and ($f_1$) hold. Let $u \in C^1([0,1])$ be a solution to ($P_\lambda$), (BC) for $\lambda \in [0,1]$. If $|u|$ achieves its maximum at $t_0 \in (0,1)$, then*

$$\left|u(t)\right| \le M \quad \text{for } t \in [0,1].$$

*Proof* Suppose on the contrary that $|u|$ achieves its maximum at $t_0 \in (0,1)$. We can assume that $u(t_0) > M$. In the case $u(t_0) \le -M$ the proof is similar. It is clear that $\dot{u}(t_0) = 0$. For

$t \in [0,1]$ we have

$$\int_{t_0}^{t} (t-\sigma)u(\sigma)\frac{d}{d\tau}\Phi'(\dot{u})(\tau)\Big|_{\tau=\sigma} d\sigma = t\int_{t_0}^{t} u\frac{d}{d\tau}\Phi'(\dot{u})\,d\sigma - \int_{t_0}^{t}\sigma u\frac{d}{d\tau}\Phi'(\dot{u})\,d\sigma.$$

Since $\Phi'(\dot{u}(t_0)) = \Phi'(0) = 0$,

$$\int_{t_0}^{t} u\frac{d}{d\tau}\Phi'(\dot{u})\,d\sigma = u\Phi'(\dot{u})\Big|_{t_0}^{t} - \int_{t_0}^{t}\dot{u}\Phi'(\dot{u})\,d\sigma = u(t)\Phi'\big(\dot{u}(t)\big) - \int_{t_0}^{t}\dot{u}\Phi'(\dot{u})\,d\sigma$$

and

$$\begin{aligned}\int_{t_0}^{t}\sigma u\frac{d}{d\tau}\Phi'(\dot{u})\,d\sigma &= \sigma u\Phi'(\dot{u})\big|_{t_0}^{t} - \int_{t_0}^{t}(u+\sigma\dot{u})\Phi'(\dot{u})\,d\sigma \\ &= tu(t)\Phi'\big(\dot{u}(t)\big) - \int_{t_0}^{t}u\Phi'(\dot{u})\,d\sigma - \int_{t_0}^{t}\sigma\dot{u}\Phi'(\dot{u})\,d\sigma.\end{aligned}$$

Combining the above, we get

$$\int_{t_0}^{t}(t-\sigma)u(\sigma)\frac{d}{d\tau}\Phi'(\dot{u})(\tau)\Big|_{\tau=\sigma} d\sigma = \int_{t_0}^{t}u\Phi'(\dot{u})\,d\sigma + \int_{t_0}^{t}(\sigma-t)\dot{u}\Phi'(\dot{u})\,d\sigma.$$

Hence, using $(P_\lambda)$, we have

$$\int_{t_0}^{t}(t-\sigma)\big(\lambda u(\sigma)f\big(\sigma,u(\sigma),\dot{u}(\sigma)\big) + \dot{u}(\sigma)\Phi'\big(\dot{u}(\sigma)\big)\big)\,d\sigma = \int_{t_0}^{t}u(\sigma)\Phi'\big(\dot{u}(\sigma)\big)\,d\sigma.$$

Note that, for $0<\lambda\le 1$, $xf(t,x,0)>0$, $|x|>M$ implies $\lambda xf(t,x,0)>0$, $|x|>M$. Thus, by assumption $(f_1)$, $\lambda u(t_0)f(t_0,u(t_0),0)>0$. The continuity of $f$, $u$, and $\dot{u}$ implies that there exists a neighborhood $N\subset(0,1)$ of $t_0$ such that

$$\lambda u(t)f\big(t,u(t),\dot{u}(t)\big)>0 \quad \text{for } t\in N.$$

Since $u\in C^1$ and achieves its maximum at $t_0$, there exist $t_0^-$ and $t_0^+$ such that

- $u(t)>M$ for $t\in(t_0^-,t_0^+)$,
- $\dot{u}(t)\ge 0$ on $(t_0^-,t_0]$,
- $\dot{u}(t)\le 0$ on $[t_0,t_0^+)$.

