

## ***T*-COLORINGS, DIVISIBILITY AND THE CIRCULAR CHROMATIC NUMBER**

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### **Abstract**

Let  $T$  be a  $T$ -set, i.e., a finite set of nonnegative integers satisfying  $0 \in T$ , and  $G$  be a graph. In the paper we study relations between the  $T$ -edge spans  $\text{esp}_T(G)$  and  $\text{esp}_{d \odot T}(G)$ , where  $d$  is a positive integer and

$$d \odot T = \{0 \leq t \leq d(\max T + 1) : d \mid t \Rightarrow t/d \in T\}.$$

We show that  $\text{esp}_{d \odot T}(G) = d \text{esp}_T(G) - r$ , where  $r$ ,  $0 \leq r \leq d - 1$ , is an integer that depends on  $T$  and  $G$ . Next we focus on the case  $T = \{0\}$  and show that

$$\text{esp}_{d \odot \{0\}}(G) = \lceil d(\chi_c(G) - 1) \rceil,$$

where  $\chi_c(G)$  is the circular chromatic number of  $G$ . This result allows us to formulate several interesting conclusions that include a new formula for the circular chromatic number

$$\chi_c(G) = 1 + \inf \{ \text{esp}_{d \odot \{0\}}(G)/d : d \geq 1 \}$$

and a proof that the formula for the  $T$ -edge span of powers of cycles, stated as conjecture in [Y. Zhao, W. He and R. Cao, *The edge span of  $T$ -coloring on graph  $C_n^d$* , Appl. Math. Lett. 19 (2006) 647–651], is true.

**Keywords:**  $T$ -coloring, circular chromatic number.

**2010 Mathematics Subject Classification:** 05C15.

## 1. INTRODUCTION

In the paper we study relations between two different generalizations of ordinary vertex colorings:  $T$ -colorings and  $(k, d)$ -colorings. Let  $G$  be a graph with  $n$ -vertex set  $V$  and edge set  $E$ . Given integers  $1 \leq d \leq k$ , by a  $(k, d)$ -coloring of  $G$  we mean any function  $c: V \rightarrow [0, k-1]$  ( $[a, b] := \{a, a+1, \dots, b\}$  for any integers  $a \leq b$ ) such that

$$d \leq |c(u) - c(v)| \leq k - d$$

whenever  $uv \in E$ . This notion may be viewed as a generalization of a  $k$ -coloring since  $(k, d)$ -colorings of  $G$  are  $k$ -colorings of  $G$  and  $(k, 1)$ -colorings are the same as  $k$ -colorings that use colors from the interval  $[0, k-1]$ . The *circular chromatic number*, introduced by Vince [12] as a generalization of the chromatic number, is defined by the formula

$$\chi_c(G) = \inf \{k/d: G \text{ has a } (k, d)\text{-coloring}\}.$$

The circular chromatic number was studied by many authors, see [14, 15] for a survey of results. It was shown for example [12] that the distance between the circular and ordinary chromatic number does not exceed 1, i.e.

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

In the same paper Vince proved two useful facts: (1)  $G$  has a  $(k, d)$ -coloring if and only if  $\chi_c(G) \leq k/d$ ; (2)  $\chi_c(G)$  is a rational number which has a form  $k/d$ , where  $k \leq n$ . We will use these observations to show that there is a relation between  $\chi_c(G)$  and  $\text{esp}_T(G)$  the  $T$ -edge span defined below. Given a  $T$ -set  $T$ , i.e., a finite set that consists of nonnegative integers and satisfies  $0 \in T$ , by a  $T$ -coloring of  $G$  we mean any function  $c: V \rightarrow \mathbb{Z}$  such that

$$|c(u) - c(v)| \notin T$$

whenever  $uv \in E$ .  $T$ -colorings were introduced as a model for the frequency assignment problem in [5]. This notion also may be viewed as a generalization of ordinary vertex colorings since  $T$ -colorings are vertex colorings and vertex

colorings are  $\{0\}$ -colorings. The  $T$ -edge span, introduced by Cozzens and Roberts [1], is defined as

$$\text{esp}_T(G) = \min\{\text{esp}(c) : c \text{ is a } T\text{-coloring of } G\},$$

where  $\text{esp}(c) = \max\{|c(u) - c(v)| : uv \in E\}$  is the *edge span* of  $c$  (if  $G$  is an empty graph then  $\text{esp}(c) = 0$ ). If we replace  $\text{esp}(c)$  by  $\text{sp}(c)$  (the *span* of  $c$ , i.e.,  $\max\{|c(u) - c(v)| : u, v \in V\}$ ) we will receive the  $T$ -span of  $G$ . Both parameters were studied by many authors, there are results concerning computational complexity of the problem of computing  $\text{sp}_T(G)$  [2, 3], the behaviour of the greedy algorithm [7] and formulas describing  $\text{sp}_T(G)$  and  $\text{esp}_T(G)$  for some  $T$ -sets  $T$  and some graphs  $G$  [8, 9, 13].

