



## Global edge alliances in graphs

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### ABSTRACT

In the paper we introduce and study a new problem of finding a minimum *global edge alliance* in a graph which is related to the global defensive alliance (Haynes et al., 2013; Hedetniemi, 2004) and the global defensive set (Lewoń et al., 2016).

We proved the  $\mathcal{NP}$ -completeness of the global edge alliance problem for subcubic graphs and we constructed polynomial time algorithms for trees. We found the exact values of the size of the minimum global edge alliance for certain classes: paths, cycles, wheels, complete  $k$ -partite graphs and complete  $k$ -ary trees. Moreover, we proved some lower bounds for arbitrary graphs.

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## 1. Introduction

In the following we consider only simple nonempty graphs, and we use standard notations of the graph theory. Let  $G$  be a graph and let  $X \subset V(G)$ . By an *open neighborhood* of  $X$  in graph  $G$  we mean the set  $\{v \in V(G) : \exists u \in X \{v, u\} \in E(G)\}$ , denoted by  $N_G(X)$ . By a *closed neighborhood* of  $X$  in graph  $G$  we mean set  $X \cup N_G(X)$ , denoted by  $N_G[X]$ . Set  $X$  is a *dominating set* of  $G$  iff  $V(G) = N_G[X]$ , and  $X$  is a *total dominating set* iff  $V(G) = N_G(X)$ . By  $\gamma_t(G)$  we denote the size of the minimum total dominating set in  $G$ .

Let  $S \subset V(G)$ . We define for any non-empty subset  $X$  of  $S$  the predicate  $SEC_{G,S}(X) = \text{true}$  iff  $|N_G[X] \cap S| \geq |N_G[X] \setminus S|$ . In the following, we use the notation  $SEC(X)$  instead of  $SEC_{G,S}(X)$  if  $G$  and  $S$  are clearly given.

By  $G[A]$ , where  $A \subset V(G)$ , we denote a subgraph of  $G$  induced by set  $A$ , and by  $G \setminus A$  we mean the graph  $G[V \setminus A]$ . For the sake of notation simplicity, we write  $N_G[v]$  and  $N_G[v, u]$  instead of  $N_G[\{v\}]$  and  $N_G[\{v, u\}]$ , respectively, and analogously,  $SEC(v)$  and  $SEC(v, u)$ . Let  $\deg_G(v) = |N_G(v)|$  be the degree of a vertex  $v \in V(G)$ . By  $n(G)$ ,  $\Delta(G)$  and  $\delta(G)$  we denote the number of vertices of  $G$ , the maximum and the minimum degree of a vertex of  $G$ , respectively. By an *isolated vertex* (in a graph  $G$ ) we mean a vertex  $v \in V(G)$  with  $\deg_G(v) = 0$ , and by an *isolated edge* (in a graph  $G$ ) we mean an edge  $\{u, v\}$  such that  $\deg_G(u) = \deg_G(v) = 1$ . Set  $X \subset V(G)$  is an *independent set* iff each vertex of  $X$  is isolated in  $G[X]$ . By a *pendant vertex* we mean a vertex of degree 1. We call each neighbor of a pendant vertex in a tree a *support vertex*. Let  $\text{diam}(G) = \max\{d_G(v, u) : u, v \in E(G)\}$ , where  $d_G(v, u)$  is the length of a shortest path in  $G$  between  $v$  and  $u$ .

Let  $S \subset V(G)$  for a given graph  $G$ . Set  $S$  is an *edge alliance* in  $G$  iff  $G[S]$  has no isolated vertices and for each edge  $e = \{v, u\} \in E(G[S])$  we have  $SEC(v, u) = \text{true}$ . An edge alliance  $S$  is a *global edge alliance* in  $G$  if it also dominates  $G$ . By  $\gamma_{ea}(G)$  we denote the size of the minimum global edge alliance in graph  $G$ .

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Fig. 1. The examples of the global alliance number and the global edge alliance number: (a)  $\gamma_a = 2 < \gamma_{ea} = 3$  and (b)  $\gamma_{ea} = 3 < \gamma_a = 5$ .

### 1.1. Related problems and our contribution

A set  $S$  is a *defensive alliance* (or *alliance*) iff for each vertex  $v \in S$  we have  $SEC(v) = true$ . If  $S$  is also a dominating set of  $G$ , we say that  $S$  is a *global defensive alliance* (or *global alliance*). By  $\gamma_a(G)$  we mean the size of the minimum global alliance in  $G$ . The concept of alliances and global alliances in graphs is due to [16] and [15]. The problem has certain interesting applications in web communities [12] or fault-tolerant computing [22].

In [15] the authors proved bounds on the minimum global alliance for general graphs (lower bounds:  $\frac{\sqrt{4n+1}-1}{2}$  and  $\frac{n}{\lceil \frac{\Delta}{2} \rceil + 1}$ , upper bound:  $n - \lceil \frac{\delta}{2} \rceil$ ), for bipartite graphs (lower bound:  $\lceil \frac{2n}{\Delta+3} \rceil$ ), and trees (lower bound:  $\frac{n+2}{4}$ , upper bound:  $\frac{3n}{5}$ ). For the other bounds on trees, see [1,2,8]. The exact values of the minimum global alliance were given in [15] for complete graphs, complete bipartite graphs, cycles, paths and wheels, for  $k$ -ary trees ( $k \in \{2, 3, 4\}$ ) in [6], and independently in [14] (for  $k \in \{2, 3\}$ ), and for star graphs in [17]. In [5] the authors proved the  $\mathcal{NP}$ -completeness of the minimum global alliance problem for general graphs, in [18] the author proved it for bipartite or chordal graphs, and in [20] the authors proved  $\mathcal{NP}$ -completeness for subcubic bipartite planar graphs. In [21] the authors study the problem of finding two disjoint global alliances in graphs.

Set  $S$  is a *defensive set* in  $G$  iff for each vertex  $v \in S$  we have:  $SEC(v) = true$  or there exists a neighbor  $u \in S$  of  $v$  (i.e.  $\{v, u\} \in E(G)$ ) such that  $SEC(v, u) = true$ . If  $S$  is also a dominating set of  $G$ , we say that  $S$  is a *global defensive set*. By  $\gamma_{ds}(G)$  we denote the size of the minimum global defensive set in  $G$ . The concept of defensive sets introduced and studied in [20] arises from the concept of alliances, but is a kind of relaxation of the alliance problem. In [20] the authors proved the  $\mathcal{NP}$ -completeness of the minimum global defensive set problem for subcubic bipartite planar graphs, they constructed polynomial time algorithm for trees, and proved some bounds on  $\gamma_{ds}$ .

Set  $S$  is a *secure set* in  $G$  iff  $\forall X \subset S, SEC(X) = true$ . The concept of secure sets was introduced in [3] and studied in certain papers, e.g., [4,9–11].

In the paper we introduce and study the global edge alliance problem. The concept of edge alliance arises from the idea of alliances and it is a restriction of defensive sets. In the alliance problem a vertex being under an attack (say  $x$ ) can be defended by itself and some of its neighbors, and it is possible iff  $SEC(x) = true$ . In the defensive set problem [20], if  $SEC(x) = false$ , we allow one of the neighbors of  $x$ , say  $y$ , to join 'the war', i.e., an attack can be simultaneously done on two vertices  $x$  and  $y$ , and in that case each attack on  $x$  and  $y$  can be defended, whenever  $SEC(x, y) = true$ . In the edge alliance problem, instead of defending the nodes, we defend the links between them, i.e., the structure being under an attack is an edge which can be defended by its end vertices and some of their neighbors. Note that an edge alliance is a defensive set.

We prove the  $\mathcal{NP}$ -completeness of the global edge alliance problem for subcubic graphs and we construct polynomial time algorithm for trees. We find the exact values of the size of the minimum global edge alliance for certain classes: paths, cycles, wheels, complete  $k$ -partite graphs and complete  $k$ -ary trees. We prove the lower bound for arbitrary graphs.

## 2. Bounds on the minimum global edge alliance

By  $\mathcal{G}_{ea}(G)$  we denote  $\{S \subset V(G) : S \text{ is a global edge alliance of } G\}$ . If  $\delta(G) \geq 1$ , then  $\mathcal{G}_{ea}(G) \neq \emptyset$ .

Let  $G$  be a graph with  $\delta(G) \geq 1$ , and let  $S \subset V(G)$  be a global edge alliance. Since  $S$  is a dominating set of  $G$  and  $G[S]$  has no isolated vertices, we have that  $S$  is a total dominating set of  $G$ . Moreover,  $S$  is a global defensive set. Thus,

**Proposition 2.1.** *Let  $G$  be a graph with  $\delta(G) \geq 1$ . Then,*

$$\gamma_{ea}(G) \geq \max\{\gamma_{ds}(G), \gamma_t(G)\}. \quad \square$$

There is no such a relation, in general, between the global alliance number and the global edge alliance number, which is shown in Fig. 1. For subcubic graphs with  $\delta \geq 2$  by [15] we have that  $\gamma_t = \gamma_a$ . Thus,

**Proposition 2.2.** *Let  $G$  be a graph with  $\delta(G) \geq 2$  and  $\Delta(G) \leq 3$ . Then,*

$$\gamma_{ea}(G) \geq \gamma_a(G). \quad \square$$

Following [20], we prove the lower bound on  $\gamma_{ea}$  for arbitrary graphs. By  $\nu(G)$  we mean the size of the maximum matching in graph  $G$  (i.e., set of edges no two of which have a common end). For a given global edge alliance  $S$  in graph  $G$ , by  $\nu_S(G)$  we mean  $\nu(G[S])$ . Finally, let us define by  $\nu_{ea}(G) = \max\{\nu_S(G) : S \in \mathcal{G}_{ea}(G) \wedge \gamma_{ea}(G) = |S|\}$ .