Hence $\Phi'(\dot{u}(t))\ge 0$ for $t\in(t_0^-,t_0]$ and $\Phi'(\dot{u}(t))\le 0$ for $t\in[t_0,t_0^+)$, since $\Phi'$ is increasing. This implies that

$$\int_{t_0}^{t}(t-\sigma)\dot{u}(\sigma)\Phi'\big(\dot{u}(\sigma)\big)\,d\sigma\ge 0 \quad \text{for } t\in\big(t_0^-,t_0^+\big)$$

and

$$\int_{t_0}^{t}u(\sigma)\Phi'\big(\dot{u}(\sigma)\big)\,d\sigma\le 0 \quad \text{for } t\in\big(t_0^-,t_0^+\big).$$

It follows that for $t$ close to $t_0$

$$0<\int_{t_0}^{t}(t-\sigma)\big(\lambda u(\sigma)f\big(\sigma,u(\sigma),\dot{u}(\sigma)\big)+\dot{u}(\sigma)\Phi'\big(\dot{u}(\sigma)\big)\big)\,d\sigma = \int_{t_0}^{t}u(\sigma)\Phi'\big(\dot{u}(\sigma)\big)\,d\sigma\le 0,$$

a contradiction. Thus $u(t_0)\le M$. □

**Lemma 2.5** *Assume that* ($\Phi_1$)–($\Phi_3$) *and* ($f_1$) *hold. Let* $u \in C^1([0,1])$ *be a solution to* $(P_\lambda)$, (BC) *for* $\lambda \in [0,1]$. *There exists a constant* $M_0 > 0$ *independent of* $\lambda$ *and* $u$ *such that*

$$\left|u(t)\right| \le M_0 \quad \textit{for } t \in [0,1].$$

*Proof* For $\lambda = 0$, problem $(P_\lambda)$ has a unique linear solution, so there is a constant $C > 0$ such that $|u(t)| \le C$ for $t \in [0,1]$. Let $0 < \lambda \le 1$. If $|u|$ achieves its maximum at $t = 0$, then $u(0)\dot{u}(0) \le 0$. The boundary conditions give

$$u(0)\bigl(A + \alpha u(0)\bigr) = \beta u(0)\dot{u}(0) \le 0,$$

and consequently $|u(0)| \le |A/\alpha|$. Similarly, $|u(1)| \le |B/a|$. If the maximum is at any $t_0 \in (0,1)$, then by Lemma 2.4 we get $|u(t)| \le M$. As a result, for $\lambda \in [0,1]$, we have

$$\left|u(t)\right| \le M_0 = \max\bigl\{M, |A/\alpha|, |B/a|\bigr\} \quad \text{for } t \in [0,1]. \qquad \square$$

Now we provide bounds for $\dot{u}$. The proof of the following theorem is based on [13].

**Lemma 2.6** *Assume that* ($\Phi_1$)–($\Phi_4$), ($f_1$), *and* ($f_2$) *hold. Let* $u \in C^1([0,1])$ *be a solution to* $(P_\lambda)$, (BC) *for* $\lambda \in [0,1]$. *There exists a constant* $M_1 > 0$, *independent of* $\lambda$ *and* $u$, *such that*

$$\left|\dot{u}(t)\right| \le M_1 \quad \textit{for } t \in [0,1].$$

*Proof* Since we have obtained a priori bounds $|u(t)| \le M_0$, it is easy to observe that there exists a constant $C \ge 0$ independent of $\lambda$ and $u$ such that

$$\left|\dot{u}(t_0)\right| \le C$$

for some $t_0 \in [0,1]$. The point $t_0$ belongs to an interval $[\mu,\nu] \subset [0,1]$ such that the sign of $\dot{u}(t)$ does not change in $[\mu,\nu]$ and $\dot{u}(\mu) = \dot{u}(t_0)$ and/or $\dot{u}(\nu) = \dot{u}(t_0)$.

Assume that $\dot{u}(\mu) = \dot{u}(t_0)$ and $\dot{u}(t) \ge 0$ for every $t \in [\mu,\nu]$. The other cases are treated similarly and the same bound is obtained.

Denote by $S_0$, $T_0$ the upper bounds of $S$ and $T$, respectively, on $[0,1] \times [-M_0, M_0]$. Since

$$\left|\lambda f(t,u,\dot{u})\right| \le S_0\bigl(\Phi'(\dot{u})\dot{u} - \Phi(\dot{u})\bigr) + T_0,$$

we have

$$\int_\mu^t \frac{S_0\dot{u}|\frac{d}{dt}\Phi'(\dot{u})|}{S_0(\Phi'(\dot{u})\dot{u} - \Phi(\dot{u})) + T_0}\,d\tau \le S_0\int_\mu^t \dot{u}\,d\tau \le 2S_0M_0.$$

