Second-order Stark effect and polarizability of a relativistic two-dimensional hydrogenlike atom in the ground state

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The second-order Stark effect for a planar Dirac one-electron atom in the ground state is analyzed within the framework of the Rayleigh-Schrödinger perturbation theory, with the use of the Sturmian series expansion of the generalized Dirac-Coulomb Green's function. A closed-form analytical expression for the static dipole polarizability of that system is found. The formula involves the generalized hypergeometric function ${}_3F_2$ with the unit argument. Numerical values of the polarizabilities for relativistic planar hydrogenic atoms with atomic numbers $1 \leqslant Z \leqslant 68$ are provided in a tabular form. A simple formula for the polarizability of a nonrelativistic two-dimensional hydrogenic atom, reported previously by several other authors, is recovered from our result in the nonrelativistic limit.

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I. INTRODUCTION

For several decades, theoreticians have been investigating properties of model planar quantum systems. Recent years have seen a growth of interest in such studies, driven primarily by the rapid progress in low-dimensional condensed matter physics and materials science. It is not surprising that the system that has attracted much interest in this context is the planar analog of the one-electron atom. Its exceptional charm is rooted in its physical simplicity, as well as in the fact that the pertinent Schrödinger, Klein-Gordon, and Dirac equations admit analytical solutions [1,2]. In consequence, a good deal of information about various properties of that particular system has been gathered over the past years. However, a somewhat astonishing asymmetry may be observed: whereas a number of works have dealt with the planar hydrogenic atom subjected to the action of a magnetic field (the reader will find a comprehensive relevant bibliography in our recent works [3,4]), much less effort has been put into considering such an atom immersed in an electric field [1,5–14] (cf. also Refs. [15–18]). Further studies on the Stark effect for planar one-electron atoms are thus desirable, and the present paper meets that demand.

In Refs. [1,6–10,15–17], a simple analytical expression for the polarizability of the two-dimensional hydrogenlike atom in the ground state has been found (or may be inferred from akin results presented therein). A common feature of all these works is that the atomic electron has been described with the use of the Schrödinger equation. In the present paper, we derive an analytical formula for the polarizability of that particular atomic system, but with the employment of the Dirac equation rather than the Schrödinger one. The calculations are carried out within the framework of the second-order Rayleigh-Schrödinger perturbation theory, with the use of the Sturmian series expansion of the generalized radial Dirac-Coulomb Green's function. The resulting formula for the polarizability appears to be much more complex than its nonrelativistic counterpart and involves an irreducible generalized hypergeometric function, ${}_3F_2$, with the unit argument. In the nonrelativistic limit, we recover the expression found in Refs. [1,6–10,15–17].

II. PROBLEM FORMULATION

We are concerned with a Dirac one-electron atom (or ion) with a pointlike, spinless, and motionless nucleus of electric charge $+Ze$. The atomic electron is constrained to move in a plane through the nucleus. It is assumed that the interaction potential between the electron and the nucleus is the one-over-distance Coulomb one. The system is immersed in a static and uniform lateral electric field of strength $\boldsymbol{F}$. It is posited that the electric field is weak, in the sense that the electron is considered to stay bounded (in other words, the probability of the occurrence of the field-ionization process is negligibly small), and field-induced energy shifts are small compared to the fine-structure splitting of the planar Dirac-Coulomb energy levels. Under the assumptions specified above, the time-independent electronic wave function is taken to be a solution to the planar Dirac equation

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta mc^2-\frac{Ze^2}{(4\pi\epsilon_0)r}+V^{(1)}(\boldsymbol{r})-E\right]\Psi(\boldsymbol{r})=0 \qquad (\boldsymbol{r}\in\mathbb{R}^2), \tag{1a}$$

subject to the standard constraint of single valuedness and the boundary conditions

$$\sqrt{r}\,\Psi(\boldsymbol{r})\overset{r\to 0}{\longrightarrow}0, \qquad \sqrt{r}\,\Psi(\boldsymbol{r})\overset{r\to\infty}{\longrightarrow}0. \tag{1b}$$

In Eq. (1a),

$$V^{(1)}(\boldsymbol{r})=e\boldsymbol{F}\cdot\boldsymbol{r} \tag{2}$$

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is the potential energy of the interaction between the electron and the perturbing electric field $\boldsymbol{F}$. Henceforth, we assume that the atomic plane is the $\{x, y\}$ one, with the Cartesian unit vectors $\boldsymbol{n}_x$ and $\boldsymbol{n}_y$, and that the field $\boldsymbol{F}$ is directed along $\boldsymbol{n}_x$, i.e.,

$$\boldsymbol{F} = F\boldsymbol{n}_x. \tag{3}$$

The two components of the Dirac vector matrix

$$\boldsymbol{\alpha} = \alpha_1\boldsymbol{n}_x + \alpha_2\boldsymbol{n}_y \tag{4}$$

and the matrix β are taken to be

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5}$$

respectively, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{6}$$

while I stands for the unit 2×2 matrix.

Since the field $\boldsymbol{F}$ has been assumed to be weak, we treat the term $V^{(1)}(\boldsymbol{r})$ as a small perturbation of the Dirac-Coulomb Hamiltonian. Proceeding within the framework of the Rayleigh-Schrödinger perturbation theory, E and $\Psi(\boldsymbol{r})$ may be sought in the forms of the series

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \cdots \tag{7a}$$

and

$$\Psi(\boldsymbol{r}) = \Psi^{(0)}(\boldsymbol{r}) + \Psi^{(1)}(\boldsymbol{r}) + \Psi^{(2)}(\boldsymbol{r}) + \cdots. \tag{7b}$$

The zeroth-order terms $E^{(0)}$ and $\Psi^{(0)}(\boldsymbol{r})$ appearing above are those solutions to the planar bound-state Dirac-Coulomb energy-eigenvalue problem

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} - E^{(0)}\right]\Psi^{(0)}(\boldsymbol{r}) = 0 \quad (\boldsymbol{r}\in\mathbb{R}^2), \tag{8a}$$

$$\sqrt{r}\,\Psi^{(0)}(\boldsymbol{r}) \overset{r\to0}{\longrightarrow} 0, \quad \sqrt{r}\,\Psi^{(0)}(\boldsymbol{r}) \overset{r\to\infty}{\longrightarrow} 0, \tag{8b}$$

from which E and $\Psi(\boldsymbol{r})$ evolve in effect of the action of the electric field.

