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Clarke duality for Hamiltonian systems with nonstandard growth

S. Acinas[a,1], J. Maksymiuk[b,*], F. Mazzone[c,2]

[a] *Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, (L6300CLB) Santa Rosa, La Pampa, Argentina*
[b] *Department of Technical Physics and Applied Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland*
[c] *Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales, Universidad Nacional de Río Cuarto, (5800) Río Cuarto, Córdoba, Argentina*

Abstract

We consider the existence of periodic solutions to Hamiltonian systems with growth conditions involving G-function. We introduce the notion of symplectic G-function and provide relation for the growth of Hamiltonian in terms of certain constant C_G associated to symplectic G-function G. We discuss an optimality of this constant for some special cases. We also provide an applications to the Φ-laplacian type systems.

Keywords: periodic solutions, Orlicz spaces, Euler-Lagrange equation, critical points, Clarke duality
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1. Introduction

In this paper we consider the problem of existence of periodic solutions to the Hamiltonian system

$$J\dot{u} = -\nabla\mathcal{H}(t, u(t)) \tag{1}$$

where the *Hamiltonian* $\mathcal{H}$ is in $C^1([0,T]\times\mathbb{R}^{2n},\mathbb{R})$, $u:[0,T]\to\mathbb{R}^{2n}$ and J denotes the canonical symplectic matrix

$$J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$$

Our work is motivated by the book by J. Mawhin and M. Willem [1] and by the paper by Y. Tian and W. Ge [2]. In [1, Theorem 3.1] the authors assume a

*corresponding author
Email addresses: sonia.acinas@gmail.com (S. Acinas), jakub.maksymiuk@pg.edu.pl (J. Maksymiuk), fmazzone@exa.unrc.edu.ar (F. Mazzone)
[1] SECyT-UNRC, FCEyN-UNLPam and UNSL
[2] SECyT-UNRC, FCEyN-UNLPam and CONICET

quadratic growth condition on $\mathcal{H}$:

$$\mathcal{H}(t,u) \leq \frac{\alpha}{2}|u|^2 + \gamma(t),$$

where $\alpha \in (0, 2\pi/T)$, $\gamma \in \mathbf{L}^2$, and a coercivity condition $\lim_{u\to\infty} \frac{1}{T}\int_0^T \mathcal{H}(t,u)\,dt = \infty$. Then they obtained, using Clarke dual action method, existence of a T-periodic solution to the equation (1). This result is further applied to show existence of periodic solution to the classical Lagrangian system (see [1, Theorem 3.5]).

These results was extended in [2], where the same methods are applied to the Hamiltonians of the following form

$$\mathcal{H}(t,u) = \frac{1}{a}F(t,u_1) + \frac{a^{q-1}}{q}|u_2|^q, \ u = (u_1, u_2) \text{ and } a > 0. \tag{2}$$

The Authors also consider Lagrangian systems. In fact, solutions corresponding to this particular Hamiltonian provide solutions of the p-laplacian equation:

$$\frac{d}{dt}(|\dot{u}_1|^{p-2}\dot{u}_1) + \nabla F(t,u_1) = 0, \ \frac{1}{p} + \frac{1}{q} = 1.$$

Among other conditions, they assume that $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following growth conditions. There exists $l \in \mathbf{L}^{2\max\{q,p-1\}}([0,T],\mathbb{R}^n)$ such that

$$F(t,y) \geq \left\langle l, |y|^{\frac{p-2}{2}} y \right\rangle, \quad y \in \mathbb{R}^n, \text{ a.e. } t \in [0,T], \tag{A1}$$

and there exists $0 < a < \min\{T^{-\frac{p}{q}}, T^{-1}\}$ and $\gamma \in \mathbf{L}^{\max\{q,p-1\}}([0,T])$ such that

$$F(t,y) \leq \frac{a^2}{p}|y|^p + \gamma(t), \quad y \in \mathbb{R}^n, \text{ a.e. } t \in [0,T], \tag{A2}$$

The objective of this paper is to extend these results. Our main result, Theorem 5.1, establish existence of periodic solutions for equation (1) under assumptions that Hamiltonian satisfies an anisotropic growth conditions given by a G-function G. We will seek for solutions in anisotropic Orlicz-Sobolev space (see Section 2).

Our theorem improves the results of [2] in several directions. Using anisotropic G-functions we can consider more general growth conditions. In particular, we allow $\mathcal{H}$ to have different growth in different directions. Moreover, we do not assume that Hamiltonians have any particular structure like $H(t,u) = H_1(t,u_1) + H_2(t,u_2)$. Our theorem also improve results of [2, Theorem 2.1], when Theorem 5.1 is applied to the Hamiltonian of the form (2) it provides better result (see Remarks 5.4 and 5.5).

The method used in [1, 2] and in the present paper involves the Clarke dual action functional. It is shown that the critical points of the dual action gives solutions to the problem (1). The Clarke duality was introduced in 1978 by F. Clarke [3], and it was developed by F. Clarke and I. Ekeland in [4, 5, 6, 7], to overcome the difficulty that appear when the Hamiltonian action is indefinite. In [8], the Clarke duality was applied to prove some result on the famous Rabinowitz conjecture.

To obtain existence result, we need to prove that the dual action for a perturbed problem with associated Hamiltonian $\mathcal{H}_\varepsilon$, $\varepsilon > 0$ small enough, is differentiable and coercive. To do this, we introduce in Section 3 a notion of symplectic and semi-symplectic G-function. We show in Section 4 that if the Hamiltonian satisfies

$$G(\lambda u) - \beta(t) \leq \mathcal{H}(t,u) \leq G(\Lambda u) + \gamma(t),$$

then the associated dual action functional is differentiable on the anisotropic Orlicz-Sobolev space $\mathbf{W}^1\,\mathbf{L}^{G^\star}$, where $G^\star$ denotes the convex conjugate of G.

To show that perturbed dual action is coercive we need estimates for the quadratic form $\int_0^T \langle J\dot{u}, v\rangle\ dt$. We show in Section 3 that for semi-symplectic function G this quadratic form is bounded on Orlicz-Sobolev space $\mathbf{W}^1_T\,\mathbf{L}^G$ and that

$$\int_0^T \langle J\dot{u}, u\rangle\ dt \geq -C_1 \int_0^T G(T\dot{u})\,dt - C_2$$

on $\mathbf{W}^1_T\,\mathbf{L}^G$.