For $\mu \le \tau \le t$, we have

$$\begin{aligned}
&\bigl(\Phi'\bigl(\dot{u}(\tau)\bigr)\dot{u}(\tau) - \Phi\bigl(\dot{u}(\tau)\bigr)\bigr) - \bigl(\Phi'\bigl(\dot{u}(\mu)\bigr)\dot{u}(\mu) - \Phi\bigl(\dot{u}(\mu)\bigr)\bigr) \\
&\quad = \int_\mu^t \frac{d}{dt}\bigl(\Phi'\bigl(\dot{u}(\tau)\bigr)\dot{u}(\tau) - \Phi\bigl(\dot{u}(\tau)\bigr)\bigr)\Big|_{t=\sigma}\,dt = \int_\mu^\tau \dot{u}\frac{d}{dt}\Phi'(\dot{u})\,d\sigma.
\end{aligned}$$

Note that since $\Phi$ is a convex differentiable function and $\Phi(0) = 0$, we have $\Phi(x) \le \Phi'(x)x$ for every $x \in \mathbb{R}$. Thus, $0 \le S_0(\Phi'(\dot{u}(\mu))\dot{u}(\mu) - \Phi(\dot{u}(\mu)))$. On the other hand, there exists $C_0 \ge 0$ such that $S_0(\Phi'(\dot{u}(\mu))\dot{u}(\mu) - \Phi(\dot{u}(\mu))) + T_0 \le C_0$. Hence,

$$0 \le S_0\bigl(\Phi'\bigl(\dot{u}(\tau)\bigr)\dot{u}(\tau) - \Phi\bigl(\dot{u}(\tau)\bigr)\bigr) + T_0 \le S_0 \int_\mu^\tau \dot{u}\left|\frac{d}{dt}\Phi'(\dot{u})\right| d\sigma + C_0 + T_0.$$

Set $g(\tau) = S_0 \int_\mu^\tau \dot{u}|\frac{d}{dt}\Phi'(\dot{u})|\,d\sigma + C_0$, then integration by substitution yields

$$\log\left(\frac{g(t) + T_0}{C_0 + T_0}\right) = \int_{C_0}^{g(t)} \frac{1}{x + T_0}\,dx = \int_\mu^t \frac{S_0 \dot{u} \frac{d}{dt}\Phi'(\dot{u})}{g(\tau) + T_0}\,d\tau \le 2S_0M_0.$$

Thus

$$g(t) \le (T_0 + C_0)e^{2S_0M_0} - T_0$$

and by $(\Phi_4)$

$$(k_\Phi - 1)\Phi\bigl(\dot{u}(t)\bigr) \le \Phi'\bigl(\dot{u}(t)\bigr)\dot{u}(t) - \Phi\bigl(\dot{u}(t)\bigr) \le \frac{1}{S_0}\bigl((T_0 + C_0)e^{2S_0M_0} - T_0\bigr).$$

The last inequality gives $|\dot{u}(t)| \le M_1$ for all $t \in [0, 1]$. □

## 3 Proof of the main theorem

Introduce the map $N\colon C^1_{\mathrm{BC}}([0, 1]) \to C^0([0, 1])$ defined by

$$N(u)(t) = f(t, u, \dot{u}),$$

and for $\lambda \in [0, 1]$ consider the composition $\widehat{K} \circ \lambda N$, where the map $\widehat{K}\colon C([0, 1]) \to C^1([0, 1])$ is well defined by Lemma 2.3. Moreover, by the Arzela–Ascoli theorem, $\hat{K}$ is compact. Since $N$ is continuous, the composition $\hat{K} \circ \lambda N$ is also compact.

The fixed points of $\widehat{K} \circ \lambda N$ are of interest to us. Instead of looking for fixed points of $K \circ N$, one can look for zeros of $Id - K \circ N$. For this we will use the Leray–Schauder degree and its homotopical invariance. Consider the homotopy $H\colon [0, 1] \times C^1_{\mathrm{BC}}([0, 1]) \to C^1_{\mathrm{BC}}([0, 1])$ given by

$$H(\lambda, u) = \bigl(Id - \widehat{K}(\lambda N)\bigr)(u).$$

Observe first that $H(0, l) = 0$, where $l = \widehat{K}(0)$ is unique. Thus, if $\overline{B}_r(0)$ is a closed ball with center 0 and radius $r$ with the property $l \in B_r(0)$, then

$$\deg\bigl(H(0, u), B_r(0)\bigr) = 1.$$

It is well known that if $r$ is such that

$$H(\lambda, u) \ne 0 \quad \text{for } \lambda \in [0, 1] \text{ and } u \in \partial B_r(0), \tag{5}$$