The remainder of the paper is organized as follows. In Section 2 we study relations between  $\text{esp}_T(G)$  and  $\text{esp}_{d \odot T}(G)$ , where  $d$  is a positive integer and  $d \odot T = \{0 \leq t \leq d(\max T + 1) : d \mid t \Rightarrow t/d \in T\}$ . We show that  $\text{esp}_{d \odot T}(G) = d \text{esp}_T(G) - r$ , where  $r$ ,  $0 \leq r \leq d - 1$ , is an integer that depends on  $T$  and  $G$ . In Section 3 we study the distance between the  $T$ -span and  $T$ -edge span and show that it cannot exceed  $\max T$ . We also give examples that prove that this bound is tight. Section 4 contains our main results. We show that if  $T$  is an interval, i.e.,  $T = [0, d - 1]$  (or equivalently  $T = d \odot \{0\}$ ), then  $(k, d)$ -colorings ( $k \geq d$ ) are nonnegative  $T$ -colorings with span bounded by  $k - 1$  and edge span bounded by  $k - d$ . We use this relation to show that

$$\text{esp}_{d \odot \{0\}}(G) = \lceil d(\chi_c(G) - 1) \rceil.$$

We also discuss whether it is possible to extend this relation to all  $T$ -sets. Using the above formula we show that

$$\chi_c(G) = 1 + \inf \{ \text{esp}_{d \odot \{0\}}(G) / d : d \geq 1 \}$$

and discuss how these formulas allow us to move known results from the world of the  $T$ -edge span to the world of the circular chromatic number and vice versa. The last section is devoted to the powers of cycles investigated in [13]. The authors conjectured and partially proved that

$$\text{esp}_{d \odot \{0\}}(C_n^p) = pd + \lceil rd/q \rceil,$$

where  $q \geq 2$  and  $r$  are the quotient and the remainder of the division of  $n$  by  $p + 1$ , respectively. We show that it is true in general.

## 2. T-EDGE SPAN AND $d \odot T$ -EDGE SPAN

The operation  $\odot$  was introduced in [6], where it was shown that  $\text{sp}_{d \odot T}(G) = d \text{sp}_T(G)$ . Below we prove a similar formula for the  $T$ -edge span, but before we proceed we need to recall the following result.

**Lemma 1** (Lemma 2.2(i) of [6]). *If  $a$  and  $b$  are real numbers, then  $\lfloor |a - b| \rfloor \leq \lfloor \lfloor a \rfloor - \lfloor b \rfloor \rfloor \leq \lceil |a - b| \rceil$ .*

**Lemma 2.** *Let  $G$  be a graph,  $T$  be a  $T$ -set and  $d$  be a positive integer.*

- (1) *If  $c$  is a  $T$ -coloring of  $G$ , then  $dc$  is a  $d \odot T$ -coloring of  $G$ .*
- (2) *If  $c$  is a  $d \odot T$ -coloring of  $G$ , then  $\lfloor c/d \rfloor$  is a  $T$ -coloring of  $G$ .*

**Proof.** Let  $uv$  be an edge of  $G$  (if  $G$  is empty, then our claim is obvious).

(1) If  $|c(u) - c(v)| \geq \max T + 1$ , then  $|dc(u) - dc(v)| \geq d(\max T + 1) = \max d \odot T + 1$ . If  $|c(u) - c(v)| < \max T + 1$  and  $|dc(u) - dc(v)| \in d \odot T$ , then the definition of  $d \odot T$  gives  $|c(u) - c(v)| \in T$ , a contradiction. Hence  $|dc(u) - dc(v)| \notin d \odot T$  in both cases.