It turns out that the constant C_1 is related to the growth condition on Hamiltonian that we consider in Theorem 5.1:

$$\mathcal{H}(t,u) \leq G(\Lambda u) + \gamma(t),$$

where $\Lambda^{-1} > T\max\{1, C_G\}$ and $\gamma \in \mathbf{L}^1$. Namely, the smaller value of C_1 gives the wider class of Hamiltonians we can consider. Therefore, it is important to determine the optimal value for C_1 (we denote it by C_G). We show that this optimal value is related to certain constrained optimization problem and we obtain the optimal value for C_G in some simple cases. In Section 3 we also discuss how the constant C_G and the given bound for Λ are related to the bounds for α imposed in [1, 2].

This paper is structured as follows. In Section 2 we present the auxiliary results. We briefly recall the notion of G-function and Orlicz-Sobolev spaces. In Section 3 we introduce the concept of symplectic G-function and we study some properties. The main result about symplectic G-function is Theorem 3.5 which establishes boundedness of certain canonical quadratic functional. In Section 4, we discuss differentiability of the dual action. In Section 5, we present our main result, which establishes existence of periodic solutions for Hamiltonian system. Finally, in Section 6 we apply the previous results to the problem of existence solutions for certain second order systems.

2. Auxiliary results

In this section we collect some auxiliary results. First, we briefly recall some facts concerning convex functions. Next, we will be concerned with the notion of G-function and Orlicz spaces. We refer the reader to [9, 1] for more comprehensive information about convex functions and to [10, 11, 12, 13, 14] for more information on anisotropic G-functions and Orlicz spaces.

2.1. Convex functions

Recall that for arbitrary convex function $G\colon \mathbb{R}^n \to \mathbb{R}$ the convex conjugate of G is defined by

$$G^\star\colon \mathbb{R}^n \to (-\infty,\infty], \quad G^\star(v) = \sup_{u\in\mathbb{R}^n}\{\langle u,v\rangle - G(u)\}.$$

In general, $G^\star$ need not be finite. Assuming $\lim_{|u|\to\infty}\frac{G(u)}{|u|} = \infty$ we get $G^\star < +\infty$. Immediately from the definition of $G^\star$ we get:

- $G_1 \le G_2 \implies G_2^\star \le G_1^\star$,
- $F(u) = aG(bu) - c \implies F^\star(v) = aG^\star(v/ab) + c$, where $a, b > 0$ and $c \in \mathbb{R}$,
- *Fenchel's inequality:*
$$\langle u,v\rangle \le G(u) + G^\star(v),$$
- let $G_i\colon \mathbb{R}^{n_i} \to \mathbb{R}$, $i = 1,2$, be continuous convex functions and define $F\colon \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ by
$$F(u) = F(u_1,u_2) = G_1(u_1) + G_2(u_2)$$
then
$$F^\star(v) = F^\star(v_1,v_2) = G_1^\star(v_1) + G_2^\star(v_2),$$
where the inner product in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ is taken as the sum of inner products in components,
- if G is a differentiable convex function, then
$$G(u_1) - G(u_1 - u_2) \le \langle \nabla G(u_1), u_2\rangle \le G(u_1+u_2) - G(u_1), \tag{3}$$
- *Young's identity:* if G is a differentiable convex function, then
$$\langle \nabla G(u), u\rangle = G(u) + G^\star(\nabla G(u)). \tag{4}$$

Definition 2.1. *Be $\Gamma(\mathbb{R}^n)$ we denote the set of all differentiable, strictly convex functions $G\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$\lim_{|u|\to\infty}\frac{G(u)}{|u|} = \infty. \tag{5}$$

It is well known that if G is in $\Gamma(\mathbb{R}^n)$ then its convex conjugate is also in $\Gamma(\mathbb{R}^n)$. Moreover, in this case relation $\nabla G^\star = (\nabla G)^{-1}$ holds. The next lemma is a generalization of Proposition 2.2 from [1].

Proposition 2.2. *Let $H\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Assume that there exists a convex function $G\colon \mathbb{R}^n \to \mathbb{R}$ satisfying* (5) *and constants β, $\gamma > 0$ such that*

$$-\beta \leq H(u) \leq G(u) + \gamma, \ \textit{for all } u \in \mathbb{R}^n. \tag{6}$$

Then for any $r > 1$

$$G^\star(\nabla H(u)) \leq \frac{1}{r-1} G(ru) + \frac{r}{r-1}(\beta + \gamma). \tag{7}$$

Proof. Conjugating (6) and using (4), we obtain

$$G^\star(\nabla H(u)) - \gamma \leq H^\star(\nabla H(u)) = \langle \nabla H(u), u \rangle - H(u).$$

From Fenchel's inequality, we get

$$\langle \nabla H(u), u \rangle = \frac{1}{r} \langle \nabla H(u), ru \rangle \leq \frac{1}{r} G^\star(\nabla H(u)) + \frac{1}{r} G(ru).$$

Combining the above inequalities and (6) we obtain

$$G^\star(\nabla H(u)) \leq \frac{1}{r} G^\star(\nabla H(u)) + \frac{1}{r} G(ru) + \beta + \gamma,$$

which implies (7). □

Remark 2.3. *Inequality* (7) *for the case of the function $G(u) = |u|^q$ is slightly better than the corresponding inequality in [1, Proposition 2.2] where the estimate $|\nabla H(v)| \leq C(|u| + \beta + \gamma + 1)^{q-1}$ is obtained. Here we obtain $|\nabla H(v)| \leq C(|u|^q + \beta + \gamma)^{(q-1)/q}$. This simple fact allows us in forthcoming results to use less restrictive hypothesis on certain functions.*

2.2. G-functions and Orlicz spaces

Definition 2.4. *A function $G\colon \mathbb{R}^n \to [0, +\infty)$ is called a G-function if G is convex and satisfies $G(u) = 0 \iff u = 0$, $G(-u) = G(u)$, $\lim_{|u|\to\infty} \frac{G(u)}{|u|} = \infty$.*

It follows that the convex conjugate of a G-function is also a G-function.