**Theorem 2.1.** Let  $G$  be a graph with  $\delta(G) \geq 1$ . Then,

$$\gamma_{ea}(G) \geq \frac{\sqrt{4n(G) + (v_{ea}(G) - 1)^2} + v_{ea}(G) - 1}{2}.$$

**Proof.** Let  $S$  be any global edge alliance such that  $|S| = \gamma_{ea}(G)$  and  $v_{ea}(G) = v_S(G)$ , and let  $V = V(G)$ ,  $n = n(G)$ ,  $s = |S|$  and  $v = v_{ea}(G)$ . Obviously,  $v > 0$ . The thesis is equivalent to  $s^2 - (v - 1)s - n \geq 0$ . Let us assume to the contrary that  $s^2 - (v - 1)s - n < 0$ . Since  $S$  is a total dominating set of  $G$  we have  $|N_G[S] \cap (V \setminus S)| = |V \setminus S| = n - s > s^2 - vs$ .

On the one hand, let  $M = \{\{v_i, u_i\} : i \in \{1, \dots, v\}\}$  be a maximum matching in  $G[S]$ , and let  $U = \bigcup_{i=1}^v \{v_i, u_i\}$ . Since  $SEC(v_i, u_i) = true$ , we have that  $|N_G[v_i, u_i] \cap (V \setminus S)| \leq |N_G[v_i, u_i] \cap S| \leq s$ . Hence, by  $|N_G[S \cup U] \cap (V \setminus S)| + \sum_{i=1}^v |N_G[v_i, u_i] \cap (V \setminus S)| \geq |(N_G[S \cup U] \cup \bigcup_{i=1}^v N_G[v_i, u_i]) \cap (V \setminus S)| = |V \setminus S| > s^2 - vs$ , we get  $|N_G[S \cup U] \cap (V \setminus S)| > s(s - 2v)$ .

On the other hand, since for every  $w \in S$  there is  $u \in S$  such that  $\{w, u\} \in E(G[S])$  and  $SEC(w, u) = true$ , we have that  $|N_G[w] \cap (V \setminus S)| \leq |N_G[w, u] \cap (V \setminus S)| \leq |N_G[w, u] \cap S| \leq s$ . Thus,  $|N_G[S \cup U] \cap (V \setminus S)| \leq \sum_{w \in S \cup U} |N_G[w] \cap (V \setminus S)| \leq s(s - 2v)$ , a contradiction.  $\square$

Following [20], let us observe that for any  $r \geq 1$  and  $l \geq r$  there is a graph  $G$  with  $n(G) = l + r + l(l + r)$  vertices and  $v_{ea}(G) = r$  such that the lower bound proved in Theorem 2.1 is tight. Let  $G = K_{r,l}^*$  be the graph obtained from the complete bipartite graph  $K_{r,l}$  by attaching  $l + r$  vertices to each vertex of the part with  $l$  vertices. It is easy to notice that  $\gamma_{ea}(G) = l + r$  and  $\gamma_{ea}^2(G) - \gamma_{ea}(G)(v_{ea}(G) - 1) = n(G)$ .

**Proposition 2.3.** Let  $G$  be a graph with  $\delta(G) \geq 1$ . Then,

$$v_{ea}(G) \geq \lceil \frac{\text{diam}(G) + 1}{5} \rceil.$$

**Proof.** Let  $d = \text{diam}(G)$ , and let  $v$  and  $u$  be vertices of  $G$  such that  $d_G(v, u) = d$ . Let  $A_i = \{w \in V(G) : d_G(v, w) = i\}$ , for  $i \in \{0, \dots, d\}$ . Obviously,  $A_0 = \{v\}$  and  $u \in A_d$ . Let us observe that for each edge  $\{x, y\} \in E(G)$ , if  $x \in A_i$  and  $y \in A_j$ , then  $|i - j| \leq 1$ .

Let  $S \subset V(G)$  be a global edge alliance of  $G$  such that  $\gamma_{ea}(G) = |S|$ . Since  $S$  is a total dominating set we have  $S \cap A_1 \neq \emptyset$  and there is  $\{v_0, u_0\} \in E(G)$  such that  $\{v_0, u_0\} \subset S \cap (A_0 \cup A_1 \cup A_2)$ . If  $d \leq 4$ , the thesis holds.

Let  $d = 5k + i$ , where  $k \geq 1$  and  $0 \leq i \leq 4$ . Let us observe that for each  $l \in \{0, \dots, d - 2\}$ ,  $S \cap (A_l \cup A_{l+1} \cup A_{l+2}) \neq \emptyset$ . Thus, for each  $j \in \{1, \dots, k - 1\}$  there is an edge  $\{v_j, u_j\} \in E(G)$  such that  $\{v_j, u_j\} \subset S \cap (A_{5j-2} \cup A_{5j-1} \cup A_{5j} \cup A_{5j+1} \cup A_{5j+2})$ . If  $j = k$ , then there is an edge  $\{v_k, u_k\} \in E(G)$  such that  $\{v_k, u_k\} \subset S \cap (A_{5k-2} \cup A_{5k-1} \cup A_{5k} \cup \dots \cup A_{5k+i})$ .

Since  $\{\{v_0, u_0\}, \dots, \{v_k, u_k\}\}$  is the matching in  $G[S]$ , we have  $v_{ea}(G) \geq v_S(G) \geq k + 1 = \lceil \frac{\text{diam}(G) + 1}{5} \rceil$ .  $\square$

**Corollary 2.4.** Let  $G$  be a graph with  $\delta(G) \geq 1$ . Then,

$$\gamma_{ea}(G) \geq \frac{\sqrt{4n(G) + (\lceil \frac{\text{diam}(G) + 1}{5} \rceil - 1)^2} + \lceil \frac{\text{diam}(G) + 1}{5} \rceil - 1}{2}. \quad \square$$

### 3. Global edge alliance of certain graph classes

In this section we give the exact formulas for the global edge alliance number of the following classes: paths, cycles, wheels, complete multipartite graphs and complete  $k$ -ary trees.

Let  $G$  be a graph with  $\delta(G) \geq 1$  and  $\Delta(G) \leq 2$ . Hence, we have that  $|N_G[u, v] \setminus \{u, v\}| \leq 2$  for each  $\{u, v\} \in E(G)$ , and so  $\gamma_{ea}(G) = \gamma_t(G)$ . Thus,

**Proposition 3.1.** Let  $G$  be a path of order  $n \geq 2$  or a cycle of order  $n \geq 3$ . Then,

$$\gamma_{ea}(G) = \begin{cases} \frac{n}{2} \text{AMZ@Pif } 4 \mid n, \\ \lfloor \frac{n}{2} \rfloor + 1 \text{AMZ@Potherwise.} \quad \square \end{cases}$$

Let us recall that by wheel  $W_k$  ( $k \geq 3$ ) we mean a graph obtained from cycle  $C_k$  by adding the central vertex  $v_c$  and joining it with all other vertices of the cycle  $C_k$ . Hence, we have  $|N_{W_k}[v_c]| = |V(W_k)| = k + 1$ ,  $\Delta(W_k) = k$  and for each  $v \in V(W_k) \setminus \{v_c\}$  we have  $\text{deg}_{W_k}(v) = 3$ .

**Proposition 3.2.** Let  $G$  be a wheel of order  $n$ . Then,

$$\gamma_{ea}(G) = \lceil \frac{n}{2} \rceil.$$

**Proof.** Let  $G$  be a wheel of order  $n$ . If  $n = 4$ , then  $\gamma_{ea}(G) = 2$ . Let  $n \geq 5$  and let  $v_c \in V(G)$  be the central vertex of wheel  $G$ . Let  $S \subset V(G)$  be any global edge alliance of  $G$ . If  $v_c \in S$ , then there is  $v \in V(G) \setminus \{v_c\}$  such that  $v \in S$ . Hence,  $N_G[v, v_c] = V(G)$ ,

so  $|S| \geq \lceil n/2 \rceil$ . Let  $v_c \notin S$ . For each  $\{v, u\} \in E(G[S])$  we have  $|N_G[v, u] \setminus \{v, u\}| = 3$ , so  $|S \cap N_G[u, v]| \geq 3$ . Hence, each connected component of graph  $G[S]$  has at least three vertices. Since  $S$  is a dominating set of  $G$ , we have  $|S| \geq \lceil 3(n-1)/5 \rceil$ . Thus,  $|S| \geq \min\{\lceil n/2 \rceil, \lceil 3(n-1)/5 \rceil\} = \lceil n/2 \rceil$ . Take any  $S \subset V(G)$  such that  $v_c \in S$  and  $|S \cap V(G)| = \lceil n/2 \rceil$ . Thus,  $S$  is a global edge alliance and  $\gamma_{ea}(G) = \lceil \frac{n}{2} \rceil$ .  $\square$

**Theorem 3.1.** Let  $G$  be a complete multipartite graph of order  $n \geq 3$ . Then,

$$\gamma_{ea}(G) = \lceil \frac{n}{2} \rceil.$$

**Proof.** Let  $G$  be a complete multipartite graph of order  $n \geq 3$ . For each edge  $\{u, v\} \in E(G)$  we have  $N_G[u, v] = V(G)$ . Let  $S \subset V(G)$  be any global edge alliance of  $G$ , and let  $\{u, v\} \in E(G[S])$ . Then,  $SEC(u, v) \Leftrightarrow |V(G) \cap S| \geq |V(G) \setminus S| \Leftrightarrow |S| \geq \lceil n/2 \rceil$ . Thus,  $\gamma_{ea}(G) \geq \lceil n/2 \rceil$ .