In this work, we shall carry out calculations of the energy corrections $E^{(1)}$ and $E^{(2)}$ in the case when $E^{(0)}$ and $\Psi^{(0)}(\boldsymbol{r})$ refer to the *ground* state of the isolated atom. The energy $E^{(0)}$ of that state is

$$E^{(0)} = 2\gamma_{1/2}mc^2, \tag{9}$$

with

$$\gamma_\kappa = \sqrt{\kappa^2 - (\alpha Z)^2}, \tag{10}$$

where $\alpha = e^2/(4\pi\epsilon_0)c\hbar$ is the Sommerfeld fine-structure constant, while the wave function $\Psi^{(0)}(\boldsymbol{r})$ is

$$\Psi^{(0)}(\boldsymbol{r}) = a^{(0)}_{1/2}\Psi^{(0)}_{1/2}(\boldsymbol{r}) + a^{(0)}_{-1/2}\Psi^{(0)}_{-1/2}(\boldsymbol{r}), \tag{11}$$

with the basis eigenfunctions chosen to be

$$\Psi^{(0)}_{m_a}(\boldsymbol{r}) = \frac{1}{\sqrt{r}}\begin{pmatrix} P^{(0)}(r)\Phi_{-1/2,m_a}(\varphi) \\ iQ^{(0)}(r)\Phi_{1/2,m_a}(\varphi) \end{pmatrix} \quad \left(m_a = \pm\tfrac{1}{2}\right) \tag{12}$$

and with the mixing coefficients $a^{(0)}_{\pm1/2}$ constrained to obey

$$\left|a^{(0)}_{1/2}\right|^2 + \left|a^{(0)}_{-1/2}\right|^2 = 1. \tag{13}$$

In Eq. (12) and hereafter, $0 \leqslant \varphi < 2\pi$ is the polar angle between the unit vector $\boldsymbol{n}_x$ and the radius vector $\boldsymbol{r}$, and

$$\Phi_{\kappa m_\kappa}(\varphi) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} \delta_{-\kappa,m_\kappa}\, e^{i(m_\kappa-1/2)\varphi} \\ \delta_{\kappa m_\kappa}\, e^{i(m_\kappa+1/2)\varphi} \end{pmatrix} \quad \left(\kappa = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots;\ m_\kappa = \pm\kappa\right) \tag{14}$$

are the axial spinors introduced by Poszwa and Rutkowski [19] and are discussed more comprehensively in Ref. [4, Appendix] (notice that the quantum number κ used in the present paper and in Ref. [4] has the opposite sign in relation to the one from Ref. [19]), while the ground-state radial functions $P^{(0)}(r)$ and $Q^{(0)}(r)$ are

$$P^{(0)}(r) = \sqrt{\frac{2Z(1+2\gamma_{1/2})}{a_0\Gamma(2\gamma_{1/2}+1)}}\left(\frac{4Zr}{a_0}\right)^{\gamma_{1/2}} e^{-2Zr/a_0} \tag{15a}$$

and

$$Q^{(0)}(r) = \sqrt{\frac{2Z(1-2\gamma_{1/2})}{a_0\Gamma(2\gamma_{1/2}+1)}}\left(\frac{4Zr}{a_0}\right)^{\gamma_{1/2}} e^{-2Zr/a_0}, \tag{15b}$$

with $a_0 = (4\pi\epsilon_0)\hbar^2/me^2$ being the Bohr radius. To ensure that $\gamma_{1/2}$ is real and positive, we impose the constraint

$$Z < \tfrac{1}{2}\alpha^{-1}. \tag{16}$$

It may be verified that the axial spinors (14) are orthonormal in the sense of

$$\int_0^{2\pi} d\varphi\ \Phi^\dagger_{\kappa m_\kappa}(\varphi)\Phi_{\kappa' m'_\kappa}(\varphi) = \delta_{\kappa\kappa'}\delta_{m_\kappa m'_\kappa} \tag{17}$$

and that the radial functions (15) are normalized to unity in the sense of

$$\int_0^\infty dr\left\{[P^{(0)}(r)]^2 + [Q^{(0)}(r)]^2\right\} = 1. \tag{18}$$

Consequently, it holds that

$$\int_{\mathbb{R}^2} d^2\boldsymbol{r}\ \Psi^{(0)\dagger}_{m_a}(\boldsymbol{r})\Psi^{(0)}_{m'_a}(\boldsymbol{r}) = \delta_{m_a m'_a} \quad \left(m_a, m'_a = \pm\frac{1}{2}\right), \tag{19}$$

and, by virtue of the constraint (13), the function (11) is normalized to unity in the sense of

$$\int_{\mathbb{R}^2} d^2\boldsymbol{r}\ \Psi^{(0)\dagger}(\boldsymbol{r})\Psi^{(0)}(\boldsymbol{r}) = 1. \tag{20}$$

III. THE FIRST-ORDER STARK EFFECT

The first-order corrections $E^{(1)}$ and $\Psi^{(1)}(\boldsymbol{r})$ appearing in Eqs. (7a) and (7b) solve the inhomogeneous system