Proposition 2.5. *Let G be a G-function. Then, for every $u \in \mathbb{R}^n$ we have*

$$\begin{aligned} 0 < s_1 \leq s_2 &\implies s_2 G(u/s_2) \leq s_1 G(u/s_1), \\ 0 < s_1, s_2 &\implies G(s_1 u) + G(s_2 u) \leq G((s_1 + s_2)u). \end{aligned}$$

The proof are straightforward. Immediately from the Fenchel inequality we get that for every $\mu, \nu > 0$ and every $u, v \in \mathbb{R}^n$

$$-\mu\nu\, G(u/\mu) - \mu\nu\, G^\star(v/\nu) \leq \langle u, v \rangle \leq \mu\nu\, G(u/\mu) + \mu\nu\, G^\star(v/\nu), \tag{8}$$

since $G(-u) = G(u)$.

We say that G-function G satisfies the Δ_2-*condition* (denoted $G \in \Delta_2$), if there exists a constant $C > 0$ such that for every $u \in \mathbb{R}^n$

$$G(2u) \leq CG(u) + 1. \tag{9}$$

Note that this definition is equivalent that the traditional one, i.e. that there exists $r_0, C > 0$ with $G(2u) \leq CG(u)$ for $|u| > r_0$. If there exists $C > 0$ such that $G(2u) \leq CG(u)$ for all $u \in \mathbb{R}^n$, then we say that G satisfies the Δ_2-*condition globally*.

Recall that $G_1 \prec G_2$ if there exist $K > 0$ and $C \geq 0$ such that $G_1(u) \leq G_2(Ku) + C$, for every $u \in \mathbb{R}^n$. Directly from the definition, if $G_1 \prec G_2$ then $G_2^\star \prec G_1^\star$.

Let G be a G-function. The Orlicz space $\mathbf{L}^G = \mathbf{L}^G([0,T], \mathbb{R}^n)$ is defined to be

$$\mathbf{L}^G = \left\{ u \colon [0,T] \to \mathbb{R}^n \colon \; u\text{-measurable}\;, \exists \lambda > 0 \int_0^T G(\lambda u)\, dt < \infty \right\}.$$

The space $\mathbf{L}^G$ equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \lambda > 0 \colon \; \int_0^T G(u/\lambda)\, dt \leq 1 \right\}$$

is a Banach space. Observe that

$$\|u\|_{\mathbf{L}^G} > 1 \implies \int_0^T G(u)\, dt \geq \|u\|_{\mathbf{L}^G}$$

and therefore for any $u \in \mathbf{L}^G$

$$\|u\|_{\mathbf{L}^G} \leq \int_0^T G(u)\, dt + 1. \tag{10}$$

A generalized form of Holder's inequality holds

$$\int_0^T \langle u, v \rangle \; dt \leq 2\|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^\star}}, \quad u \in \mathbf{L}^G, v \in \mathbf{L}^{G^\star}.$$

The subspace $\mathbf{E}^G = \mathbf{E}^G([0,T], \mathbb{R}^n)$ is defined to be the closure of $\mathbf{L}^\infty$ in $\mathbf{L}^G$. The equality $\mathbf{E}^G = \mathbf{L}^G$ holds if and only if $G \in \Delta_2$. It is known that $\mathbf{E}^{G^\star}$ is separable and $\mathbf{L}^G = \left(\mathbf{E}^{G^\star}\right)^\star$. Hence $\mathbf{L}^G$ can be equipped with weak$\star$ topology induced from $\mathbf{E}^{G^\star}$.

We define the anisotropic Orlicz-Sobolev space of vector valued functions $\mathbf{W}^1\,\mathbf{L}^G = \mathbf{W}^1\,\mathbf{L}^G([0,T], \mathbb{R}^n)$ by

$$\mathbf{W}^1\,\mathbf{L}^G = \{u \in \mathbf{L}^G \colon \dot{u} \text{ absolutely continuous and } \dot{u} \in \mathbf{L}^G\}.$$

The space $\mathbf{W}^1\,\mathbf{L}^G$ is a Banach space when equipped with the norm

$$\|u\|_{\mathbf{W}^1\,\mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}.$$

As usual, for a function $u \in \mathbf{L}^1([0,T],\mathbb{R}^n)$ we will write $u = \widetilde{u} + \overline{u}$, where $\overline{u} = \frac{1}{T}\int_0^T u\,dt$. One can show that

$$\|u\|'_{\mathbf{W}^1\,\mathbf{L}^G} = |\overline{u}| + \|\dot{u}\|_{\mathbf{L}^G} \tag{11}$$

is an equivalent norm to $\|\cdot\|_{\mathbf{W}^1\,\mathbf{L}^G}$ (see [10, Remark 1]). We also set

$$\mathbf{W}^1_T\,\mathbf{L}^G := \Big\{u \in \mathbf{W}^1\,\mathbf{L}^G : u(0) = u(T)\Big\}$$

and

$$\widetilde{\mathbf{W}}^1_T\,\mathbf{L}^G = \left\{v \in \mathbf{W}^1_T\,\mathbf{L}^G : \int_0^T v(t)\,dt = 0\right\}.$$

In the space $\mathbf{W}^1\,\mathbf{L}^G$ an anisotropic version of Poincaré-Wirtinger inequality holds (see [10] or [12]):

$$G(\widetilde{u}) \leq \frac{1}{T}\int_0^T G(T\dot{u})\,dt.$$

Integrating both sides, we get

$$\int_0^T G(\widetilde{u})\,dt \leq \int_0^T G(T\,\dot{u})\,dt. \tag{12}$$

We will also use the following simple lemma.

Lemma 2.6 (see [10, Corollary 2.5])**.** *If u_k is a bounded sequence in Orlicz-Sobolev space then u_k has a uniformly convergent subsequence.*

3. Symplectic G-functions

Definition 3.1. *We say that a G-function $G\colon \mathbb{R}^{2n} \to [0,\infty)$ is* symplectic *if $G^\star(Ju) = G(u)$ for all $u \in \mathbb{R}^{2n}$.*

It is obvious that if G is symplectic then $G^\star$ is also symplectic. On the other hand, if a symplectic function satisfies Δ_2-condition then its conjugate also satisfies this condition. Note that if a G-function G is differentiable and symplectic, then $G^\star$ is also differentiable and

$$\nabla G(u) = J\nabla G^\star(Ju). \tag{13}$$

Definition 3.2. *We say that a G-function $G\colon \mathbb{R}^{2n} \to [0,\infty)$ is* semi-symplectic *if $G^\star(J\cdot) \prec G$.*

Obviously, every symplectic function is semi-symplectic.