Let  $V(G) = V_1 \cup \dots \cup V_k$ , where  $k \geq 2$ , and all  $V_i$  are maximal independent sets and pairwise disjoint. Let  $r_i = |V_i|$ , for each  $i \in \{1, \dots, k\}$ . If for each  $i \in \{1, \dots, k\}$  an integer  $r_i$  is even, then let  $S \subset V(G)$  be any set such that  $|S \cap V_i| = r_i/2$ . Thus,  $S$  is a global edge alliance and  $|S| = n/2$ .

Without loss of generality, let us assume that  $r_1, \dots, r_l$  are odd integers, and  $r_{l+1}, \dots, r_k$  are even integers, for some  $1 \leq l \leq k$ . Take any  $S \subset V(G)$  such that  $|S \cap V_i| = \lceil r_i/2 \rceil$  for each  $i \in \{1, \dots, \lceil l/2 \rceil\}$ ,  $|S \cap V_i| = \lfloor r_i/2 \rfloor$  for each  $i \in \{\lceil l/2 \rceil + 1, \dots, l\}$ , and  $|S \cap V_i| = r_i/2$  for each  $i \in \{l+1, \dots, k\}$ . Hence,  $\delta(G[S]) \geq 1$  and  $|S| = \lceil n/2 \rceil$ . Thus,  $S$  is a global edge alliance.  $\square$

### 3.1. Complete $k$ -ary trees

Let us remind that the exact formulas for the minimum global alliance of complete  $k$ -ary trees are known only for  $k \in \{2, 3, 4\}$  [6]. In this section we give the exact formulas for the minimum global edge alliance of complete  $k$ -ary trees for arbitrary  $k \geq 2$ .

Let  $\delta(G) \geq 1$  and  $A \subset V(G)$ . By  $\gamma_{ea}(G, A)$  we mean  $\min\{|S \cap A| : S \in \mathcal{G}_{ea}(G)\}$ .

**Observation 3.3.** Let  $G$  be a graph with  $\delta(G) \geq 1$ , and let  $A_i \subset V(G)$ , for  $i \in \{1, \dots, p\}$ , where  $p \geq 1$ . If sets  $A_1, \dots, A_p$  are pairwise disjoint, then

$$\sum_{i=1}^p \gamma_{ea}(G, A_i) \leq \gamma_{ea}(G, \bigcup_{i=1}^p A_i) \leq \gamma_{ea}(G). \quad \square$$

By a complete  $k$ -ary tree of height  $h$ ,  $k \geq 2$ ,  $h \geq 1$  (both integers), denoted by  $T_k^h$ , we mean a tree with a vertex set and an edge set, respectively,

$$V(T_k^h) = \{v_1^0, v_1^1, \dots, v_k^1, v_1^2, \dots, v_{k^2}^2, \dots, v_1^{h-1}, \dots, v_{k^{h-1}}^{h-1}, v_1^h, \dots, v_{k^h}^h\},$$

$$E(T_k^h) = \bigcup_{l=0}^{h-1} \bigcup_{i=1}^{k^l} \bigcup_{j=(i-1)k+1}^{i \cdot k} \{\{v_i^l, v_j^{l+1}\}\}.$$

Let  $T = T_k^h$ . Let us observe that  $\deg_T(v_1^0) = k$ , and for each  $i \in \{1, \dots, h\}$  and  $j \in \{1, \dots, k^i\}$ ,  $d_T(v_1^0, v_j^i) = i$  and, if  $i \neq h$ ,  $\deg_T(v_j^i) = k+1$ . Set of all leaves in  $T$  is  $\{v_1^h, \dots, v_{k^h}^h\}$ , and  $d_T(v_1^0, v_j^h) = h$ , for each  $j \in \{1, \dots, k^h\}$ . By  $r(T)$  we mean the only vertex of degree  $k$  in  $T$ .

For each  $l \in \{0, \dots, h\}$ , let us define  $L_l = \{v_1^l, \dots, v_{k^l}^l\}$  (obviously,  $L_l = \{v \in V(T) : d_T(v, r(T)) = l\}$ ) and let  $L_{h+i} = \emptyset$ , for  $i \geq 1$ . Let  $v \in L_p$ , for some  $p \in \{0, \dots, h\}$ . By  $C(T, v)$  we mean  $N_T(v) \cap L_{p+1}$ , and for each  $A \subset V(G)$ , by  $C(T, A)$  we mean  $\bigcup_{v \in A} C(T, v)$ . Obviously,  $|C(T, v)| = k$ , and  $C(T, L_h) = \emptyset$ . Let us define  $C^0(T, v) = \{v\}$ , and for each  $l \geq 1$ , let  $C^l(T, v) = C(T, C^{l-1}(T, v))$ . Let  $T_v^l = T[\bigcup_{i=0}^l C^i(T, v)]$ , and  $T_v = T_v^{h-p}$ .

**Lemma 3.4.** Let  $T = T_k^h$ ,  $h \geq 2$ ,  $k \geq 2$ , and let  $l \in \{1, 2, 3\}$ . Then,

$$\gamma_{ea}(T, V(T_r^{l-1})) \geq \frac{k^{l-1} - 1}{k - 1}.$$

**Proof.** Let  $r = r(T)$ . If  $l \leq 2$ , then the thesis is obvious. Let  $l = 3$  and  $S \in \mathcal{G}_{ea}(T)$ . If  $r \in S$ , then there is  $u \in L_1 \cap S$ . Since  $SEC(r, u) = true$  and  $|N_T[\{r, u\}]| = 2k+1$ ,  $|N_T[\{r, u\}] \cap S| \geq k+1$ . Hence,  $\gamma_{ea}(T, V(T_r^2)) \geq k+1$ . If  $r \notin S$ , then for each  $u \in L_1$ ,  $\gamma_{ea}(T, V(T_u^1)) \geq 1$ . Since  $N_T[r] \cap S \neq \emptyset$ , there is  $u \in L_1 \cap S$ . Since  $N_T[u] \cap S \neq \emptyset$ ,  $|S \cap V(T_u^1)| \geq 2$ . Thus, by **Observation 3.3**,  $\gamma_{ea}(T, V(T_r^2)) \geq \sum_{u \in L_1} \gamma_{ea}(T, V(T_u^1)) \geq k+1$ .  $\square$

**Lemma 3.5.** Let  $T = T_k^h$ ,  $h \geq 2$ ,  $k \geq 2$ . Then,

$$\gamma_{ea}(T, L_{h-2} \cup L_{h-1} \cup L_h) \geq (k+1)k^{h-2}.$$

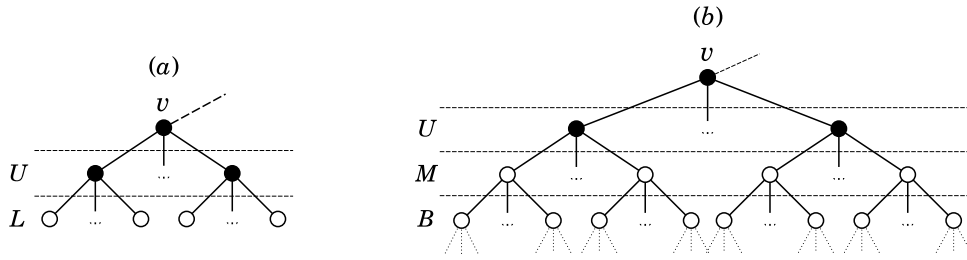


Fig. 2. Subgraphs of  $T_k^h$  discussed in Lemmas 3.5 (a) and 3.6 (b).

**Proof.** Let  $v \in L_{h-2}$ , and let  $L = C^2(T, v)$  and  $U = C^1(T, v)$  (see Fig. 2(a)). Let  $S \in \mathcal{G}_{ea}(T)$ . Since  $L \subset L_h$  and  $N_T(L) = U$ ,  $U \subset S$ . If  $v \in S$ , then  $|S \cap V(T_v)| \geq k + 1$ . If  $v \notin S$ , then for each  $u \in U$ ,  $|S \cap V(T_u)| \geq 2$ . Thus,  $\gamma_{ea}(T, V(T_v)) \geq k + 1$ .

Since  $|L_{h-2}| = k^{h-2}$ , and  $\bigcup_{v \in L_{h-2}} V(T_v) = L_{h-2} \cup L_{h-1} \cup L_h$ , by Observation 3.3 we get the thesis.  $\square$

**Lemma 3.6.** Let  $T = T_k^h$ ,  $h \geq 5$ ,  $k \geq 2$ . For each  $l \in \{0, \dots, h - 5\}$ ,

$$\gamma_{ea}(T, L_l \cup L_{l+1} \cup L_{l+2} \cup L_{l+3}) \geq (k + 1)k^l.$$

**Proof.** Let  $l \in \{0, \dots, h - 5\}$  and  $v \in L_l$ . Let  $B = C^3(T, v)$ ,  $M = C^2(T, v)$  and  $U = C^1(T, v)$  (see Fig. 2(b)). Let  $S \in \mathcal{G}_{ea}(T)$ . Since  $N_T(M) = B \cup U$ , for each  $u \in U$ ,  $|S \cap V(T_u^2)| \geq 1$ . If  $v \in S$ , then  $|S \cap V(T_v)| \geq k + 1$ . If  $v \notin S$ , then for each  $u \in U$ ,  $|S \cap V(T_u^2)| \geq 2$ . Thus,  $\gamma_{ea}(T, V(T_v)) \geq k + 1$ .