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} - E^{(0)}\right]\Psi^{(1)}(\boldsymbol{r}) = -[V^{(1)}(\boldsymbol{r}) - E^{(1)}]\Psi^{(0)}(\boldsymbol{r}), \tag{21a}$$

$$\sqrt{r}\,\Psi^{(1)}(\boldsymbol{r}) \overset{r\to0}{\longrightarrow} 0, \quad \sqrt{r}\,\Psi^{(1)}(\boldsymbol{r}) \overset{r\to\infty}{\longrightarrow} 0, \tag{21b}$$

subject to the orthogonality constraint

$$\int_{\mathbb{R}^2} d^2\boldsymbol{r}\ \Psi^{(0)\dagger}_{m_a}(\boldsymbol{r})\Psi^{(1)}(\boldsymbol{r}) = 0 \quad \left(m_a = \pm\frac{1}{2}\right). \tag{22}$$

Inserting Eq. (11) into the right-hand side of Eq. (21a) and then projecting the resulting equation from the left onto the unperturbed eigenfunctions $\Psi^{(0)}_{\pm 1/2}(\boldsymbol{r})$ yields the algebraic system

$$\sum_{m'_a=\pm 1/2} \left[V^{(1)}_{m_a m'_a} - E^{(1)}\delta_{m_a m'_a}\right] a^{(0)}_{m'_a} = 0 \quad \left(m_a = \pm\frac{1}{2}\right), \tag{23}$$

with

$$V^{(1)}_{m_a m'_a} = \int_{\mathbb{R}^2} d^2\boldsymbol{r}\ \Psi^{(0)\dagger}_{m_a}(\boldsymbol{r}) V^{(1)}(\boldsymbol{r})\Psi^{(0)}_{m'_a}(\boldsymbol{r}) \quad \left(m_a, m'_a = \pm\frac{1}{2}\right). \tag{24}$$

Since the perturbation operator (2) may be written in the form

$$V^{(1)}(\boldsymbol{r}) = eFr\cos\varphi, \tag{25}$$

using Eq. (12) and the integral identity

$$\int_0^{2\pi} d\varphi\ \cos\varphi\ \Phi^{\dagger}_{\kappa m_\kappa}(\varphi)\Phi_{\kappa' m'_\kappa}(\varphi) = \frac{1}{2}\delta_{m_\kappa/\kappa, m'_\kappa/\kappa'}(\delta_{\kappa,\kappa'+1} + \delta_{\kappa,\kappa'-1}), \tag{26}$$

we infer that

$$V^{(1)}_{m_a m'_a} = 0 \quad \left(m_a, m'_a = \pm\frac{1}{2}\right). \tag{27}$$

Consequently, the first-order contribution to the energy eigenvalue E vanishes:

$$E^{(1)} = 0, \tag{28}$$

and the mixing coefficients $a^{(0)}_{\pm 1/2}$ remain undetermined at this stage.

With the result (28) taken into account, a formal solution to the system (21) is

$$\Psi^{(1)}(\boldsymbol{r}) = -\int_{\mathbb{R}^2} d^2\boldsymbol{r}'\ \hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}')V^{(1)}(\boldsymbol{r}')\Psi^{(0)}(\boldsymbol{r}'), \tag{29}$$

where $\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}')$—the generalized planar Dirac-Coulomb Green's function associated with the unperturbed energy level (9)—is a solution to the inhomogeneous system

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} - E^{(0)}\right]\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}') = \delta^{(2)}(\boldsymbol{r}-\boldsymbol{r}')\mathcal{I} - \sum_{m_a=\pm 1/2}\Psi^{(0)}_{m_a}(\boldsymbol{r})\Psi^{(0)\dagger}_{m_a}(\boldsymbol{r}') \quad (\boldsymbol{r},\boldsymbol{r}'\in\mathbb{R}^2), \tag{30a}$$

$$\sqrt{r}\,\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}') \xrightarrow{r\to 0} 0, \quad \sqrt{r}\,\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}') \xrightarrow{r\to\infty} 0 \tag{30b}$$

(here $\mathcal{I}$ is the unit 4×4 matrix), subject to the disambiguating orthogonality constraint

$$\int_{\mathbb{R}^2} d^2\boldsymbol{r}\ \Psi^{(0)\dagger}_{m_a}(\boldsymbol{r})\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}') = 0 \quad \left(m_a = \pm\frac{1}{2}\right). \tag{31}$$

The expression for $\Psi^{(1)}(\boldsymbol{r})$ given in Eq. (29) is used in the next section, where the second-order Stark effect is analyzed.

IV. THE SECOND-ORDER STARK EFFECT AND THE ATOMIC POLARIZABILITY

The second-order corrections $E^{(2)}$ and $\Psi^{(2)}(\boldsymbol{r})$ are solutions to the inhomogeneous system

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} - E^{(0)}\right]\Psi^{(2)}(\boldsymbol{r}) = -[V^{(1)}(\boldsymbol{r}) - E^{(1)}]\Psi^{(1)}(\boldsymbol{r}) + E^{(2)}\Psi^{(0)}(\boldsymbol{r}), \tag{32a}$$

$$\sqrt{r}\,\Psi^{(2)}(\boldsymbol{r}) \xrightarrow{r\to 0} 0, \quad \sqrt{r}\,\Psi^{(2)}(\boldsymbol{r}) \xrightarrow{r\to\infty} 0, \tag{32b}$$

augmented with the orthogonality condition

$$\int_{\mathbb{R}^2} d^2\boldsymbol{r}\ \Psi^{(0)\dagger}_{m_a}(\boldsymbol{r})\Psi^{(2)}(\boldsymbol{r}) = 0 \quad \left(m_a = \pm\frac{1}{2}\right). \tag{33}$$