Example 3.3. *If $\Phi : \mathbb{R}^n \to [0, +\infty)$ is a G-function, then $G(u_1, u_2) = \Phi(u_1) + \Phi^\star(u_2)$ is symplectic. A typical example of such a function is $G(u_1, u_2) = |u_1|^p + |u_2|^q$, $1/p + 1/q = 1$.*

If $\Phi_1, \Phi_2 \colon \mathbb{R}^n \to [0, \infty)$ satisfy $\Phi_1^\star \prec \Phi_2$ then the function of the form

$$G(u) = \Phi_1(u_1) + \Phi_2(u_2),$$

is semi-symplectic but not necessary symplectic.

A more involved example is provided by $F(u) = G(Au)$, where G is a symplectic G-function and A is a symplectic matrix, i.e. $A^{-T}J = JA$. In order to prove the symplecticity of F, note that $F^\star(v) = G^\star(A^{-T}v)$. Consequently, $F^\star(J\cdot) = G^\star(A^{-T}J\cdot) = G^\star(JA\cdot) = G(A\cdot) = F(\cdot)$. In this way, we can produce more examples of symplectic G-functions than those given previously. For example,

$$G(u_1, u_2) = \Phi(u_1 + u_2) + \Phi^\star(u_1 + 2u_2)$$

is a symplectic G-function.

Note that the Orlicz space generated by the function $G(u_1, u_2) = \Phi(u_1) + \Phi^\star(u_2)$ is a product of Orlicz spaces $\mathbf{L}^\Phi$ and $\mathbf{L}^{\Phi^\star}$. This is exactly the case considered in [2] (see the definition of the space X therein). However, the Orlicz space corresponding to $G(u_1, u_2) = \Phi(u_1 + u_2) + \Phi^\star(u_1 + 2u_2)$ is not the product of Orlicz spaces (cf. [12, Example 3.7]).

Proposition 3.4. *If G is semi-symplectic then J induces embedding*

$$u \mapsto Ju, \quad \mathbf{L}^G([0, T], \mathbb{R}^{2n}) \hookrightarrow \mathbf{L}^{G^\star}([0, T], \mathbb{R}^{2n}).$$

Moreover, for any $K > 0$, $C \geq 0$ such that $G^\star(Ju) \leq G(Ku) + C$, for all $u \in \mathbb{R}^{2n}$, we have

$$\|Ju\|_{\mathbf{L}^{G^\star}} \leq K(CT + 1)\|u\|_{\mathbf{L}^G}.$$

Proof. Fix $K > 0$, $C \geq 0$ such that $G^\star(Ju) \leq G(Ku) + C$, for all $u \in \mathbb{R}^{2n}$. Let $u \in \mathbf{L}^G$. Then there exists $\lambda > 0$ such that $\int_0^T G(u/\lambda)\, dt < \infty$ and

$$\int_0^T G^\star\left(\frac{Ju}{K\lambda}\right) dt \leq CT + \int_0^T G(u/\lambda)\, dt < \infty.$$

So that $Ju \in \mathbf{L}^{G^\star}$. Suppose that $\|u\|_{\mathbf{L}^G} = 1$. Then $\int_0^T G(u)\, dt \leq 1$ and hence

$$\int_0^T G^\star\left(\frac{Ju}{K(CT + 1)}\right) dt \leq \frac{1}{CT + 1}\int_0^T G^\star\left(\frac{Ju}{K}\right) dt \leq 1.$$

This inequality implies that

$$\|Ju\|_{\mathbf{L}^{G^\star}} \leq K(CT + 1)$$

and the result follows. ∎

Let $G\colon \mathbb{R}^{2n} \to [0,\infty)$ be a semi-symplectic G-function. From Proposition 3.4, it follows that the bilinear form

$$\int_0^T \langle Jv, u\rangle \, dt, \tag{14}$$

is well-defined and it is bounded on $\mathbf{L}^G([0,T],\mathbb{R}^{2n}) \times \mathbf{L}^G([0,T],\mathbb{R}^{2n})$.

It is proved in [1, Proposition 3.2] that for every $u \in \mathbf{W}_T^{1,2}([0,T],\mathbb{R}^{2n})$

$$\int_0^T \langle J\dot{u}, u\rangle \, dt \geq -\frac{T}{2\pi}\int_0^T |\dot{u}|^2 \, dt \tag{15}$$

Similar estimate was obtained in [2] for $G(u_1,u_2) = |u_1|^p/p + |u_2|^q/q$, $1/p+1/q = 1$. Below we show that the analogous estimate can be obtained for Orlicz-Sobolev space induced by any semi-symplectic G-function.

Theorem 3.5. *Let G be a semi-symplectic G-function. Then there exist constants C_1, $C_2 >$ depending only on G and T such that for every function $u \in \mathbf{W}_T^1\mathbf{L}^G([0,T],\mathbb{R}^{2n})$ we have*

$$\int_0^T \langle J\dot{u}, u\rangle \, dt \geq -C_1\int_0^T G\left(T\dot{u}\right) \, dt - C_2. \tag{16}$$

Proof. Let $u \in \mathbf{W}_T^1\mathbf{L}^G([0,T],\mathbb{R}^{2n})$ and fix $K>0$, $C \geq 0$ such that $G^\star(Ju) \leq G(Ku)+C$, for all $u \in \mathbb{R}^{2n}$. By the Fenchel's inequality (8), the fact that G is a semi-symplectic and inequality (12), we obtain

$$\int_0^T \langle J\dot{u}, u\rangle \, dt = \frac{K}{T}\int_0^T \left\langle \frac{T}{K}J\dot{u}, \widetilde{u}\right\rangle \, dt \geq$$
$$\geq -\frac{K}{T}\left\{\int_0^T G^\star\left(J\frac{T\dot{u}}{K}\right) \, dt + \int_0^T G(\widetilde{u})\, dt\right\} \geq -\frac{K}{T}\left\{2\int_0^T G(T\dot{u})\, dt + C\right\}.$$