Since  $|L_l| = k^l$ , and  $\bigcup_{v \in L_l} V(T_v^3) = L_l \cup L_{l+1} \cup L_{l+2} \cup L_{l+3}$ , by Observation 3.3 we get the thesis.  $\square$

**Lemma 3.7.** Let  $T = T_k^h$ ,  $h = 4p + 2$ ,  $p \geq 0$ ,  $k \geq 2$ . Then,

$$\gamma_{ea}(T) = (k + 1) \frac{k^{h+2} - 1}{k^4 - 1}.$$

**Proof.** Let  $A_i = L_{4i} \cup L_{4i+1} \cup L_{4i+2} \cup L_{4i+3}$ , for each  $i \in \{0, \dots, p - 1\}$ , and let  $A_p = L_{4p} \cup L_{4p+1} \cup L_{4p+2}$ . By Observation 3.3 and by Lemmas 3.5 and 3.6,  $\gamma_{ea}(T) \geq \sum_{i=0}^p \gamma_{ea}(T, A_i) = (k + 1) \sum_{i=0}^p k^{4i} = (k + 1) \frac{k^{h+2} - 1}{k^4 - 1}$ .

Let  $S = \bigcup_{i=0}^p (L_{4i} \cup L_{4i+1})$ . Since  $S \in \mathcal{G}_{ea}(T)$  and  $|S| = (k + 1) \frac{k^{h+2} - 1}{k^4 - 1}$ ,  $\gamma_{ea}(T) = |S|$ .  $\square$

**Lemma 3.8.** Let  $T = T_k^h$ ,  $k \geq 2$ , and let  $h = q + 4p + 2 \geq 2$ , where  $q \in \{1, 2, 3\}$  and  $p \geq 0$ . Then,

$$\gamma_{ea}(T) = \frac{k^{q-1} - 1}{k - 1} + (k + 1) \frac{k^{h+2} - k^q}{k^4 - 1}.$$

**Proof.** Let  $h = q + 4p + 2 \geq 2$ , where  $q \in \{1, 2, 3\}$  and  $p \geq 0$ . Let  $r = r(T)$ . Let  $A_i = L_{q+4i} \cup L_{q+4i+1} \cup L_{q+4i+2} \cup L_{q+4i+3}$ , for each  $i \in \{0, \dots, p - 1\}$ , and let  $A_p = L_{q+4p} \cup L_{q+4p+1} \cup L_{q+4p+2}$ . Since  $V(T) = V(T_r^{q-1}) \cup \bigcup_{i=0}^p A_i$ , by Observation 3.3 and by Lemmas 3.4 and 3.7 we get  $\gamma_{ea}(T) \geq \frac{k^{q-1} - 1}{k - 1} + (k + 1) \frac{k^{h+2} - k^q}{k^4 - 1}$ .

Let  $\hat{S} = \bigcup_{i=0}^p (L_{q+4i} \cup L_{q+4i+1})$ . Let  $S_1 = \emptyset$ ,  $S_2 = \{u\}$ , where  $u \in N_T(r)$ , and  $S_3 = V(T_r^1)$ . Let us observe that for each  $q \in \{1, 2, 3\}$ ,  $S = \hat{S} \cup S_q \in \mathcal{G}_{ea}(T)$ , and  $|S| = \frac{k^{q-1} - 1}{k - 1} + (k + 1) \frac{k^{h+2} - k^q}{k^4 - 1}$ . Thus,  $\gamma_{ea}(T) = |S|$ .  $\square$

By Lemmas 3.7 and 3.8 we conclude

**Theorem 3.2.** Let  $T = T_k^h$ ,  $h \geq 2$ ,  $k \geq 2$ . Then,

1. if  $h \equiv 0 \pmod{4}$ , then  $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - k^2) + 1$ ,
2. if  $h \equiv 1 \pmod{4}$ , then  $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - k^3) + k + 1$ ,
3. if  $h \equiv 2 \pmod{4}$ , then  $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - 1)$ ,
4. if  $h \equiv 3 \pmod{4}$ , then  $\gamma_{ea}(T) = \frac{k+1}{k^4-1}(k^{h+2} - k)$ .  $\square$

#### 4. $\mathcal{NP}$ -completeness for subcubic graphs

In this section we prove the  $\mathcal{NP}$ -completeness of the global edge alliance problem for subcubic graphs, by the reduction from the  $\mathcal{NP}$ -complete 3DM problem [13].

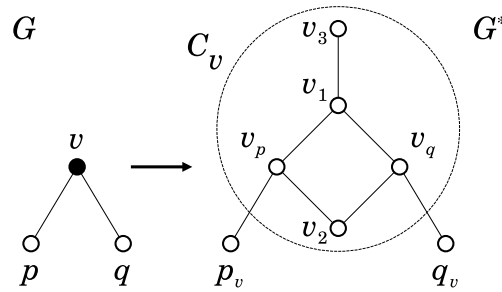


Fig. 3. Graph  $C_v$  replacing vertex  $v \in V$  of degree 2. Note that  $p_v, q_v \notin V(C_v)$ , but  $p_v \in V(H_p), q_v \in V(H_q)$ .

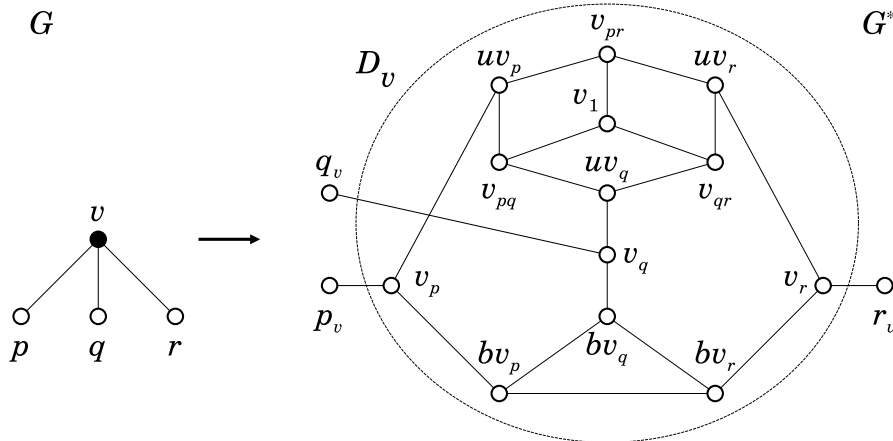


Fig. 4. Graph  $D_v$  replacing vertex  $v \in V$  of degree 3. Note that  $p_v, q_v, r_v \notin V(D_v)$ , but  $p_v \in V(H_p), q_v \in V(H_q), r_v \in V(H_r)$ .

3DM

**Instance:** A subcubic bipartite graph  $G = (V \cup Q, E)$  without pendant vertices, where  $V$  and  $Q$  is a bipartition of  $G, V = X \cup Y \cup Z, |X| = |Y| = |Z| = m, |V| = 3m$ . For each vertex  $q \in Q, \deg_G(q) = 3$  and  $q$  is adjacent to exactly one vertex from each of the sets  $X, Y$  and  $Z$ , and for each vertex  $v \in V, \deg_G(v) \in \{2, 3\}$ .

**Question:** Is there a subset  $Q' \subset Q$  of cardinality  $|Q'| = m$  dominating all vertices in  $V$ , i.e.,  $N_G(Q') = V$ ?

**Theorem 4.1.** The global edge alliance problem for subcubic graphs is  $\mathcal{NP}$ -complete.

**Proof.** The proof proceeds by the reduction from the problem  $\overline{3DM}$ . Let  $G = (V \cup Q, E)$  be a subcubic bipartite graph with bipartition  $V$  and  $Q$ , such that  $V = X \cup Y \cup Z, |X| = |Y| = |Z| = m, |V| = 3m$ , and  $|Q| = t$ . For each vertex  $q \in Q, \deg_G(q) = 3$  and  $q$  is adjacent to exactly one vertex from each of the sets  $X, Y$  and  $Z$ , and for each vertex  $v \in V, \deg_G(v) \in \{2, 3\}$ . Let  $V_i = \{v \in V : \deg_G(v) = i\}$ , and  $m_i = |V_i|$ , for  $i \in \{2, 3\}$ . We construct a subcubic graph  $G^*$  such that there is a subset  $Q' \subset Q$  of cardinality  $|Q'| = m$  dominating all vertices in  $V$  iff there is a global edge alliance  $S$  in graph  $G^*$  such that  $|S| \leq 2m_2 + 5m_3 + 9t$ .