then also

$$\deg\bigl(H(1,u),B_r(0)\bigr)=1.$$

It is not hard to check that $u\in C^1_{\mathrm{BC}}([0,1])$ is a zero of $Id-\widehat{K}\circ\lambda N$ if and only if $u$ is a solution to BVP $(P_\lambda)$, (BC). Thus, each zero $u\in C^1_{\mathrm{BC}}([0,1])$ of $Id-\widehat{K}\circ\lambda N$ satisfies the bound

$$\|u\|_{C^1([0,1])}<K,$$

where $K=\max\{M_0,M_1\}+1$, where $M_0$ and $M_1$ are the constants from Lemmas 2.5 and 2.6. Clearly, (5) holds for $r=K$ and so, since in particular $l\in B_K(0)$, we have

$$\deg\bigl(H(1,u),B_K(0)\bigr)=1.$$

This means that $Id-\widehat{K}\circ\lambda N$ has at least one zero $u_0\in B_K(0)$, which is a $C^1([0,1])$-solution to BVP of family $(P_\lambda)$ arisen when $\lambda=1$, that is, $u_0$ is a $C^1([0,1])$-solution to (P), (BC). However, as a solution of (P), $u_0$ is such that, for some constant $c$, we have

$$u_0'(t)=\bigl(\Phi'\bigr)^{-1}\biggl(\int_0^t f\bigl(\tau,u_0(\tau),u_0'(\tau)\,d\tau\bigr)+c\biggr),$$

from where, keeping in mind $(\Phi_3)$ and the continuity of $f$, we get $u_0\in C^2([0,1])$.

## 4 Examples

*Example* 4.1 Let $\Phi:\mathbb{R}\to\mathbb{R}$, $\Phi(x)=\frac{1}{p}|x|^p$, $1<p\le 2$. It is easy to see that the function $\Phi$ satisfies assumptions $(\Phi_1)$–$(\Phi_3)$. Moreover, since $\frac{1}{p}|x|^p\le x^2|x|^{p-2}$, one can take $K_\Phi=p$ in $(\Phi_4)$.

Define the function $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by the formula

$$f(t,x,v)=\frac{(x^3-x)(1+|v|^{\frac{p+1}{2}})}{1+t^2}.$$

One can easily check that

$$xf(t,x,0)=\frac{x^4-x^2}{1+t^2}>0$$

for $|x|>1$. Since $\Phi'(v)v-\Phi(v)=\frac{1-p}{p}|v|^p$ and

$$\bigl|f(t,x,v)\bigr|\le\frac{p|x^3-x|}{(p-1)(1+t^2)}\cdot\frac{1-p}{p}|v|^p+\frac{|x^3-x|}{1+t^2},$$

assumption $(f_2)$ is satisfied with $S(t,x)=\frac{p|x^3-x}{(p-1)(1+t^2)}$ and $T(t,x)=\frac{|x^3-x|}{1+t^2}$.

Assume that $\Phi$ satisfies assumptions $(\Phi_1)$–$(\Phi_4)$ and that the functions $S,T:\mathbb{R}\times\mathbb{R}\to(0,\infty)$ are continuous and such that $xT(t,x)>0$ for $|x|>M$. Then the function

$$f(t,x,v)=S(t,x)\bigl(\Phi'(v)v-\Phi(v)\bigr)+T(t,x)$$

satisfies our assumptions.

*Example* 4.2 Let $\Phi(x) = \sum_{i=1}^{n} \frac{1}{p_i}|x|^{p_i}$, $1 < p_i \leq 2$, for $i = 1, 2, \dots, n$. Then $\Phi$ satisfies assumptions $(\Phi_1)$–$(\Phi_3)$. Assumption $(\Phi_4)$ is satisfied with $K_\Phi = \min\{p_1, \dots, p_n\}$.

*Example* 4.3 Let $\Phi(x) = \frac{1}{p}|x|^p \log(1 + x^2)$. The function $\Phi$ satisfies all assumptions. In particular, as $\Phi'(x) = x|x|^{p-2} \log(1 + x^2) + \frac{1}{p}|x|^p \frac{2x}{1+x^2}$ and

$$p\frac{1}{p}|x|^p \log\bigl(1 + x^2\bigr) \leq x^2|x|^{p-2} \log\bigl(1 + x^2\bigr) + \frac{1}{p}\frac{2x}{1 + x^2},$$

we can take $K_\Phi = p$. One can also consider functions of the form $\Phi(x) = \frac{1}{p}|x|^p \log^r(1 + |x|^s)$ for suitable choice of $p$, $r$, $s$.

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**Authors' contributions**
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