(2) If  $|c(u) - c(v)| \geq \max d \odot T + 1 = d(\max T + 1)$ , then  $|\lfloor c(u)/d \rfloor - \lfloor c(v)/d \rfloor| \geq \lfloor |c(u) - c(v)|/d \rfloor \geq \max T + 1$  by Lemma 1. If  $|c(u) - c(v)| < \max d \odot T + 1$ , then the definition of  $d \odot T$  gives  $d \mid |c(u) - c(v)|$  and, by Lemma 1,  $|\lfloor c(u)/d \rfloor - \lfloor c(v)/d \rfloor| = |c(u) - c(v)|/d \notin T$ . Hence  $|\lfloor c(u)/d \rfloor - \lfloor c(v)/d \rfloor| \notin T$  in both cases. ■

**Theorem 3.** *Let  $G$  be a graph,  $T$  be a  $T$ -set and  $d$  be a positive integer. There is an integer  $0 \leq r \leq d - 1$  such that  $\text{esp}_{d \odot T}(G) = d \text{esp}_T(G) - r$ .*

**Proof.** Let  $c$  be a  $T$ -coloring of  $G$  such that  $\text{esp}(c) = \text{esp}_T(G)$ . By Lemma 2,  $dc$  is a  $d \odot T$ -coloring of  $G$ . Hence

$$(1) \quad \text{esp}_{d \odot T}(G) \leq \text{esp}(dc) = d \text{esp}(c) = d \text{esp}_T(G).$$

Let  $c'$  be a  $d \odot T$ -coloring of  $G$  such that  $\text{esp}(c') = \text{esp}_{d \odot T}(G)$ . By Lemma 2,  $\lfloor c'/d \rfloor$  is a  $T$ -coloring of  $G$ . Let  $uv$  be an edge of  $G$  such that  $\text{esp}(\lfloor c'/d \rfloor) = |\lfloor c'(u)/d \rfloor - \lfloor c'(v)/d \rfloor|$  (if  $G$  is empty our claim is obvious). Then

$$(2) \quad \begin{aligned} d \text{esp}_T(G) - d &\leq d \text{esp}(\lfloor c'/d \rfloor) - d = d |\lfloor c'(u)/d \rfloor - \lfloor c'(v)/d \rfloor| - d \\ &\leq d \lceil |c'(u) - c'(v)|/d \rceil - d \leq d \lceil \text{esp}(c')/d \rceil - d \\ &= d \lceil \text{esp}_{d \odot T}(G)/d \rceil - d < \text{esp}_{d \odot T}(G). \end{aligned}$$

To complete the proof it suffices to combine (1) with (2). ■

The open problem is a formula for  $r$ . Later we will show how to compute  $r$  provided that  $T = \{0\}$  and that  $r$  can be any integer from  $[0, d - 1]$ .

**Corollary 4.** *Let  $G$  be a graph,  $T$  be a  $T$ -set and  $d$  be a positive integer. Then  $\text{esp}_T(G) = \lceil \text{esp}_{d \odot T}(G)/d \rceil$ .*

3. THE DISTANCE BETWEEN THE  $T$ -SPAN AND  $T$ -EDGE SPAN

It is known [1] that  $\text{esp}_T(G) \leq \text{sp}_T(G)$ . We are going to show that  $\text{sp}_T(G) \leq \text{esp}_T(G) + \max T$  and give examples in which the difference  $\text{sp}_T(G) - \text{esp}_T(G)$  equals  $\max T$ .

**Lemma 5.** *Let  $G$  be a graph and  $T$  be a  $T$ -set. If  $c' : V \rightarrow \mathbb{Z}$  is a  $T$ -coloring of  $G$  and  $c : V \rightarrow \mathbb{Z}$  is the remainder of the division of  $c'$  by  $\text{esp}(c') + \max T + 1$ , i.e.,  $c(v) = c'(v) \bmod (\text{esp}(c') + \max T + 1)$  for  $v \in V$ , then*

- (1)  $c$  is a  $T$ -coloring of  $G$ ;
- (2)  $\text{sp}(c) \leq \text{esp}(c') + \max T$ ;
- (3)  $\text{esp}(c) \leq \text{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T)$ .

**Proof.** Observe that (2) follows immediately from the definition of  $c$ . To prove (1) and (3), take an edge  $uv$  of  $G$  (if  $G$  is empty, our claim is obvious). Let  $q$  be the quotient of the division of  $c'$  by  $\text{esp}(c') + \max T + 1$ . Without loss of generality we may assume that  $q(u) \geq q(v)$ . It is easy to see that  $q(u) \leq q(v) + 1$  since otherwise

$$\begin{aligned} \text{esp}(c') &\geq |c'(u) - c'(v)| \\ &= |(\text{esp}(c') + \max T + 1)(q(u) - q(v)) + c(u) - c(v)| \\ &\geq (\text{esp}(c') + \max T + 1)|q(u) - q(v)| - |c(u) - c(v)| \\ &\geq 2(\text{esp}(c') + \max T + 1) - \text{esp}(c') - \max T \\ &= \text{esp}(c') + \max T + 2 > \text{esp}(c'). \end{aligned}$$

Hence there are two cases to consider.