We shall transform (in polynomial time) graph  $G$  into graph  $G^*$  in four steps:

- (S<sub>1</sub>) each  $v \in V_2$ , where  $N_G(v) = \{p, q\} \subset Q$ , replace with graph  $C_v$  (Fig. 3),
- (S<sub>2</sub>) each  $v \in V_3$ , where  $N_G(v) = \{p, q, r\} \subset Q$ , replace with graph  $D_v$  (Fig. 4),
- (S<sub>3</sub>) each  $q \in Q$ , where  $N_G(q) = \{x, y, z\} \subset V$  and  $x \in X, y \in Y, z \in Z$ , replace with graph  $H_q$  (Fig. 5),
- (S<sub>4</sub>) each  $\{v, q\} \in E(G)$ , where  $v \in V, q \in Q$ , replace with edge  $\{v_q, q_v\} \in E(G^*)$ .

Formally, all graphs  $\{C_v\}_{v \in V_2}$  are isomorphic and pairwise vertex disjoint (analogously,  $\{D_v\}_{v \in V_3}$  and  $\{H_q\}_{q \in Q}$ ), so

$$V(G^*) = \bigcup_{v \in V_2} V(C_v) \cup \bigcup_{v \in V_3} V(D_v) \cup \bigcup_{q \in Q} V(H_q)$$

$$E(G^*) = \bigcup_{v \in V_2} E(C_v) \cup \bigcup_{v \in V_3} E(D_v) \cup \bigcup_{q \in Q} E(H_q) \cup \bigcup_{v \in V, q \in Q, \{v, q\} \in E(G)} \{\{v_q, q_v\}\}$$



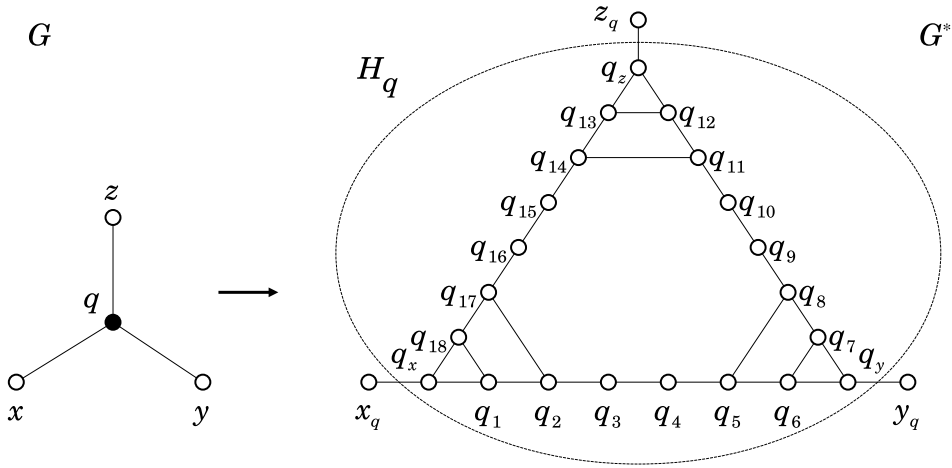


Fig. 5. Graph  $H_q$  replacing vertex  $q \in Q$ . Note that  $x_q, y_q, z_q \notin V(H_q)$ .

It is easy to observe that  $\Delta(G^*) \leq 3$ .

( $\Rightarrow$ ) Suppose that  $Q' \subset Q$  dominates all vertices in  $V$  and  $|Q'| = m$ . Since  $|X| = |Y| = |Z| = m$ , we have  $|N_G(v) \cap Q'| = 1$ , for each  $v \in V$ . Hence, let  $\{q(v)\} = N_G(v) \cap Q'$ , for each  $v \in V$ . Moreover, if  $v \in V_2$ , then let  $\{p(v)\} = N_G(v) \cap (Q \setminus Q')$  and if  $v \in V_3$ , then let  $\{p(v), r(v)\} = N_G(v) \cap (Q \setminus Q')$ . For each  $q \in Q$ , if  $N_G(q) = \{x, y, z\} \subset V$  and  $x \in X, y \in Y, z \in Z$ , then let  $Q_q^+ = \{q_x, q_y, q_z, q_3, q_4, q_9, q_{10}, q_{15}, q_{16}\} \subset V(H_q)$  and  $Q_q^- = \{q_1, q_2, q_3, q_7, q_8, q_9, q_{13}, q_{14}, q_{15}\} \subset V(H_q)$  (see Fig. 5). Let

$$S = \bigcup_{v \in V_2} \{v_1, v_{q(v)}\} \cup \bigcup_{v \in V_3} \{v_{q(v)}, bv_{q(v)}, uv_{p(v)}, v_{p(v)r(v)}, uv_{r(v)}\} \cup \bigcup_{q \in Q'} Q_q^+ \cup \bigcup_{q \in Q \setminus Q'} Q_q^-.$$

For each  $v \in V_2$ , set  $\{v_1, v_{q(v)}\} \subset V(C_v)$  is a dominating set of  $C_v$ , and for each  $v \in V_3$ , set  $\{v_{q(v)}, bv_{q(v)}, uv_{p(v)}, v_{p(v)r(v)}, uv_{r(v)}\} \subset V(D_v)$  is a dominating set of  $D_v$ . If  $q \in Q'$ , then  $Q_q^+$  is a dominating set of  $H_q$ , and if  $q \in Q \setminus Q'$ , then  $Q_q^-$  is a dominating set of  $H_q$ . Thus,  $S$  is a dominating set of  $G^*$ .

Each edge of  $G^*[S]$  belongs to the one graph from  $\{C_v\}_{v \in V_2} \cup \{D_v\}_{v \in V_3} \cup \{H_q\}_{q \in Q}$ , or is equal to  $\{v_q, q_v\}$  for some  $q \in Q'$  and  $v \in V$ .

Let  $e \in E(G^*[S])$ . If  $e$  is one of the edges  $\{q_3, q_4\}, \{q_9, q_{10}\}$  or  $\{q_{15}, q_{16}\}$  for some  $q \in Q'$ , then  $SEC(e) = true$ . Otherwise,  $e$  is not an isolated edge in  $G^*[S]$ . Thus, by  $\Delta(G^*) \leq 3$ , we have that  $S$  is a global edge alliance, and  $|S| \leq 2m_2 + 5m_3 + 9|Q'| + 9|Q \setminus Q'| = 2m_2 + 5m_3 + 9t$ .

( $\Leftarrow$ ) Let  $S$  be any global edge alliance of  $G^*$  such that  $|S| \leq 2m_2 + 5m_3 + 9t$ . In the following we refer to the notation from Figs. 3–5.

**Claim 4.1.** For each  $v \in V_2, |S \cap V(C_v)| \geq 2$ . Moreover, if  $|S \cap V(C_v)| = 2$ , then  $|S \cap \{v_p, v_q\}| = 1$ , and,  $\{v_1, v_p, v_q\} \subset S$  or  $\{v_1, v_q, q_v\} \subset S$ , where  $N_G(v) = \{p, q\}$ .

**Proof.** Let  $v \in V_2$ , and let  $N_G(v) = \{p, q\}$ . Since  $S$  is a total dominating set of  $G^*$ ,  $v_1 \in S$ . If  $v_3 \in S$ , then  $|N_{G^*}[v_2] \cap S| \geq 1$  and  $|S \cap V(C_v)| \geq 3$ . Otherwise,  $v_p \in S$  or  $v_q \in S$ . Since  $S$  is an edge alliance,  $p_v \in S$  and  $q_v \in S$ . Thus,  $\{v_1, v_p, p_v\} \subset S$  or  $\{v_1, v_q, q_v\} \subset S$ .  $\square$

**Claim 4.2.** For each  $v \in V_3, |S \cap V(D_v)| \geq 5$ . Moreover, if  $|S \cap V(D_v)| = 5$ , then  $|S \cap \{v_p, v_q, v_r\}| = 1$ , and,  $\{bv_p, v_p, p_v\} \subset S$  or  $\{bv_q, v_q, q_v\} \subset S$  or  $\{bv_r, v_r, r_v\} \subset S$ , where  $N_G(v) = \{p, q, r\}$ .

**Proof.** Let  $v \in V_3$  and  $N_G(v) = \{p, q, r\}$ . Let us remind that  $S$  is a global edge alliance.

Suppose  $\{bv_p, bv_q, bv_r\} \cap S = \emptyset$ . Hence,  $\{v_p, v_q, v_r\} \subset S$ . If  $v_1 \in S$ , then there is  $u \in N_{G^*}[v_1] \cap S$  such that  $|S \cap N_{G^*}[u]| \geq 3$ . Thus,  $|S \cap V(D_v)| \geq 6$ . Let  $v_1 \notin S$ , and let us assume without loss of generality that  $v_{pr} \in S$ . If  $|N_{G^*}[v_{pr}] \cap S| \geq 3$ , then  $|S \cap V(D_v)| \geq 6$ . Let  $|N_{G^*}[v_{pr}] \cap S| = 2$ . If  $uv_r \in S$  ( $uv_p \in S$ , analogously), then  $N_{G^*}[v_{pq}] \cap S \neq \emptyset$ . Thus,  $|S \cap V(D_v)| \geq 6$ .

Suppose  $S \cap \{bv_p, bv_q, bv_r\} \neq \emptyset$ . If  $|S \cap \{bv_p, bv_q, bv_r\}| \geq 2$ , then analogously to the case  $S \cap \{bv_p, bv_q, bv_r\} = \emptyset$  we get  $|S \cap V(D_v)| \geq 6$ . Thus, without loss of generality, let us assume that  $bv_q \in S$  and  $S \cap \{bv_p, bv_r\} = \emptyset$ . Hence,  $v_q \in S$  and we consider two cases  $uv_q \in S$  or  $q_v \in S$ . Let  $U = N_{G^*}[v_{pr}]$  and  $U_1 = N_{G^*}[U]$ .