□

If G is symplectic, instead of semi-symplectic, following the same lines as the proof of Theorem 3.5, we can prove that inequality (16) is satisfied with $C_1(T) = 2/T$ and $C_2 = 0$. In addition, after the change of variable $t = Ts$, inequality (16) takes the form

$$\int_0^1 \langle J\dot{u}, u\rangle \, dt \geq -2\int_0^1 G\left(\dot{u}\right) dt.$$

The value of the constant C_1 in Theorem 3.5 imposes restrictions on the results obtained in the following sections. A smaller constant C_1 results in a more inclusive estimate for Λ in Theorem 5.1 and Proposition 5.2. Therefore, it is useful to obtain the smallest possible value for C_1. For example, in [1] it is proved that $C_1 = 1/\pi$ when $G(u) = |u|^2/2$. In this case, we can see that the optimal constant is far from 2.

Definition 3.6. *For a symplectic G-function G we define*

$$C_G(T) = -\inf\left\{\frac{\displaystyle\int_0^T \langle J\dot{u}, u\rangle\ dt}{\displaystyle\int_0^T G\left(T\dot{u}\right)\ dt}\ :\ u \in \mathbf{W}^1_T\,\mathbf{L}^G([0,T],\mathbb{R}^{2n})\right\} \tag{17}$$

The rest of this section is devoted to the problem of optimality of $C_G(T)$. We relate this problem to the constrained optimization problem and we obtain exact values in some special cases. Note that the change of variable $t = Ts$ implies that $C_G(T) = C_G(1)/T$. Therefore, from now on in this section we will assume that $T = 1$ and G is a symplectic function. For simplicity, we put $C_G := C_G(1)$.

Proposition 3.7. *The relation $C_G = C_{G^\star}$ holds for every symplectic function G.*

Proof. For $v \in \mathbf{W}^1_T\,\mathbf{L}^{G^\star}$ and $u = Jv$, we have

$$\frac{\displaystyle\int_0^1 \langle J\dot{u}, u\rangle dt}{\displaystyle\int_0^1 G(\dot{u})dt} = \frac{\displaystyle\int_0^1 \langle -\dot{v}, Jv\rangle dt}{\displaystyle\int_0^1 G(J\dot{v})dt} = \frac{\displaystyle\int_0^1 \langle J\dot{v}, v\rangle dt}{\displaystyle\int_0^1 G^\star(\dot{v})dt}$$

Using the fact that $u \mapsto Ju$ is invertible from $\mathbf{W}^1_T\,\mathbf{L}^G([0,1],\mathbb{R}^{2n})$ onto $\mathbf{W}^1_T\,\mathbf{L}^{G^\star}([0,1],\mathbb{R}^{2n})$, the statement follows. $\square$

Consider the following constrained optimization problem on $\mathbf{W}^1_T\,\mathbf{L}^G\left([0,1],\mathbb{R}^{2n}\right)$:

$$\begin{cases} \text{minimize } f(u) \\ \text{subject to } g(u) = \gamma \end{cases} \tag{P}$$

where $f, g : \mathbf{W}^1_T\,\mathbf{L}^G \to \mathbb{R}$ are given by

$$f(u) = \int_0^1 \langle J\dot{u}, u\rangle\, dt, \quad g(u) = \int_0^1 G\left(\dot{u}\right)\ dt.$$

It is obvious that f is C^1 map. Moreover, if $G^\star$ satisfies Δ_2 then g is also C^1 map. For $\gamma > 0$ set

$$A(\gamma) = \inf\left\{f(u) :\ u \in \mathbf{W}^1_T\,\mathbf{L}^G\left([0,1],\mathbb{R}^{2n}\right),\quad g(u) = \gamma\right\}.$$

With this notation we have

$$C_G = -\inf_{\gamma>0} \gamma^{-1} A(\gamma). \tag{18}$$

Lemma 3.8. *Assume that G, $G^\star \in \Delta_2$. The problem* (P) *has a solution $u_\gamma \in \mathbf{W}^1_T\,\mathbf{L}^G\left([0,1],\mathbb{R}^{2n}\right)$.*

Proof. Note that if $u(t)$ is an admissible function for the problem (P) (i.e. $u \in \mathbf{W}^1_T\,\mathbf{L}^G\left([0,1],\mathbb{R}^{2n}\right)$ and $g(u)=\gamma$) then $v(t)=u(1-t)$ is also admissible. Hence $f(u)$ and $f(v)$ have different sign and consequently $A(\gamma)<0$.

Let u_n be a minimizing sequence for (P). We can assume that $f(u_n)<0$. Since $f(u+c)=f(u)$ for every $c\in\mathbb{R}^{2n}$, we can suppose that $\overline{u}=0$. It follows that u_n is bounded.

This implies that there exists a subsequence (denoted u_n again) and $u_\gamma \in \mathbf{W}^1_T\,\mathbf{L}^G$ such that $u_u \to u_\gamma$ uniformly and $\dot{u}_u \rightharpoonup \dot{u}_\gamma$. Thus, by definition, $A(\gamma)=f(u_\gamma)$. Since $A(\gamma)<0$, we have $\dot{u}_\gamma\neq 0$ and $g(u_\gamma)>0$. Since g is weakly lsc, we have that $g(u_\gamma)\leq\gamma$.

If $g(u_\gamma)<\gamma$, then there would be a $\lambda>1$ with $g(\lambda u_\gamma)=\gamma$. But then $f(\lambda u_\gamma)=\lambda^2 f(u_\gamma)<f(u_\gamma)=A(\gamma)$ which is a contradiction. This implies that u_γ is admissible and the proof is finished.

□

Theorem 3.9. *Let G be a differentiable and strictly convex symplectic function satisfying Δ_2 condition. Then*

$$C_G = \sup \frac{1}{T_u}\frac{\displaystyle\int_0^{T_u}\langle\nabla G(u),u\rangle\,dt}{\displaystyle\int_0^{T_u}G^\star(\nabla G(u))dt}, \tag{19}$$

where the supremum is taken among all periodic solutions of the Hamiltonian system $J\dot{u}(t)=-\nabla G(u(t))$ and the constant T_u denotes a period of u.