(a)  $q(u) = q(v) + 1$ . Then  $|c'(u) - c'(v)| = |(\text{esp}(c') + \max T + 1)(q(u) - q(v)) + (c(u) - c(v))| = |\text{esp}(c') + \max T + 1 + (c(u) - c(v))|$ . Since  $\text{esp}(c') + \max T + 1 > \text{esp}(c') \geq |c'(u) - c'(v)|$  and  $|c(u) - c(v)| \leq \text{esp}(c') + \max T$ , we have  $|c(u) - c(v)| = \text{esp}(c') + \max T + 1 - |c'(u) - c'(v)|$ . This gives  $|c(u) - c(v)| \geq \max T + 1$  and  $|c(u) - c(v)| \leq \text{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T)$  since  $|c'(u) - c'(v)| \notin T$  implies  $|c'(u) - c'(v)| \geq \min(\mathbb{N} \setminus T)$ .

(b)  $q(u) = q(v)$ . Then  $|c'(u) - c'(v)| = |c(u) - c(v)|$ , which gives  $|c(u) - c(v)| \notin T$  and  $|c(u) - c(v)| \leq \text{esp}(c') \leq \text{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T)$ . ■

**Corollary 6.** *Let  $G$  be a graph and  $T$  be a  $T$ -set. Then*

- (1) *There is a  $T$ -coloring  $c$  of  $G$  such that  $\text{sp}(c) \leq \text{esp}_T(G) + \max T$  and  $\text{esp}(c) \leq \text{esp}_T(G) + \max T + 1 - \min(\mathbb{N} \setminus T)$ .*
- (2) *If  $T$  is an interval, then there is a  $T$ -coloring  $c$  of  $G$  such that  $\text{esp}(c) = \text{esp}_T(G)$  and  $\text{sp}(c) \leq \text{esp}_T(G) + \max T$ .*
- (3)  $\text{esp}_T(G) \leq \text{sp}_T(G) \leq \text{esp}_T(G) + \max T$ .

**Proof.** (1) Let  $c'$  be a  $T$ -coloring of  $G$  satisfying  $\text{esp}(c') = \text{esp}_T(G)$  and  $c$  be the remainder of the division of  $c'$  by  $\text{esp}_T(G) + \max T + 1$ . The claim follows from Lemma 5.

(2) Follows from (1) since  $\min(\mathbb{N} \setminus T) = \max T + 1$  if  $T$  is an interval.

(3) Follows from (1) and the definition of the  $T$ -span. ■

The above inequalities are tight. It is known [1] that  $\text{esp}_T(G) = \text{sp}_T(G)$  for all weakly perfect graphs and all  $T$ -sets  $T$ . It is also easy to see that if  $T$  is an interval, then  $\text{sp}_T(C_{2n+1}) = 2 \max T + 2$  ( $\text{sp}_T(G) = (\max T + 1)(\chi(G) - 1)$  if  $T$  is an interval, see [1]) and  $\text{esp}_T(C_{2n+1}) = \lceil (\max T + 1)(1 + 1/n) \rceil$  (see Theorem 8) which gives  $\text{sp}_T(C_{2n+1}) = \text{esp}_T(C_{2n+1}) + \max T$  provided that  $n \geq \max T + 1$ .

#### 4. THE RELATION BETWEEN $(k, d)$ -COLORINGS AND $T$ -COLORINGS

Now we are ready to prove that there is a relation between  $(k, d)$ -colorings and  $T$ -colorings provided that  $T$  is an interval.

**Lemma 7.** *Let  $G$  be a graph and  $d$  be a positive integer. If  $T = [0, d - 1]$ , then for every function  $c: V \rightarrow \mathbb{Z}$  and every integer  $k \geq d$  the following conditions are equivalent:*

- (1)  $c$  is a  $T$ -coloring of  $G$  such that  $\text{sp}(c) \leq k - 1$  and  $\text{esp}(c) \leq k - d$ ;
- (2)  $c - \min c(V)$  is a  $(k, d)$ -coloring of  $G$ .