Proceeding as in the preceding section, after making use of the results (28) and (29), one arrives at the following algebraic system for the thus far undetermined mixing coefficients $a^{(0)}_{\pm 1/2}$:

$$\sum_{m'_a=\pm 1/2}\left[V^{(1,1)}_{m_a m'_a} - E^{(2)}\delta_{m_a m'_a}\right] a^{(0)}_{m'_a} = 0 \quad \left(m_a = \pm\frac{1}{2}\right), \tag{34}$$

with

$$V^{(1,1)}_{m_a m'_a} = -\int_{\mathbb{R}^2} d^2\boldsymbol{r}\int_{\mathbb{R}^2} d^2\boldsymbol{r}'\ \Psi^{(0)\dagger}_{m_a}(\boldsymbol{r})V^{(1)}(\boldsymbol{r})\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}')V^{(1)}(\boldsymbol{r}')\Psi^{(0)}_{m'_a}(\boldsymbol{r}') \quad \left(m_a, m'_a = \pm\frac{1}{2}\right). \tag{35}$$

Plugging Eqs. (12) and (25), and also the following multipole representation of $\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}')$,

$$\hat{\mathcal{G}}^{(0)}(\boldsymbol{r},\boldsymbol{r}') = \sum_{\kappa=-\infty-1/2}^{+\infty+1/2}\sum_{m_\kappa=\pm\kappa}\frac{1}{\sqrt{rr'}}\begin{pmatrix} \hat{g}^{(0)}_{(++)\kappa}(r,r')\Phi_{\kappa m_\kappa}(\varphi)\Phi^{\dagger}_{\kappa m_\kappa}(\varphi') & -i\,\hat{g}^{(0)}_{(+-)\kappa}(r,r')\Phi_{\kappa m_\kappa}(\varphi)\Phi^{\dagger}_{-\kappa m_\kappa}(\varphi') \\ i\,\hat{g}^{(0)}_{(-+)\kappa}(r,r')\Phi_{-\kappa m_\kappa}(\varphi)\Phi^{\dagger}_{\kappa m_\kappa}(\varphi') & \hat{g}^{(0)}_{(--)\kappa}(r,r')\Phi_{-\kappa m_\kappa}(\varphi)\Phi^{\dagger}_{-\kappa m_\kappa}(\varphi') \end{pmatrix}, \tag{36}$$

into the right-hand side of Eq. (35) and then carrying out angular integrations with the aid of Eq. (26) casts the matrix element $V^{(1,1)}_{m_a m'_a}$ into the form

$$V^{(1,1)}_{m_a m'_a} = -\delta_{m_a m'_a}\tfrac{1}{4}e^2F^2\left[R^{(1,1)}_{1/2}(P^{(0)},Q^{(0)};P^{(0)},Q^{(0)}) + R^{(1,1)}_{-3/2}(P^{(0)},Q^{(0)};P^{(0)},Q^{(0)})\right] \quad \left(m_a, m'_a = \pm\frac{1}{2}\right), \tag{37}$$

with

$$R_\kappa^{(1,1)}(P^{(0)}, Q^{(0)}; P^{(0)}, Q^{(0)}) = \int_0^\infty dr \int_0^\infty dr' \begin{pmatrix} P^{(0)}(r) & Q^{(0)}(r) \end{pmatrix} r \hat{G}_\kappa^{(0)}(r, r') r' \begin{pmatrix} P^{(0)}(r') \\ Q^{(0)}(r') \end{pmatrix}, \tag{38}$$

where

$$\hat{G}_\kappa^{(0)}(r, r') = \begin{pmatrix} \hat{g}_{(++)\kappa}^{(0)}(r, r') & \hat{g}_{(+-)\kappa}^{(0)}(r, r') \\ \hat{g}_{(-+)\kappa}^{(0)}(r, r') & \hat{g}_{(--)\kappa}^{(0)}(r, r') \end{pmatrix} \tag{39}$$

is the generalized radial Dirac-Coulomb Green's function associated with the unperturbed ground-state energy level (9). It is seen from Eq. (37) that the matrix with the elements $V_{m_a m_a'}^{(1,1)}$ is a multiple of the 2×2 unit matrix; in effect the secular equation for the algebraic system (34) has the double root

$$E^{(2)} = -\tfrac{1}{4} e^2 F^2 \big[R_{1/2}^{(1,1)}(P^{(0)}, Q^{(0)}; P^{(0)}, Q^{(0)}) + R_{-3/2}^{(1,1)}(P^{(0)}, Q^{(0)}; P^{(0)}, Q^{(0)}) \big], \tag{40}$$

and the mixing coefficients $a_{\pm1/2}^{(0)}$ again remain undetermined.

To complete the task of calculation of the second-order energy correction $E^{(2)}$, we have to evaluate the double radial integral (38) for $\kappa = 1/2$ and for $\kappa = -3/2$. For that purpose, we shall exploit the following series representation of the generalized Green's function $\hat{G}_\kappa^{(0)}(r, r')$:

$$\hat{G}_\kappa^{(0)}(r, r') = \sum_{n_r=-\infty}^{\infty} \frac{1}{\mu_{n_r\kappa}^{(0)} - 1} \begin{pmatrix} S_{n_r\kappa}^{(0)}(r) \\ T_{n_r\kappa}^{(0)}(r) \end{pmatrix} \begin{pmatrix} \mu_{n_r\kappa}^{(0)} S_{n_r\kappa}^{(0)}(r') & T_{n_r\kappa}^{(0)}(r') \end{pmatrix} \quad \left(\kappa \neq -\frac{1}{2} \right), \tag{41}$$

with

$$\mu_{n_r\kappa}^{(0)} = \frac{|n_r| + \gamma_\kappa + N_{n_r\kappa}}{\gamma_{1/2} + \frac{1}{2}}, \tag{42}$$

involving the pertinent radial Dirac-Coulomb Sturmian functions (cf. Ref. [4, Sec. 3]) evaluated at the energy (9):