Let  $\{bv_q, v_q, uv_q\} \subset S$ . If  $v_{pr} \in S$ , then  $|S \cap U_1| \geq 3$ . Since  $\{bv_q, v_q, uv_q\} \cap U_1 = \emptyset$ , we have  $|S \cap V(D_v)| \geq 6$ . So, let  $v_{pr} \notin S$ . Let us observe that  $S \cap \{v_{pq}, v_{qr}\} \neq \emptyset$ . If  $\{v_{pq}, v_{qr}, v_1\} \subset S$ , then  $|S \cap V(D_v)| \geq 6$ . If  $v_{qr} \notin S$  ( $v_{pq} \notin S$ , analogously), then  $v_{pq} \in S$  and  $|S \cap N_{G^*}[\{v_{pr}, uv_r\}]| \geq 2$ . Thus,  $|S \cap V(D_v)| \geq 6$ . So, let  $S \cap \{v_{pq}, v_{qr}, v_1\} = \{v_{pq}, v_{qr}\}$ . Since  $S \cap U \neq \emptyset$ , we have  $|S \cap V(D_v)| \geq 6$ .

Let  $\{bv_q, v_q, q_v\} \subset S$ , and let  $U_2 = N_{C^*}[U_1] \cap V(D_v)$ . Since  $S \cap U \neq \emptyset$  and  $U_2 \cap \{bv_q, v_q\} = \emptyset$ , we have  $|S \cap U_2| \geq 3$ . Thus,  $|S \cap V(D_v)| \geq 5$ .  $\square$

**Claim 4.3.** For each  $q \in Q$ ,  $|S \cap V(H_q)| \geq 9$ .

**Proof.** Let  $q \in Q$  and  $N_C(q) = \{x, y, z\}$ , where  $x \in X, y \in Y, z \in Z$ . Let us remind that  $S$  is a global edge alliance. We prove that  $|S \cap \{q_x, q_1, q_2, q_3, q_{16}, q_{17}, q_{18}\}| \geq 3$ . Let  $q_x \in S$  and  $|S \cap \{q_2, q_{17}\}| \leq 1$ . If  $q_2 \in S$  ( $q_{17} \in S$ , analogously), then  $q_1 \in S$  or  $q_3 \in S$ . If  $S \cap \{q_2, q_{17}\} = \emptyset$ , then  $S \cap \{q_1, q_3\} \neq \emptyset$  and  $S \cap \{q_{16}, q_{18}\} \neq \emptyset$ . Let  $q_x \notin S$ . If  $q_1 \in S$  ( $q_{18} \in S$ , analogously), then there is  $u \in N_{C^*}[q_1]$  such that  $|S \cap N_{C^*}[u]| \geq 3$ . If  $S \cap \{q_1, q_{18}\} = \emptyset$ , then  $\{q_2, q_{17}\} \subset S$ , and so  $S \cap \{q_3, q_{16}\} \neq \emptyset$ . Analogously, we have  $|S \cap \{q_y, q_4, q_5, q_6, q_7, q_8, q_9\}| \geq 3$  and  $|S \cap \{q_z, q_{10}, q_{11}, q_{12}, q_{13}, q_{14}, q_{15}\}| \geq 3$ . Thus,  $|S \cap V(H_q)| \geq 9$ .  $\square$

**Claim 4.4.**  $|S| = 2m_2 + 5m_3 + 9t$ , and

- (i) For each  $v \in V_2, |S \cap V(C_v)| = 2$ . Moreover,  $|S \cap \{v_p, v_q\}| = 1$  and  $\{v_u, u_v\} \subset S$  for exactly one  $u \in N_C(v) = \{p, q\}$ .
- (ii) For each  $v \in V_3, |S \cap V(D_v)| = 5$ . Moreover,  $|S \cap \{v_p, v_q, v_r\}| = 1$  and  $\{v_u, u_v\} \subset S$  for exactly one  $u \in N_C(v) = \{p, q, r\}$ .
- (iii) For each  $q \in Q, |S \cap V(H_q)| = 9$ . Moreover,  $\{q_x, q_y, q_z, x_q, y_q, z_q\} \subset S$  or  $S \cap \{q_x, q_y, q_z, x_q, y_q, z_q\} = \emptyset$ , where  $N_C(q) = \{x, y, z\}$ .

**Proof.** Since  $|S| \leq 2m_2 + 5m_3 + 9t$ , by Claims 4.1–4.3 we have that  $|S| = 2m_2 + 5m_3 + 9t$ , and the properties (i) and (ii) hold.

(iii). Let  $q \in Q$  and  $N_C(q) = \{x, y, z\}$ , where  $x \in X, y \in Y, z \in Z$ . Since  $|S| = 2m_2 + 5m_3 + 9t$ , we have by properties (i) and (ii) that  $|S \cap V(H_q)| = 9$ . From the proof of Claim 4.3 we have

$$|S \cap \{q_x, q_1, q_2, q_3, q_{16}, q_{17}, q_{18}\}| = 3 \tag{1}$$

$$|S \cap \{q_y, q_4, q_5, q_6, q_7, q_8, q_9\}| = 3 \tag{2}$$

$$|S \cap \{q_z, q_{10}, q_{11}, q_{12}, q_{13}, q_{14}, q_{15}\}| = 3 \tag{3}$$

By properties (i) and (ii) we have that

$$\text{for each } v \in N_C(q), \text{ if } v_q \in S, \text{ then } q_v \in S. \tag{4}$$

In the following we prove that

$$\text{for each } v \in N_C(q), \text{ if } q_v \in S, \text{ then } S \cap N_{C^*}(q_v) = \{v_q\}. \tag{5}$$

Without loss of generality, let  $v = x$  (see Fig. 5). Let us assume to the contrary that  $q_x \in S$  and  $S \cap \{q_1, q_{18}\} \neq \emptyset$ . Hence by Eq. (1), we have that  $|S \cap \{q_2, q_3, q_{16}, q_{17}\}| \leq 1$ . Thus, we consider three cases: (a)  $|S \cap \{q_2, q_{17}\}| = 1$ , or (b)  $|S \cap \{q_3, q_{16}\}| = 1$ , or (c)  $|S \cap \{q_1, q_{18}\}| = 2$ .

Case (a). Let  $q_{17} \in S$  ( $q_2 \in S$ , analogously). Hence,  $q_{18} \in S$  and  $S \cap \{q_2, q_3, q_{16}\} = \emptyset$ , and so,  $\{q_4, q_5, q_{14}\} \subset S$ . If  $q_y \in S$ , then by Eq. (2) we have  $SEC(\{q_4, q_5\}) = \text{false}$ , a contradiction. Thus,  $q_y \notin S$ . Hence by Eq. (4), we have that  $y_q \notin S$ , implying that  $q_6 \in S$ . Further, by (2), we have that  $S \cap \{q_8, q_9\} = \emptyset$ , and so  $\{q_{10}, q_{11}\} \subset S$ . Since  $q_{14} \in S$ , by (3) we get  $S \cap \{q_{12}, q_{13}, q_z\} = \emptyset$ , which contradicts (4).

Case (b). Let  $q_3 \in S$  ( $q_{16} \in S$ , analogously). Hence, we have that  $q_{18} \in S$ , otherwise  $S \cap N_{C^*}[q_{17}] = \emptyset$ . So,  $\{q_4, q_{14}, q_{15}\} \subset S$ , and by (3) and (4) we get  $q_{13} \in S$ . Thus,  $\{q_8, q_9\} \subset S$ , and by Eq. (2) we get  $SEC(\{q_8, q_9\}) = \text{false}$ , a contradiction.

Case (c). Let  $\{q_1, q_{18}\} \subset S$ . Hence,  $S \cap \{q_2, q_3, q_{16}\} = \emptyset$ , and so,  $\{q_4, q_5, q_{14}\} \subset S$ . The proof goes analogously to case (a). Thus, we proved property (5).

In the following we prove that  $\{q_x, q_y, q_z\} \subset S$  or  $S \cap \{q_x, q_y, q_z\} = \emptyset$ . It suffices to prove that if  $q_x \in S$ , then  $q_y \in S$ . Let us assume to the contrary that  $q_x \in S$  and  $q_y \notin S$ . By (1) and (5), we have that  $x_q \in S, S \cap \{q_1, q_{18}\} = \emptyset$  and  $|S \cap \{q_2, q_3, q_{16}, q_{17}\}| = 2$ . If  $\{q_2, q_{17}\} \subset S$ , then by (1) we have  $SEC(\{q_2, q_{17}\}) = \text{false}$ , implying that  $|S \cap \{q_2, q_{17}\}| \leq 1$  and  $|S \cap \{q_3, q_{16}\}| \geq 1$ . If  $q_3 \notin S$ , then  $\{q_{16}, q_{17}\} \subset S$ , and so  $q_4 \in S$ . If  $q_3 \in S$ , then  $q_{17} \notin S$ , and so  $q_4 \in S$ . Analogously, we have  $q_{15} \in S$ . Thus,  $\{q_4, q_{15}\} \subset S$ . Since  $q_y \notin S$ , we have by (4) that  $y_q \notin S$ . Hence by (2), we have that  $1 \leq |S \cap \{q_6, q_7\}| \leq 2$ , and so  $\{q_4, q_5, q_6\} \subset S$ . Further,  $S \cap \{q_8, q_9\} = \emptyset$ , and so  $\{q_{10}, q_{11}\} \subset S$ . By (3), we have that  $SEC(\{q_{10}, q_{11}\}) = \text{false}$ , a contradiction.