**Proof.** Let  $uv$  be an edge of  $G$  (our claim is obvious if  $G$  is empty) and  $c' = c - \min c(V)$ .

( $\Rightarrow$ )  $c$  is a  $T$ -coloring of  $G$  and  $T$  is an interval, so  $|c'(u) - c'(v)| = |c(u) - c(v)| \geq d$ . Moreover,  $|c'(u) - c'(v)| = |c(u) - c(v)| \leq \text{esp}(c) \leq k - d$  and  $c'(V) \subseteq [0, \text{sp}(c)] \subseteq [0, k - 1]$ .

( $\Leftarrow$ )  $c'$  is a  $(k, d)$ -coloring of  $G$ , so  $|c(u) - c(v)| = |c'(u) - c'(v)| \geq d$  and  $|c(u) - c(v)| = |c'(u) - c'(v)| \leq k - d$ . This proves that  $c$  is a  $T$ -coloring and gives  $\text{esp}(c) \leq k - d$ . To complete the proof it suffices to observe that  $c'(V) \subseteq [0, k - 1]$  implies  $\text{sp}(c) = \text{sp}(c') \leq k - 1$ . ■

**Theorem 8.** *Let  $G$  be a graph and  $d$  be a positive integer. If  $T = [0, d - 1]$ , then*

$$\text{esp}_T(G) = \lceil d(\chi_c(G) - 1) \rceil.$$

**Proof.** Without loss of generality we assume that  $G$  is not empty. Then  $k = \lceil d\chi_c(G) \rceil - 1 \geq d$ . If  $\text{esp}_T(G) \leq k - d$ , then, by Corollary 6, there is a  $T$ -coloring  $c$  of  $G$  such that  $\text{esp}(c) = \text{esp}_T(G) \leq k - d$  and  $\text{sp}(c) \leq \text{esp}_T(G) + d - 1 \leq k - 1$ .



Lemma 7 implies now that  $c - \min c(V)$  is a  $(k, d)$ -coloring, which finally gives  $d\chi_c(G) \leq k$ , a contradiction. Hence

$$\text{esp}_T(G) \geq k - d + 1.$$

On the other hand,  $(k + 1)/d \geq \chi_c(G)$  so there exists a  $(k + 1, d)$ -coloring  $c$  of  $G$ . Without loss of generality we assume that  $\min c(V) = 0$ . By Lemma 7,  $c$  has to be a  $T$ -coloring of  $G$  with  $\text{esp}(c) \leq k - d + 1$ . This gives

$$\text{esp}_T(G) \leq k - d + 1.$$

Combining these inequalities together, we get  $\text{esp}_T(G) = k - d + 1 = \lceil d\chi_c(G) \rceil - d = \lceil d(\chi_c(G) - 1) \rceil$ . ■

Since  $T$  is an interval, we know that  $|T| = \max T + 1$  and the above formula may be expressed as

$$\text{esp}_T(G) = \lceil |T|(\chi_c(G) - 1) \rceil.$$

This resembles Tesman's inequality  $\text{sp}_T(G) \leq |T|(\chi(G) - 1)$  which holds for all  $T$ -sets  $T$  and all graphs  $G$  [11], so it is interesting to ask the following question.

Does  $\text{esp}_T(G) \leq \lceil |T|(\chi_c(G) - 1) \rceil$  for all  $T$ -sets  $T$  and all graphs  $G$ ?

Unfortunately, the answer is negative even for odd cycles. To show this, let us consider integers  $1 \leq k \leq n - 1$  and set  $T = \{0, 2, \dots, 2k\}$  and  $G = C_{2n+1}$ . Then  $\lceil |T|(\chi_c(G) - 1) \rceil = \lceil (k + 1)(1 + 1/n) \rceil = k + 2$  and  $\text{esp}_T(C_{2n+1}) \geq 2k + 2$  since otherwise the differences of colors assigned to adjacent vertices of  $G$  in any  $T$ -coloring of  $G$  with minimal edge span would be odd and their sum would not be 0, a contradiction.

Theorem 8 shows also that the value of integer  $r$  of Theorem 3 can be arbitrary. Indeed, if we take  $0 \leq r \leq d - 1$  and a planar graph  $G$  such that  $\chi_c(G) = 3 - r/d$  (which exists by [10]), then  $\chi(G) = 3$  and  $\text{esp}_{d \circ \{0\}}(G) = \lceil d(\chi_c(G) - 1) \rceil = \lceil d(2 - r/d) \rceil = 2d - r = d(\chi(G) - 1) - r = d \text{esp}_{\{0\}}(G) - r$ . The open question is if this is true for all  $T$ -sets  $T$ .