$$S_{n_r\kappa}^{(0)}(r) = \sqrt{\frac{4\pi\epsilon_0}{e^2} \frac{(1 + 2\gamma_{1/2})|n_r|!(|n_r| + 2\gamma_\kappa)}{4Z N_{n_r\kappa}(N_{n_r\kappa} - \kappa)\Gamma(|n_r| + 2\gamma_\kappa)}} \left(\frac{4Zr}{a_0} \right)^{\gamma_\kappa} e^{-2Zr/a_0} \left[L_{|n_r|-1}^{(2\gamma_\kappa)} \left(\frac{4Zr}{a_0} \right) - \frac{N_{n_r\kappa} - \kappa}{|n_r| + 2\gamma_\kappa} L_{|n_r|}^{(2\gamma_\kappa)} \left(\frac{4Zr}{a_0} \right) \right] \tag{43a}$$

and

$$T_{n_r\kappa}^{(0)}(r) = -\sqrt{\frac{4\pi\epsilon_0}{e^2} \frac{(1 - 2\gamma_{1/2})|n_r|!(|n_r| + 2\gamma_\kappa)}{4Z N_{n_r\kappa}(N_{n_r\kappa} - \kappa)\Gamma(|n_r| + 2\gamma_\kappa)}} \left(\frac{4Zr}{a_0} \right)^{\gamma_\kappa} e^{-2Zr/a_0} \left[L_{|n_r|-1}^{(2\gamma_\kappa)} \left(\frac{4Zr}{a_0} \right) + \frac{N_{n_r\kappa} - \kappa}{|n_r| + 2\gamma_\kappa} L_{|n_r|}^{(2\gamma_\kappa)} \left(\frac{4Zr}{a_0} \right) \right]. \tag{43b}$$

Here $L_n^{(\alpha)}(\rho)$ is the generalized Laguerre polynomial [20, Sec. 5.5] [we define $L_{-1}^{(\alpha)}(\rho) \equiv 0$], and

$$N_{n_r\kappa} = \pm\sqrt{n_r^2 + 2|n_r|\gamma_\kappa + \kappa^2}, \tag{44}$$

where one chooses the positive sign for $n_r > 0$ and the negative sign for $n_r < 0$; if $n_r = 0$, then the positive sign is to be chosen for $\kappa \leqslant -\frac{1}{2}$ and the negative one for $\kappa \geqslant \frac{1}{2}$, i.e., it holds that $N_{0\kappa} = -\kappa$. The functions (43) and the expansion (41) may be constructed proceeding along the route analogous to the one taken by us in Ref. [21] for the three-dimensional Dirac-Coulomb problem.

Insertion of the expansion (41) into Eq. (38) transforms the latter into

$$\begin{aligned} R_\kappa^{(1,1)}(P^{(0)}, Q^{(0)}; P^{(0)}, Q^{(0)}) = \sum_{n_r=-\infty}^{\infty} \frac{1}{\mu_{n_r\kappa}^{(0)} - 1} \int_0^\infty dr\, r \big[P^{(0)}(r) S_{n_r\kappa}^{(0)}(r) + Q^{(0)}(r) T_{n_r\kappa}^{(0)}(r) \big] \\ \times \int_0^\infty dr'\, r' \big[\mu_{n_r\kappa}^{(0)} P^{(0)}(r') S_{n_r\kappa}^{(0)}(r') + Q^{(0)}(r') T_{n_r\kappa}^{(0)}(r') \big]. \end{aligned} \tag{45}$$

The two radial integrals which enter the summand may be evaluated with the use of Eqs. (15), (42), and (43), together with the known integration formula

$$\int_0^\infty d\rho \rho^\gamma e^{-\rho} L_n^{(\alpha)}(\rho) = \frac{\Gamma(\gamma + 1)\Gamma(n + \alpha - \gamma)}{n!\Gamma(\alpha - \gamma)} \quad (\mathrm{Re}\,\gamma > -1). \tag{46}$$

The results are

$$\int_0^\infty dr r\big[P^{(0)}(r)S^{(0)}_{n_r\kappa}(r)+Q^{(0)}(r)T^{(0)}_{n_r\kappa}(r)\big] = -\frac{\sqrt{4\pi\epsilon_0}\,a_0^{3/2}}{e}\frac{(N_{n_r\kappa}-\kappa)[(|n_r|+\gamma_\kappa-\gamma_{1/2}-2)-2\gamma_{1/2}(N_{n_r\kappa}+\kappa)]}{8Z^2\sqrt{2|n_r|!N_{n_r\kappa}(N_{n_r\kappa}-\kappa)}\Gamma(2\gamma_{1/2}+1)\Gamma(|n_r|+2\gamma_\kappa+1)}\frac{\Gamma(\gamma_\kappa+\gamma_{1/2}+2)\Gamma(|n_r|+\gamma_\kappa-\gamma_{1/2}-2)}{\Gamma(\gamma_\kappa-\gamma_{1/2}-1)} \tag{47a}$$

and

$$\int_0^\infty dr r\big[\mu^{(0)}_{n_r\kappa}P^{(0)}(r)S^{(0)}_{n_r\kappa}(r)+Q^{(0)}(r)T^{(0)}_{n_r\kappa}(r)\big] = -\frac{\sqrt{4\pi\epsilon_0}\,a_0^{3/2}}{e}\frac{\big(\mu^{(0)}_{n_r\kappa}-1\big)(N_{n_r\kappa}-\kappa)}{16Z^2\sqrt{2|n_r|!N_{n_r\kappa}(N_{n_r\kappa}-\kappa)}\Gamma(2\gamma_{1/2}+1)\Gamma(|n_r|+2\gamma_\kappa+1)}\frac{\Gamma(\gamma_\kappa+\gamma_{1/2}+2)\Gamma(|n_r|+\gamma_\kappa-\gamma_{1/2}-2)}{\Gamma(\gamma_\kappa-\gamma_{1/2}-1)}$$
$$\times\left\{2\gamma_{1/2}(|n_r|+\gamma_\kappa-\gamma_{1/2}-2)-(N_{n_r\kappa}+\kappa)+\frac{N_{n_r\kappa}+\frac{1}{2}}{|n_r|+\gamma_\kappa-\gamma_{1/2}}[(|n_r|+\gamma_\kappa-\gamma_{1/2}-2)-2\gamma_{1/2}(N_{n_r\kappa}+\kappa)]\right\}. \tag{47b}$$