By (4) and (5), we have that for each  $v \in N_C(q), v_q \in S \Leftrightarrow q_v \in S$ . Thus,  $\{q_x, q_y, q_z, x_q, y_q, z_q\} \subset S$  or  $S \cap \{q_x, q_y, q_z, x_q, y_q, z_q\} = \emptyset$ .  $\square$

For each  $q \in Q$ , let us define  $N_C(q) = \{x(q), y(q), z(q)\}$ , where  $x(q) \in X, y(q) \in Y, z(q) \in Z$ . Let

$$Q' = \{q \in Q : \{q_{x(q)}, q_{y(q)}, q_{z(q)}\} \subset S\}.$$

Let  $v \in V$ . By Claim 4.4 (i) and (ii) there is  $q \in N_C(v)$  such that  $\{v_q, q_v\} \subset S$ . Hence by Claim 4.4 (iii), we have that  $q \in Q'$ . Thus, set  $Q'$  dominates  $V$ , and so  $3|Q'| = \sum_{q \in Q'} |N_C(q)| \geq |N_C(Q')| = |V| = 3m$ . By Claim 4.4, we have that for every  $p, q \in Q'$  and  $p \neq q, N_C(p) \cap N_C(q) = \emptyset$ . Thus,  $|Q'| = m$ . This establishes Theorem 4.1.  $\square$



### 5. $O(n\Delta^2 \log \Delta)$ -time algorithm for trees

In [6] the authors constructed  $O(n \log \Delta)$ -time algorithm for finding the minimum global alliance in trees. In this section we present  $O(n\Delta^2 \log \Delta)$ -time algorithm for finding a minimum global edge alliance in trees.

We construct the optimal solution for a given tree  $T$  using the bottom-up technique in accordance with a defined orientation of  $T$ . First, we orient all edges of  $T$  in an *in-tree* manner with a leaf as root, i.e., we choose any leaf  $r$  as root and orient all edges of tree  $T$  towards the root  $r$ . As a result, for each vertex  $v \in V(T) \setminus \{r\}$ , there is exactly one oriented edge outcoming from a vertex  $v$  towards  $r$ , let us denote this edge by  $e_v = \{v, r_v\}$ . By  $T_v$  we denote a subtree of  $T$  rooted at  $v$  and consisting of all (oriented) edges that lead to vertex  $v$ . By  $T_v^*$  we mean the tree  $T_v$  with an attached edge  $e_v$ , i.e.,  $T_v^* = T_v \cup e_v$ . Let  $p(v) = \deg_T(v) - 1$  and let  $N_v^b = \{v_1, \dots, v_{p(v)}\}$  be the set of vertices adjacent to  $v$  and different from  $r_v$ .

The key idea of the approach is to use the recursive scheme, in which we build a data structure  $A_v$ , related to the vertex  $v$ , from data structures  $A_{v_1}, \dots, A_{v_{p(v)}}$  related to the children of vertex  $v$  (i.e.,  $N_v^b$ ). We will use some auxiliary data structure ( $B_v$ ) to clarify the process of building  $A_v$  from  $A_{v_1}, \dots, A_{v_{p(v)}}$ . It is important to ensure that one can apply the data structures associated with all children of vertex  $v$  to build  $A_v$ . The algorithm goes as follows:

1. Starting from leaves, first build  $A_v$ , and go towards root  $r$ .
2. Traversing tree  $T$  for each vertex  $v \neq r$ :
  - (i) construct an *auxiliary* data structure  $B_v$  using  $A_{v_1}, \dots, A_{v_{p(v)}}$ ,
  - (ii) construct  $A_v$  from  $B_v$ .
3. Use  $A_s$ , where  $s$  is the only neighbor of root  $r$ , to find an optimal solution.

The total time complexity of the algorithm depends on the time complexity of the construction of structures  $A_v$  and  $B_v$ . In fact, by this schema we calculate the size of the optimal solution. The construction of the optimal solution may be possible by using additional data structures for saving the appropriate information while building structures  $A_v$  and  $B_v$ , which, however, does not change the time complexity of the algorithm.

In the following, for the sake of notation simplicity, we shall use *gea* instead of *global edge alliance*, and *ea* instead of *edge alliance*. We use the symbol  $\infty$  to denote illegal cases, and assume that  $\infty \geq a$ ,  $\infty \pm a = \infty$  and  $\min\{\infty, a\} = a$ , where  $a$  is a number or  $\infty$ .

**Theorem 5.1.** *There exists  $O(n\Delta^2 \log \Delta)$  time algorithm finding the minimum global edge alliance for trees with at most  $n$  vertices and the maximum degree bounded by  $\Delta$ .*

**Proof.** Let  $v \in V(T) \setminus \{r\}$ ,  $p = \deg_T(v) - 1$  and  $q = \deg_T(r_v) - 1$ . We define a tree  $T_v^l$  obtained from  $T_v^*$  by attaching  $l \geq 0$  pendant vertices  $L_l = \{u_1, \dots, u_l\}$  to vertex  $r_v$ . Note that  $T_v^0 = T_v^*$ .

Let us define  $A_v = (a_v^{00}, a_v^{01}, a_v^{10}, A_v^{11})$ , where  $a_v^{jh}$  is an integer or  $\infty$ , for  $(j, h) \in \{(0, 0), (0, 1), (1, 0)\}$ , and  $A_v^{11}$  is a matrix of the size  $(q + 1) \times 1$ , all described as follows:

$$\begin{aligned} a_v^{00} &= \min\{|S \setminus \{r_v\}| : S \text{ is a } \textit{gea} \text{ in } T_v \wedge v \notin S \wedge r_v \notin S\}, \\ a_v^{01} &= \min\{|S \setminus \{r_v\}| : S \setminus \{r_v\} \text{ is a } \textit{gea} \text{ in } T_v \setminus \{v\} \text{ and an } \textit{ea} \text{ in } T_v \wedge \\ &\quad v \notin S \wedge r_v \in S\}, \\ a_v^{10} &= \min\{|S \setminus \{r_v\}| : S \text{ is a } \textit{gea} \text{ in } T_v^* \wedge v \in S \wedge r_v \notin S\}. \end{aligned}$$

For each  $k \in \{0, \dots, q\}$ , if  $k < q/2$ , then let

$$A_v^{11}[k] = \min\{|S \setminus \{r_v\}| : S \text{ is a } \textit{gea} \text{ in } T_v^{q-2k} \wedge \{v, r_v\} \subset S \wedge L_{q-2k} \cap S = \emptyset\},$$

and if  $k \geq q/2$ , then let

$$A_v^{11}[k] = \min\{|S \setminus (L_{2k-q} \cup \{r_v\})| : S \text{ is a } \textit{gea} \text{ in } T_v^{2k-q} \wedge \{v, r_v\} \subset S \wedge L_{2k-q} \subset S\}.$$

Let us observe that for  $a_v^{jh}$ , we have  $j = 1$  iff  $v \in S$ , and  $h = 1$  iff  $r_v \in S$ . If any  $\min(\cdot)$  cannot be legally defined, we preset the value as  $\infty$ .

In the next steps we construct  $A_v$  in accordance with the given definitions.

Let  $v$  be a leaf. Then, by definition we initially put  $a_v^{00} = \infty$ ,  $a_v^{01} = 0$ ,  $a_v^{10} = \infty$  and  $A_v^{11}[k] = 1$  for  $2k + 2 \geq q$ , and  $A_v^{11}[k] = \infty$  for  $2k + 2 < q$ .

Let  $v$  be a vertex that is not a leaf. Let us define  $B_v = (B_v^0, B_v^1)$ , where  $B_v^0$  is a matrix of the size  $p \times 3$  and  $B_v^1$  is a matrix of the size  $(p + 1) \times p \times 4$ . Let us remind that  $N_v^b = \{v_1, \dots, v_p\}$ .

For each  $i \in \{1, \dots, p\}$  we set  $B_v^0[i, 0] = i$ , and if  $a_{v_i}^{10} > a_{v_i}^{00}$ , then let  $B_v^0[i, 1] = 1$ , otherwise, let  $B_v^0[i, 1] = 0$ . Finally,  $B_v^0[i, 2] = |a_{v_i}^{10} - a_{v_i}^{00}|$ . Let us observe that if  $a_{v_i}^{10} = \infty$  or  $a_{v_i}^{00} = \infty$ , then  $B_v^0[i, 2] = \infty$ . The matrix  $B_v^0$  can be constructed in  $O(p)$  time.

For each  $k \in \{0, \dots, p\}$  and  $i \in \{1, \dots, p\}$  we set  $B_v^1[k, i, 0] = i$ , and if  $A_{v_i}^{11}[k] > a_{v_i}^{01}$ , then let  $B_v^1[k, i, 1] = 1$ , otherwise, let  $B_v^1[k, i, 1] = 0$ . We put  $B_v^1[k, i, 2] = \min\{A_{v_i}^{11}[k], a_{v_i}^{01}\}$  and finally,  $B_v^1[k, i, 3] = |A_{v_i}^{11}[k] - a_{v_i}^{01}|$ . The matrix  $B_v^1$  can be constructed in  $O(p^2)$  time.