Proof. Using Lemma 3.8, we obtain a function $u_\gamma\in\mathbf{W}^1_T\,\mathbf{L}^G\left([0,1],\mathbb{R}^{2n}\right)$ satisfying $f(u_\gamma)=A(\gamma)$. Applying the Lagrange multiplier rule, we find $\lambda\in\mathbb{R}$ such that

$$f'(u_\gamma)=2\lambda g'(u_\gamma).$$

Consequently, for any $w\in\mathbf{W}^1_T\,\mathbf{L}^G([0,1],\mathbb{R}^{2n})$ we have that

$$0=2\int_0^1\langle J\dot{u}_\gamma,w\rangle\;dt-2\lambda\int_0^1\langle\nabla G(\dot{u}_\gamma),\dot{w}\rangle\;dt.$$

Integrating by parts we get

$$0=\int_0^1\left\langle J\dot{u}_\gamma+\lambda\frac{d}{dt}\nabla G\left(\dot{u}_\gamma\right),w\right\rangle\,dt-\lambda\left\langle\nabla G\left(\dot{u}_\gamma\right),w\right\rangle\Big|_0^1 \tag{20}$$

We deduce that for every $w\in C_0^\infty([0,1],\mathbb{R}^{2n})$

$$0=\int_0^1\left\langle J\dot{u}_\gamma+\lambda\frac{d}{dt}\nabla G\left(\dot{u}_\gamma\right),w\right\rangle\,dt$$

Hence $J\dot{u}_\gamma+\lambda\frac{d}{dt}\nabla G\left(\dot{u}_\gamma\right)=0$ a.e. $t\in[0,1]$. Consequently, (20) implies that for every $w\in\mathbf{W}^1_T\,\mathbf{L}^G([0,1],\mathbb{R}^{2n})$

$$0=\left\langle\nabla G\left(\dot{u}_\gamma\right),w\right\rangle\Big|_0^1=\left\langle\nabla G\left(\dot{u}_\gamma(1)\right)-\nabla G\left(\dot{u}_\gamma(0)\right),w(0)\right\rangle.$$

Since $w(0)$ is arbitrary and ∇G is a one-to-one map, we have $\dot{u}_\gamma(1) = \dot{u}_\gamma(0)$. Hence, u_γ solves

$$\begin{cases} J\dot{u}_\gamma + \lambda \frac{d}{dt}\nabla G(\dot{u}_\gamma) = 0, \quad \text{a.e } t \in [0,1] \\ u_\gamma(0) - u_\gamma(1) = \dot{u}_\gamma(0) - \dot{u}_\gamma(1) = 0. \end{cases} \tag{21}$$

Integration by parts and (21) yields

$$A(\gamma) = f(u_\gamma) = \int_0^1 \langle J\dot{u}_\gamma, u_\gamma\rangle \, dt = -\lambda \int_0^1 \left\langle \frac{d}{dt}\nabla G(\dot{u}_\gamma), u_\gamma \right\rangle dt = \lambda \int_0^1 \langle \nabla G(\dot{u}_\gamma), \dot{u}_\gamma\rangle \, dt.$$

Since $A(\gamma) < 0$ (see proof of Lemma 3.8) and $\langle \nabla G(\dot{u}_\gamma), \dot{u}_\gamma\rangle > 0$, we get $\lambda < 0$.

Define $u(s) := J\nabla G\left(\frac{du_\gamma}{dt}|_{t=\lambda s}\right)$. Note that $u(s) = -\nabla G^\star\left(J\frac{du_\gamma}{dt}|_{t=\lambda s}\right)$ by (13). We have

$$\begin{aligned} \nabla G(u(s)) = \nabla G\left(-\nabla G^\star\left(J\frac{du_\gamma}{dt}|_{t=\lambda s}\right)\right) &= -\left. J\frac{du_\gamma}{dt}\right|_{t=\lambda s} = \\ &= \lambda\frac{d}{dt}\nabla G\left(\frac{du_\gamma}{dt}|_{t=\lambda s}\right) = -\lambda J\frac{d}{dt}u(s) = -J\frac{d}{ds}u(s) \end{aligned}$$

Hence u solves $Jdu/ds = -\nabla G(u(s))$. Since u solves an autonomous system and $u(0) = u(\lambda^{-1})$, the function $u(s)$ is defined for every $s \in \mathbb{R}$ and is T_u-periodic with $T_u = -\lambda^{-1}$.

Performing the change of variable $t = \lambda s$ we obtain

$$\begin{aligned} A(\gamma) = -\lambda^2 \int_{\lambda^{-1}}^0 \left\langle \nabla G\left(\frac{du_\gamma}{dt}|_{t=\lambda s}\right), \frac{du_\gamma}{dt}|_{t=\lambda s}\right\rangle ds = \\ = -\lambda^2 \int_{\lambda^{-1}}^0 \langle Ju(s), J\nabla G(u(s))\rangle \, ds = \\ = -\lambda^2 \int_{\lambda^{-1}}^0 \langle u(s), \nabla G(u(s))\rangle \, ds = -\frac{1}{T_u^2}\int_0^{T_u} \langle u(s), \nabla G(u(s))\rangle \, ds \end{aligned}$$

Using the fact that $\nabla G(u(s)) = -\left. J\frac{du_\gamma}{dt}\right|_{t=\lambda s}$ and that G is symplectic we obtain

$$\begin{aligned} \gamma = \int_0^1 G(\dot{u}_\gamma)dt = -\lambda \int_{\lambda^{-1}}^0 G\left(\frac{du_\gamma}{dt}|_{t=\lambda s}\right) ds = \\ = -\lambda \int_{\lambda^{-1}}^0 G(J\nabla G(u(s)))ds = \frac{1}{T_u}\int_0^{T_u} G^\star(\nabla G(u(s)))ds. \end{aligned}$$

Thus, we have just proved that for every $\gamma > 0$ there exists a T_u-periodic function u such that $\dot{u} = -\nabla G(u)$ and

$$\frac{A(\gamma)}{\gamma} = -\frac{1}{T_u}\frac{\displaystyle\int_0^{T_u} \langle u(s), \nabla G(u(s))\rangle \, ds}{\displaystyle\int_0^{T_u} G^\star(\nabla G(u(s)))ds}$$