Plugging Eqs. (47) and (42) into the right-hand side of Eq. (45), collecting then the terms corresponding to n_r and $-n_r$, after some algebra one arrives at the following representation of the double integral (38):

$$R^{(1,1)}_\kappa\big(P^{(0)},Q^{(0)};P^{(0)},Q^{(0)}\big) = \frac{(4\pi\epsilon_0)a_0^3}{e^2}\frac{\Gamma^2(\gamma_\kappa+\gamma_{1/2}+2)}{64Z^4\Gamma(2\gamma_{1/2}+1)\Gamma(2\gamma_\kappa+1)}\left\{\frac{\gamma_{1/2}[(2\kappa+1)\gamma_{1/2}+4]}{\gamma_\kappa-\gamma_{1/2}+1}\right.$$
$$\times{}_3F_2\left(\begin{matrix}\gamma_\kappa-\gamma_{1/2}-1,\ \gamma_\kappa-\gamma_{1/2}-1,\ \gamma_\kappa-\gamma_{1/2}+1\\ \gamma_\kappa-\gamma_{1/2}+2,\ 2\gamma_\kappa+1\end{matrix};1\right)-\frac{\gamma_\kappa+\gamma_{1/2}}{2\kappa+1}$$
$$\left.\times{}_3F_2\left(\begin{matrix}\gamma_\kappa-\gamma_{1/2}-1,\ \gamma_\kappa-\gamma_{1/2}-1,\ \gamma_\kappa-\gamma_{1/2}\\ \gamma_\kappa-\gamma_{1/2}+1,\ 2\gamma_\kappa+1\end{matrix};1\right)\right\}\quad\left(\kappa=\frac{1}{2}\ \text{or}\ \kappa=-\frac{3}{2}\right). \tag{48}$$

Here and hereafter, ${}_3F_2(\cdots)$ denotes the generalized hypergeometric function

$${}_3F_2\left(\begin{matrix}a_1,\ a_2,\ a_3\\ b_1,\ b_2\end{matrix};z\right)=\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\sum_{k=0}^\infty\frac{\Gamma(a_1+k)\Gamma(a_2+k)\Gamma(a_3+k)}{\Gamma(b_1+k)\Gamma(b_2+k)}\frac{z^k}{k!}. \tag{49}$$

Application of the identity

$${}_3F_2\left(\begin{matrix}a_1,\ a_2,\ a_3\\ a_3+1,\ b\end{matrix};1\right)=\frac{\Gamma(b)\Gamma(b-a_1-a_2+1)}{(b-a_3-1)\Gamma(b-a_1)\Gamma(b-a_2)}$$
$$-\frac{(a_1-a_3-1)(a_2-a_3-1)}{(a_3+1)(b-a_3-1)}{}_3F_2\left(\begin{matrix}a_1,\ a_2,\ a_3+1\\ a_3+2,\ b\end{matrix};1\right)\quad[\mathrm{Re}(b-a_1-a_2)>-1] \tag{50}$$

brings the expression (48) to the final general form:

$$R^{(1,1)}_\kappa\big(P^{(0)},Q^{(0)};P^{(0)},Q^{(0)}\big) = -\frac{(4\pi\epsilon_0)a_0^3}{e^2}\frac{(\gamma_{1/2}+1)(2\gamma_{1/2}+1)(2\gamma_{1/2}+3)}{32Z^4(2\kappa+1)}\left\{1-\frac{[(2\kappa+1)\gamma_{1/2}+2]^2\Gamma^2(\gamma_\kappa+\gamma_{1/2}+2)}{(\gamma_\kappa-\gamma_{1/2}+1)\Gamma(2\gamma_{1/2}+4)\Gamma(2\gamma_\kappa+1)}\right.$$
$$\left.\times{}_3F_2\left(\begin{matrix}\gamma_\kappa-\gamma_{1/2}-1,\ \gamma_\kappa-\gamma_{1/2}-1,\ \gamma_\kappa-\gamma_{1/2}+1\\ \gamma_\kappa-\gamma_{1/2}+2,\ 2\gamma_\kappa+1\end{matrix};1\right)\right\}\quad\left(\kappa=\frac{1}{2}\ \text{or}\ \kappa=-\frac{3}{2}\right). \tag{51}$$

For $\kappa=1/2$, the hypergeometric series in Eq. (51) is a truncating one and may be expressed in terms of elementary algebraic functions. In that case one has

$$R^{(1,1)}_{1/2}(P^{(0)},Q^{(0)};P^{(0)},Q^{(0)}) = \frac{(4\pi\epsilon_0)a_0^3}{e^2}\frac{\gamma_{1/2}(\gamma_{1/2}+1)(2\gamma_{1/2}+1)(4\gamma_{1/2}+5)}{64Z^4}, \tag{52a}$$

while for $\kappa=-3/2$ Eq. (51) becomes

$$R^{(1,1)}_{-3/2}\big(P^{(0)},Q^{(0)};P^{(0)},Q^{(0)}\big) = \frac{(4\pi\epsilon_0)a_0^3}{e^2}\frac{(\gamma_{1/2}+1)(2\gamma_{1/2}+1)(2\gamma_{1/2}+3)}{64Z^4}\left\{1-\frac{4(\gamma_{1/2}-1)^2\Gamma^2(\gamma_{3/2}+\gamma_{1/2}+2)}{(\gamma_{3/2}-\gamma_{1/2}+1)\Gamma(2\gamma_{1/2}+4)\Gamma(2\gamma_{3/2}+1)}\right.$$
$$\left.\times{}_3F_2\left(\begin{matrix}\gamma_{3/2}-\gamma_{1/2}-1,\ \gamma_{3/2}-\gamma_{1/2}-1,\ \gamma_{3/2}-\gamma_{1/2}+1\\ \gamma_{3/2}-\gamma_{1/2}+2,\ 2\gamma_{3/2}+1\end{matrix};1\right)\right\}. \tag{52b}$$