With all the values  $B_v, a_v, b_v$  and  $c_v$  already calculated, now we can determine  $A_v$ . Let  $a_v^0 = \sum_{i=1}^p (1 - B_v^0[i, 1])$  and  $b_v^0 = \sum_{i=1}^p \min\{a_{v_i}^{10}, a_{v_i}^{00}\}$ . If for some  $i \in \{1, \dots, p\}$  we have  $a_{v_i}^{10} = a_{v_i}^{00} = \infty$ , then  $b_v^0 = \infty$ . Let  $c_v^0 = \min\{B_v^0[i, 2] : i \in \{1, \dots, p\} \wedge B_v^0[i, 1] = 1\}$ . The values  $a_v^0, b_v^0$  and  $c_v^0$  can be calculated in  $O(p)$  time.

**Claim 5.1.** The value  $a_v^{00}$  can be determined in  $O(p)$  time.

**Proof.** We have to ensure that  $v$  is dominated by at least one  $v_i$ , where  $i \in \{1, \dots, p\}$ . If  $a_v^0 > 0$ , then  $a_v^{00} = b_v^0$ , otherwise,  $a_v^{00} = b_v^0 + c_v^0$ .  $\square$

**Claim 5.2.** The value  $a_v^{01}$  can be determined in  $O(p)$  time.

**Proof.** Since  $v$  is dominated by  $r_v$ , just take the best solution:  $a_v^{01} = b_v^0$ .  $\square$

**Claim 5.3.** The value  $a_v^{10}$  can be determined in  $O(p^2 \log p)$  time.

**Proof.** We have to ensure that for each  $v_i \in S$  the edge alliance property holds for an edge  $\{v, v_i\}$ . For every  $k \in \{1, \dots, p\}$ , let us define

$$s_k = \min\{|S \setminus \{r_v\}| : S \text{ is a } gea \text{ in } T_v^* \wedge v \in S \wedge r_v \notin S \wedge |N_v^b \cap S| = k\},$$

or  $s_k = \infty$ , if there is no such  $S$ . Obviously,  $a_v^{10} = \min\{s_1, \dots, s_p\}$ .

For  $k \in \{1, \dots, p\}$  we calculate  $s_k$  or prove that there is  $l > k$ , such that  $s_l \leq s_k$ . Let  $a = \sum_{i=1}^p (1 - B_v^1[k - 1, i, 1])$ , and  $b = \sum_{i=1}^p B_v^1[k - 1, i, 2]$ . We have to ensure that exactly  $k$  edges  $\{v, v_i\}$  satisfy the edge alliance condition for  $i \in \{1, \dots, p\}$ . The rest of vertices  $v_i$  are outside the edge alliance.

If  $a > k$ , then it is easy to observe that for some  $l \geq a$  we have  $s_l \leq s_k$ . Thus, without loss of generality we can assume that  $a \leq k$ .

If  $a = k$ , then we can put  $s_k = b + 1$ .

If  $a < k$ , then we do the following: let  $\hat{B}_v$  be a matrix of the size  $p \times 4$  obtained from  $B_v^1[k - 1]$  by sorting rows  $B_v^1[k - 1, i]$  (for  $i \in \{1, \dots, p\}$ ) in a non-decreasing order with respect to the value  $B_v^1[k - 1, i, 3]$ . Thus, we get  $\hat{B}[1, 3] \leq \hat{B}[2, 3] \leq \dots \leq \hat{B}[p, 3]$ . The construction of  $\hat{B}_v$  can be done in  $O(p \log p)$  time. Let  $k_0$  be the smallest integer such that  $k - a = \sum_{i=1}^{k_0} \hat{B}_v[i, 1]$ , and let  $c = \sum_{i=1}^{k_0} \hat{B}_v[i, 3] \cdot \hat{B}_v[i, 1]$ . Hence, we put  $s_k = b + c + 1$ . Thus, we constructed  $a_v^{10}$  in  $O(p^2 \log p)$  time.  $\square$

**Claim 5.4.** The matrix  $A_v^{11}$  can be constructed in  $O(qp^2 \log p)$  time.

**Proof.**

Now, for any  $l \in \{0, \dots, q\}$  we construct  $A_v^{11}[l]$  in time  $O(p^2 \log p)$ .

The main difference between the construction of  $a_v^{10}$  and  $A_v^{11}$  is that we have to ensure that edge  $\{v, r_v\}$  satisfies the edge alliance property (i.e.,  $SEC(v, r_v) = true$ ). The proof goes analogously as for  $a_v^{10}$ . For every  $k \in \{0, \dots, p\}$  and  $l \in \{0, \dots, q\}$ ,

- if  $l < q/2$ , then let  $s_{k,l} = \min\{|S \setminus \{r_v \cup L_{q-2l}\}| : S \text{ is a } gea \text{ in } T_v^{q-2l} \wedge v \in S \wedge r_v \in S \wedge |N_v^b \cap S| = k \wedge L_{q-2l} \cap S = \emptyset\}$ ,
- if  $l \geq q/2$ , then let  $s_{k,l} = \min\{|S \setminus \{r_v \cup L_{2l-q}\}| : S \text{ is a } gea \text{ in } T_v^{2l-q} \wedge v \in S \wedge r_v \in S \wedge |N_v^b \cap S| = k \wedge L_{2l-q} \subset S\}$ ,
- and in both cases, if there is no such  $S$ , then we put  $s_{k,l} = \infty$ .

Analogously as in the construction of  $a_v^{10}$ , we have to ensure that exactly  $k$  edges  $\{v, v_i\}$  satisfy the edge alliance property for  $i \in \{1, \dots, p\}$ . Let us observe that  $SEC(v, r_v) = true$  iff  $2k + 2l + 2 \geq q + p$ . Thus, if  $2k + 2l + 2 \geq q + p$ , then we calculate the value  $s_{k,l}$  analogously as for  $a_v^{10}$ , otherwise, we put  $s_{k,l} = \infty$ . Finally,  $A_v^{11}[l] = \min\{s_{1,l}, \dots, s_{p,l}\}$ . The construction of matrix  $A_v^{11}$  can be done in  $O(qp^2 \log p)$  time.  $\square$

**Claim 5.5.** The equality  $\gamma_{ea}(T) = \min\{a_s^{10}, A_s^{11}[0] + 1\}$  holds, where  $\{s\} = N_T(r)$

**Proof.** The root  $r$  needs to be dominated, so for any global edge alliance  $S$  it is true that  $\{r, s\} \cap S \neq \emptyset$ . Thus  $a_s^{00} = \infty$ . The vertex  $s$  is the only neighbor of the root  $r$ , so a global edge alliance  $S$  such that  $r \in S$  and  $s \notin S$  cannot exist. Thus  $a_s^{01} = \infty$ .

Since  $q(s) = 0$ , therefore the matrix  $A_s^{11}$  has only one element, i.e.  $A_s^{11}[0]$ . Thus  $\gamma_{ea}(T) = \min\{a_s^{10}, A_s^{11}[0] + 1\}$ .  $\square$

To sum up, the algorithm that gives the size of the minimum global edge alliance goes as follows:

(1) For every leaf  $l$  in tree build  $A_v$  using the following values:

- (i)  $a_l^{00} = \infty, a_l^{01} = 0, a_l^{10} = \infty$ ,
- (ii)  $A_l^{11}[k] = 1$  for each  $k$  such that  $2k + 2 \geq q$ ,
- (iii)  $A_l^{11}[k] = \infty$  for each  $k$  such that  $2k + 2 < q$ .

(2) Traversing tree  $T$  towards root for each vertex  $v \neq r$  and  $v$  is not a leaf:

- (i) construct an *auxiliary* data structure  $B_v$  using  $A_{v_1}, \dots, A_{v_{p(v)}}$ ,
- (ii) applying [Claims 5.1–5.4](#) construct  $A_v$  from  $B_v$ :

(3) By [Claim 5.5](#) the value of an optimal solution is  $\min\{a_s^{10}, A_s^{11}[0] + 1\}$ , where  $s$  is the only neighbor of root  $r$ .

By [Claims 5.1–5.4](#) we can deduce that the construction of data structure  $A_v$  can be done in  $O(qp^2 \log p)$ . Therefore, the time complexity of the algorithm is  $O(n\Delta^2 \log \Delta)$ .

As mentioned before, the construction of an optimal solution may be possible in the same time complexity by using additional data structures for saving the appropriate information while building structures  $A_v$  and  $B_v$ .  $\square$

## 6. Future work and open questions

Recently, in the paper [19], the authors proved the upper bound on the edge alliance number for trees, i.e.,  $\gamma_{ea}(T) \leq 2n/3$ , and characterized the class of trees reaching this upper bound.

In the papers [7] and [8] the authors proved the upper bound for trees on the minimum total domination number, and the minimum global alliance problem, respectively. Precisely, let  $s(T)$  be the number of support vertices in a tree  $T$ . If  $T$  is a tree of order  $n(T) \geq 3$ , then  $\gamma_t(T) \leq \frac{n(T)+s(T)}{2}$  [7] and  $\gamma_a(T) \leq \frac{n(T)+s(T)}{2}$  [8]. In the paper [19] the authors proved that if  $T$  is a tree of order  $n(T) \geq 2$ , then  $\gamma_{ea}(T) \leq \frac{n(T)+s(T)}{2}$ .

The challenging problem is to give the complexity of the problem of finding the minimum global edge alliance in the class of cubic (bipartite) graphs.

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