**Theorem 9.** *Let  $G$  be a graph. Then*

$$\chi_c(G) = 1 + \inf \{ \text{esp}_{d \circ \{0\}}(G)/d : d \geq 1 \}.$$

*Moreover, if  $\chi_c(G) = k/d$  ( $1 \leq d \leq k$ ), then  $\chi_c(G) = 1 + \text{esp}_{d \circ \{0\}}(G)/d$ .*

**Proof.**  $\chi_c(G) - 1 \leq \text{esp}_{d \circ \{0\}}(G)/d$  by Theorem 8. To complete the proof it suffices to observe that if  $\chi_c(G) = k/d$ , then the same theorem gives  $\chi_c(G) - 1 = \text{esp}_{d \circ \{0\}}(G)/d$ . ■

Theorems 8 and 9 have two important consequences. Firstly, if we know a formula for  $\chi_c(G)$ , then we can easily obtain a formula for  $\text{esp}_T(G)$  for all  $T$ -sets  $T$  that are intervals. For example, Fan [4] proved that  $\chi_c(G) = \chi(G)$  if the complement of  $G$  is non-Hamiltonian, which gives

**Corollary 10.** *If  $G$  is a graph whose complement is non-Hamiltonian, then*

$$\text{esp}_{d \circ \{0\}}(G) = d(\chi(G) - 1) = \text{sp}_{d \circ \{0\}}(G)$$

for every  $d \geq 1$ .

Secondly, if the problem of computing  $\chi_c(G)$  for graphs  $G$  from a certain class  $\mathcal{G}$  is polynomially solvable, then we can compute  $\text{esp}_T(G)$  for  $G \in \mathcal{G}$  and any interval  $T$  in a polynomial time, too.

## 5. POWERS OF CYCLES

Let  $p \geq 1$  and  $n \geq 2p + 2$  be integers. Let  $q$  and  $r$  are the quotient and the remainder of the division of  $n$  by  $p + 1$ , respectively.

Zhao *et al.* in [13] proved the following theorem.

**Theorem 11.** *If  $q = pl + t$  for  $l \geq 0$ ,  $0 \leq t \leq p - 1$  such that  $p \geq td$ , then*

$$\text{esp}_{d \circ \{0\}}(C_n^p) = pd + \lceil rd/q \rceil.$$

Moreover, they conjectured that this equality holds for any  $n \geq 2p + 2$ , not only when  $p \geq td$ . We will show that it is true. Recall that it is known that if  $G$  is a  $n$ -vertex graph, then  $\chi_c(G) \geq n/\alpha(G)$ , where  $\alpha(G)$  is the independence number of  $G$ .

**Theorem 12.**  $\chi_c(C_n^p) = n/q$ .

**Proof.** Let  $v_0, v_1, \dots, v_{n-1}$  be a cyclic ordering of vertices of  $C_n^p$ . We claim that a function given by

$$c(v_i) = (iq) \bmod n$$

is a  $(n, q)$ -coloring of  $C_n^p$ . Indeed, the definition of  $c$  gives  $0 \leq c \leq n - 1$  and, if  $v_i v_j$  ( $i > j$ ) is an edge of  $C_n^p$ , then either  $1 \leq i - j \leq p$  and  $|c(v_i) - c(v_j)| = (i - j)q$  or  $1 \leq n + j - i \leq p$  and  $|c(v_i) - c(v_j)| = (n - i + j)q$ . In both cases it is easy to verify that  $q \leq |c(v_i) - c(v_j)| \leq qp \leq n - q$ .

To complete the proof it suffices to observe that  $\alpha(C_n^p) \leq q$  and use inequality  $\chi_c(G) \geq n/\alpha(G)$ . ■

**Theorem 13.**  $\text{esp}_{d \circ \{0\}}(C_n^p) = pd + \lceil rd/q \rceil$ .

**Proof.** Follows immediately from Theorems 8 and 12. ■



## 6. CONCLUSION

We proved the general relation between the circular chromatic number and  $T$ -edge span for  $T = d \odot \{0\}$ . Moreover, we applied it to solve an open conjecture concerning the  $T$ -edge span for powers of cycles  $C_n^p$ .

Possible further fields of research include for example finding the necessary conditions for  $\text{esp}_T(G) \leq \lceil |T|(\chi_c(G) - 1) \rceil$ , or analyzing dependence between  $\text{esp}_T(G)$  and  $\chi_c(G)$  on the structure of a set  $T$ .

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Received 19 March 2018  
Revised 16 October 2018  
Accepted 7 December 2018