TABLE I. Z^4-scaled polarizabilities for planar Dirac one-electron atoms in the ground state, computed from the analytical formula in Eq. (56). The number in parentheses following each entry is an uncertainty in its last two digits and stems from the one-standard-deviation uncertainty (equal to 31) in the last two digits of the value of the inverse of the fine-structure constant $\alpha^{-1} = 137.035\,999\,139$ (from CODATA 2014 [22]) used in calculations. The nonrelativistic limit of $Z^4\alpha_1(Z)$ is independent of Z and equals $Z^4\alpha_1^{\rm NR}(Z) = 0.164\,062\,5\ a_0^3$ [cf. Eq. (62)].

Z	$Z^4\alpha_1(Z)$ (units of a_0^3)	Z	$Z^4\alpha_1(Z)$ (units of a_0^3)	Z	$Z^4\alpha_1(Z)$ (units of a_0^3)
1	0.164 031 922 357 129 (14)	24	0.146 540 774 615 7 (79)	47	0.097 649 864 741 (30)
2	0.163 940 192 827 883 (55)	25	0.145 058 299 164 6 (86)	48	0.094 824 238 984 (31)
3	0.163 787 321 605 63 (12)	26	0.143 516 593 431 0 (92)	49	0.091 938 523 941 (32)
4	0.163 573 325 663 45 (22)	27	0.141 915 791 499 (10)	50	0.088 991 948 689 (34)
5	0.163 298 228 730 23 (35)	28	0.140 256 027 891 (11)	51	0.085 983 547 673 (35)
6	0.162 962 061 256 62 (50)	29	0.138 537 436 674 (11)	52	0.082 912 115 141 (37)
7	0.162 564 860 370 78 (68)	30	0.136 760 150 441 (12)	53	0.079 776 146 327 (38)
8	0.162 106 669 823 04 (88)	31	0.134 924 299 151 (13)	54	0.076 573 760 514 (40)
9	0.161 587 539 919 0 (11)	32	0.133 030 008 805 (14)	55	0.073 302 598 760 (41)
10	0.161 007 527 440 5 (14)	33	0.131 077 399 917 (15)	56	0.069 959 685 549 (43)
11	0.160 366 695 552 3 (17)	34	0.129 066 585 774 (16)	57	0.066 541 237 770 (45)
12	0.159 665 113 695 7 (20)	35	0.126 997 670 418 (17)	58	0.063 042 394 713 (46)
13	0.158 902 857 464 6 (23)	36	0.124 870 746 321 (18)	59	0.059 456 825 787 (48)
14	0.158 080 008 465 4 (27)	37	0.122 685 891 688 (19)	60	0.055 776 141 692 (51)
15	0.157 196 654 156 8 (31)	38	0.120 443 167 320 (20)	61	0.051 988 975 236 (53)
16	0.156 252 887 668 9 (35)	39	0.118 142 612 955 (21)	62	0.048 079 475 600 (56)
17	0.155 248 807 597 6 (40)	40	0.115 784 242 975 (22)	63	0.044 024 687 441 (59)
18	0.154 184 517 773 3 (45)	41	0.113 368 041 358 (23)	64	0.039 789 613 916 (63)
19	0.153 060 126 999 5 (50)	42	0.110 893 955 708 (24)	65	0.035 316 860 204 (68)
20	0.151 875 748 757 7 (55)	43	0.108 361 890 163 (25)	66	0.030 501 216 714 (75)
21	0.150 631 500 875 0 (61)	44	0.105 771 696 922 (26)	67	0.025 108 988 013 (88)
22	0.149 327 505 149 1 (66)	45	0.103 123 166 068 (27)	68	0.018 331 850 81 (13)
23	0.147 963 886 924 7 (72)	46	0.100 416 013 257 (28)		

Hence, after Eqs. (52a) and (52b) are plugged into Eq. (40), the second-order correction to energy is found to be

$$E^{(2)} = -\frac{(\gamma_{1/2}+1)^2(2\gamma_{1/2}+1)(4\gamma_{1/2}+3)}{256Z^4} \times \left\{1 - \frac{4(\gamma_{1/2}-1)^2\Gamma^2(\gamma_{3/2}+\gamma_{1/2}+2)}{(\gamma_{1/2}+1)(4\gamma_{1/2}+3)(\gamma_{3/2}-\gamma_{1/2}+1)\Gamma(2\gamma_{1/2}+3)\Gamma(2\gamma_{3/2}+1)} \times {}_3F_2\left(\begin{matrix}\gamma_{3/2}-\gamma_{1/2}-1,\ \gamma_{3/2}-\gamma_{1/2}-1,\ \gamma_{3/2}-\gamma_{1/2}+1\\ \gamma_{3/2}-\gamma_{1/2}+2,\ 2\gamma_{3/2}+1\end{matrix};1\right)\right\}\frac{F^2}{F_0^2}\frac{e^2}{(4\pi\epsilon_0)a_0}, \tag{53}$$

where

$$F_0 = \frac{e}{(4\pi\epsilon_0)a_0^2} \simeq 5.14\times10^{11}\ \mathrm{V/m} \tag{54}$$

is the atomic unit of the electric field.