On the other hand, let $u : \mathbb{R} \to \mathbb{R}^{2n}$ be a periodic solution of $J\dot{u}(s) = -\nabla G(u(s))$ and let T_u be a period of u. Set $u_0(t) = T_u^{-1} u(T_u t)$. Then $u_0 \in \mathbf{W}_T^1 \mathbf{L}^G\left([0,1], \mathbb{R}^{2n}\right)$ and

$$\inf_{\gamma>0} \frac{A(\gamma)}{\gamma} = -C_G \leq \frac{\displaystyle\int_0^1 \langle J\dot{u}_0, u_0\rangle \, dt}{\displaystyle\int_0^1 G(\dot{u}_0) dt} =$$

$$= \frac{1}{T_u} \frac{\displaystyle\int_0^1 \langle J\dot{u}(T_u t), u(T_u t)\rangle \, dt}{\displaystyle\int_0^1 G(\dot{u}(T_u t)) dt} = -\frac{1}{T_u} \frac{\displaystyle\int_0^{T_u} \langle \nabla G(u), u\rangle \, dt}{\displaystyle\int_0^{T_u} G^\star(\nabla G(u)) dt},$$

From this assertion we obtain the desired result. $\square$

Example 3.10. *If $G(u) = |u|^2/2$, then the equation $J\dot{u} = -\nabla G(u)$ is equivalent to the harmonic oscillator equation $\ddot{v} + v = 0$. Here, we have that $T_u = 2k\pi$ with $k \in \mathbb{N}$ and for every u. On the other hand, $\langle \nabla G(u), u\rangle = 2G^\star(\nabla G(u))$. Therefore $C_G = 1/\pi$ (cf. [1, Proposition 3.2]).*

Let us adapt to anisotropic G-functions the definition of Simonenko indices (see [15], cf. [11, 16]) :

$$p(G) = \inf_{u\neq 0} \frac{\langle u, \nabla G(u)\rangle}{G(u)}, \quad q(G) = \sup_{u\neq 0} \frac{\langle u, \nabla G(u)\rangle}{G(u)}$$

It is known that $q(G) < \infty$ if and only if G is globally Δ_2 and $p(G) > 1$ if and only if $G^\star$ is globally Δ_2 (see [17, Theorem 5.1]). Note that if we write $v = \nabla G(u)$ then

$$\frac{\langle u, \nabla G(u)\rangle}{G^\star(\nabla G(u))} = \frac{\langle v, \nabla G^\star(v)\rangle}{G^\star(v)} \leq q(G^\star).$$

On the other hand, if G is symplectic then $p(G^\star) = p(G)$ and $q(G^\star) = q(G)$. The previous reasoning proves the next result.

Corollary 3.11. *Let G be a differentiable and strictly convex symplectic function satisfying the Δ_2 condition globally, then*

$$p(G)(\inf T_u)^{-1} \leq C_G \leq q(G)(\inf T_u)^{-1},$$

where the infimum is taken among all periods of functions u which solve the Hamiltonian system $J\dot{u} = -\nabla G(u)$.

Next, we apply the previous results to some particular symplectic function G.

Theorem 3.12. *Suppose that $n = 1$ and $G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ given by $G(u_1, u_2) = |u_1|^p/p + |u_2|^q/q$, with $1 < p < \infty$ and $q = p/(p-1)$. Then*

$$C_G = \frac{p \sin\left(\frac{\pi}{p}\right)}{2\pi(p-1)^{1/p}}.$$

Proof. It is easy to see that the equation $J\dot{u} = -\nabla G(u)$ is equivalent to p-Laplacian equation

$$\frac{d}{ds}|\dot{u_1}(s)|^{p-2}\dot{u_1}(s) + |u_1(s)|^{p-2}u_1(s) = 0, \quad s \in \mathbb{R}. \tag{22}$$

It is well known that the 1-dimensional p-Laplacian equation is isochronous, i.e. all solutions are periodic with the same minimal period given by

$$T_p = 4(p-1)^{-1/q} B\left(1 + \frac{1}{q}, \frac{1}{p}\right) = \frac{4\pi(p-1)^{1/p}}{p \sin\left(\frac{\pi}{p}\right)},$$

where B denotes the Beta Function (see [18] for the proof).

If u is a solution of the equation $J\dot{u} = -\nabla G(u)$, then for every $\lambda > 0$ the function $\overline{u} = (\lambda u_1, \lambda^{p-1}u_2)$ is also a solution. This observation implies that the quotient

$$\frac{1}{T_p}\frac{\displaystyle\int_0^{T_p} \langle \nabla G(u), u\rangle \, dt}{\displaystyle\int_0^{T_p} G^\star(\nabla G(u))dt}$$

is independent of the solution. Consequently, we can take the solution of $J\dot{u} = -\nabla G(u)$ satisfying $G(u(0)) = 1$. Since p-Laplacian equation (22) has gives rise to an autonomous Hamiltonian system (with Hamiltonian function $-G$), we have that $G(u(t)) \equiv 1$ for every $t \in [0, 1]$.

Let C be the closed simple curve parametrized by $u(t) = (u_1(t), u_2(t))$ and let D be the region inside C whose area is denoted by $A(D)$. Note that C is traveled in clockwise direction. From Green's Theorem

$$\begin{aligned}\int_0^{T_p} \langle u, \nabla G(u)\rangle \, dt &= \int_0^{T_p} \langle u, -J\dot{u}\rangle \, dt = \\ &= \int_0^{T_p} \langle Ju, \dot{u}\rangle \, dt = \oint_C u_2 du_1 - u_1 du_2 = 2\iint_D dA = 2A(D).\end{aligned}$$

Using that the curve C is given implicitly by the equation $G(u(t)) = 1$ and performing the change of variable $r = 1 - s^p/p$, we have that

$$\begin{aligned}A(D) &= 4q^{1/q}\int_0^{p^{1/p}} \left(1 - \frac{s^p}{p}\right)^{1/q} ds = \\ &= 4(p-1)^{-1/q}\int_0^1 r^{1/q}(1-r)^{-1/q}dr = 4(p-1)^{-1/q}B\left(\frac{1}{q} + 1, \frac{1}{p}\right) = T_p.\end{aligned}$$