The relationship between the second-order energy correction and the strength of the perturbing electric field may be written in the form

$$E^{(2)} = -\tfrac{1}{2}(4\pi\epsilon_0)\alpha_1 F^2, \tag{55}$$

which defines the polarizability α_1 of the system under study. Comparison of Eqs. (55) and (53) yields the following closed-form expression for the polarizability of the planar Dirac one-electron atom in the ground state:

$$\alpha_1 = \frac{a_0^3}{Z^4}\frac{(\gamma_{1/2}+1)^2(2\gamma_{1/2}+1)(4\gamma_{1/2}+3)}{128}\left\{1 - \frac{4(\gamma_{1/2}-1)^2\Gamma^2(\gamma_{3/2}+\gamma_{1/2}+2)}{(\gamma_{1/2}+1)(4\gamma_{1/2}+3)(\gamma_{3/2}-\gamma_{1/2}+1)\Gamma(2\gamma_{1/2}+3)\Gamma(2\gamma_{3/2}+1)} \times {}_3F_2\left(\begin{matrix}\gamma_{3/2}-\gamma_{1/2}-1,\ \gamma_{3/2}-\gamma_{1/2}-1,\ \gamma_{3/2}-\gamma_{1/2}+1\\ \gamma_{3/2}-\gamma_{1/2}+2,\ 2\gamma_{3/2}+1\end{matrix};1\right)\right\}. \tag{56}$$

Numerical results for the scaled polarizabilities $Z^4\alpha_1(Z)$ for planar hydrogenic atoms with $1 \leqslant Z \leqslant 68$, computed from Eq. (56), are listed in Table I. The value of the inverse of the fine-structure constant used in calculations has been $\alpha^{-1} = 137.035\,999\,139$ (from CODATA 2014 [22]). The data are displayed in the form which also shows an estimated error in last

two digits of each entry, resulting from the declared one-standard-deviation uncertainty (equal to 31) in the last two digits of the value of α^{-1} given above.

It remains to investigate the expression in Eq. (56) in the quasirelativistic limit $\alpha Z \ll 1$. Using the approximations

$$\gamma_\kappa \simeq |\kappa| - \frac{(\alpha Z)^2}{2|\kappa|} + O[(\alpha Z)^4], \tag{57}$$

$$\Gamma(a\gamma_\kappa + a'\gamma_{\kappa'} + b) \simeq \Gamma(a|\kappa| + a'|\kappa'| + b)\left[1 - \frac{(\alpha Z)^2}{2}\left(\frac{a}{|\kappa|} + \frac{a'}{|\kappa'|}\right)\psi(a|\kappa| + a'|\kappa'| + b)\right] + O[(\alpha Z)^4], \tag{58}$$

where $\psi(z)$ is the digamma function defined as

$$\psi(z) = \frac{1}{\Gamma(z)}\frac{d\Gamma(z)}{dz}, \tag{59}$$

and

$${}_3F_2\left(\begin{matrix}\gamma_{3/2} - \gamma_{1/2} - 1,\ \gamma_{3/2} - \gamma_{1/2} - 1,\ \gamma_{3/2} - \gamma_{1/2} + 1 \\ \gamma_{3/2} - \gamma_{1/2} + 2,\ 2\gamma_{3/2} + 1\end{matrix}; 1\right) \simeq 1 + O[(\alpha Z)^4], \tag{60}$$

after straightforward but somewhat lengthy algebraic manipulations one arrives at the following quasirelativistic estimate of the polarizability:

$$\alpha_1 \simeq \alpha_1^{\rm NR}\left[1 - \tfrac{7}{2}(\alpha Z)^2\right] + O[(\alpha Z)^4]. \tag{61}$$

Here

$$\alpha_1^{\rm NR} = \frac{21}{128}\frac{a_0^3}{Z^4} \tag{62}$$

is the polarizability of the nonrelativistic planar one-electron atom in the ground state. The expression in Eq. (62) is identical to the one derived independently, from purely nonrelativistic considerations, by several other authors [1,6–10,15–17].

It is instructive to compare the formulas in Eqs. (56), (61), and (62) with their counterparts for the three-dimensional one-electron atom, which are provided in the Appendix.

APPENDIX: POLARIZABILITY OF A RELATIVISTIC THREE-DIMENSIONAL HYDROGENIC ATOM IN THE GROUND STATE

The polarizability of the three-dimensional Dirac one-electron atom in the ground state is

$$\alpha_1 = \frac{a_0^3}{Z^4}\frac{(\gamma_1+1)(2\gamma_1+1)\left(4\gamma_1^2+13\gamma_1+12\right)}{36}\left\{1 - \frac{2(\gamma_1-2)^2\Gamma^2(\gamma_2+\gamma_1+2)}{(\gamma_1+1)\left(4\gamma_1^2+13\gamma_1+12\right)(\gamma_2-\gamma_1+1)\Gamma(2\gamma_1+2)\Gamma(2\gamma_2+1)}\right.$$
$$\left.\times\, {}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-1,\ \gamma_2-\gamma_1-1,\ \gamma_2-\gamma_1+1 \\ \gamma_2-\gamma_1+2,\ 2\gamma_2+1\end{matrix}; 1\right)\right\} \tag{A1}$$

(cf. Ref. [23, Eq. (3.24)], Ref. [24, Eq. (16)], and Ref. [25, Eq. (3.42)]), with γ_κ defined as in Eq. (10). The quasirelativistic limit of the expression displayed in Eq. (A1) is

$$\alpha_1 \simeq \alpha_1^{\rm NR}\left[1 - \tfrac{28}{27}(\alpha Z)^2\right] + O[(\alpha Z)^4], \tag{A2}$$

where

$$\alpha_1^{\rm NR} = \frac{9}{2}\frac{a_0^3}{Z^4} \tag{A3}$$

is the polarizability of the nonrelativistic three-dimensional hydrogenic atom in the ground state.

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