On the other hand, using Young's identity

$$\int_0^{T_p} G^\star(\nabla G(u))dt = \int_0^{T_p} \langle u, \nabla G(u)\rangle \, dt - \int_0^{T_p} G(u)dt = T_p.$$

Collecting all computations, we obtain the result of the theorem. □

Remark 3.13. *In the case $n > 1$, the vector p-Laplacian equation* (22) *was studied in several articles (see [19] for a survey on the subject). If we write $u_1 = (u_{1,1}, 0, \ldots, 0)$ being $u_{1,1} : \mathbb{R} \to \mathbb{R}$ a periodic solution of the scalar p-Laplacian equation* (22)*, we obtain a solution of the vector p-Laplacian equation. This simple observation shows that $C_G \geq p\sin(\pi/p)/2\pi(p-1)^{1/p}$. However, as it is pointed out in [19], the vector p-Laplacian equation has other periodic solutions with periods incommensurable with T_p. More precisely, the following function*

$$u_1(t) = u_0 \cos t + v_0 \sin t,$$

where $u_0, v_0 \in \mathbb{R}^n$ are fixed vectors with $\langle u_0, v_0\rangle = 0$ and $|u_0| = |v_0|$, is solution. These functions satisfy that $|u_1(t)| := a$ is constant and $T_{u_1} = 2\pi$. Recalling that $u_2 = |u_1|^{p-2}u_1$, we have

$$\frac{\displaystyle\int_0^{2\pi} \langle \nabla G(u), u\rangle \, dt}{\displaystyle\int_0^{2\pi} G^\star(\nabla G(u))\, dt} = \frac{\displaystyle\int_0^{2\pi} |u_1|^p + |u_2|^q \, dt}{\displaystyle\int_0^{2\pi} \frac{|u_1|^p}{q} + \frac{|u_2|^q}{p}\, dt} = 2.$$

Consequently $C_G \geq 1/\pi$, but it is not a new result because $p\sin(\pi/p)/2\pi(p-1)^{1/p} \geq 1/\pi$.

It is asked in [19] if the previous ones are essentially all periodic solutions of the vector p-Laplacian equations. As far as we know, this question remains an open question.

4. Differentiability of Hamiltonian dual action

In this section, we establish the differentiability of the dual action.

Theorem 4.1. *Suppose that $G\colon \mathbb{R}^{2n} \to [0, +\infty)$ is a differentiable G-function with $G^\star$ semi-symplectic. Additionally, we assume that*

1) *$\mathcal{H} : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is measurable in t, continuously differentiable with respect to u and such that $\mathcal{H}(t, \cdot) \in \Gamma(\mathbb{R}^{2n})$.*
2) *there exist $\beta, \gamma \in \mathbf{L}^1([0, T], \mathbb{R})$, $\Lambda > \lambda > 0$ such that*

$$G(\lambda u) - \beta(t) \leq \mathcal{H}(t, u) \leq G(\Lambda u) + \gamma(t) \tag{23}$$

Then, the dual action

$$\chi(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + \mathcal{H}^\star(t, \dot{v})\, dt \tag{24}$$

is Gâteaux differentiable on $\mathbf{W}^1_T\,\mathbf{L}^{G^\star}([0, T], \mathbb{R}^{2n}) \cap \{v | d(\dot{v}, \mathbf{L}^\infty) < \lambda\}$.

Moreover, if v is a critical point of χ with $d(\dot{v}, \mathbf{L}^\infty) < \lambda$, then the function defined by $u = \nabla\mathcal{H}^\star(t, \dot{v})$ belongs to $\mathbf{W}^1\,\mathbf{L}^G([0, T], \mathbb{R}^{2n})$, solves

$$\begin{cases} \dot{u} &= J\nabla\mathcal{H}(t, u) \\ u(0) &= u(T), \end{cases}$$

and the relation $\dot{u} = J\dot{v}$ holds.

Proof. First, we conjugate (23) and we obtain

$$-\gamma(t) \le G^\star\left(\frac{v}{\Lambda}\right) - \gamma(t) \le \mathcal{H}^\star(t,v) \le G^\star\left(\frac{v}{\lambda}\right) + \beta(t). \tag{25}$$

Assumption 1) guarantees that $\mathcal{H}^\star$ is continuously differentiable with respect to v. Applying Proposition 2.2 to $\mathcal{H}^\star$ and $G^\star(v/\lambda)$ instead of $\mathcal{H}$ and G, for any $r > 1$ we get

$$G\left(\lambda\nabla\mathcal{H}^\star(t,v)\right) \le \frac{1}{r-1}G^\star\left(r\frac{v}{\lambda}\right) + \frac{r}{r-1}(\beta+\gamma). \tag{26}$$

Consider the Lagrangian function $\mathcal{L} : [0,T] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$\mathcal{L}(t,v,\xi) = \frac{1}{2}\langle J\xi, v\rangle + \mathcal{H}^\star(t,\xi). \tag{27}$$

In [10, Theorem 4.5], it was proved that if there exist $\Lambda_0, \lambda_0 > 0$ and functions $a \in C(\mathbb{R}^{2n},\mathbb{R})$ and $b \in \mathbf{L}^1([0,T],\mathbb{R})$ such that

$$|\mathcal{L}| + |\nabla_v\mathcal{L}| + G\left(\frac{\nabla_\xi\mathcal{L}}{\lambda_0}\right) \le a(v)\left(b(t) + G^\star\left(\frac{\xi}{\Lambda_0}\right)\right), \tag{28}$$

then χ, which is the action functional corresponding to $\mathcal{L}$, is Gâteaux differentiable on the set $\mathbf{W}^1_T\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n}) \cap \{v | d(\dot{v},\mathbf{L}^\infty) < \Lambda_0\}$.

In order to show that an inequality like (28) holds, first we provide an estimation for $\mathcal{L}$. From (25) and since J is orthogonal, we have

$$|\mathcal{L}| \le \frac{1}{2}|\xi||v| + G^\star\left(\frac{\xi}{\lambda}\right) + \beta(t).$$

Since $\frac{G^\star(v)}{|v|} \to \infty$ as $|v| \to \infty$, there exists $C > 0$ such that $|v| \le G^\star(v) + C$ for all $v \in \mathbb{R}^{2n}$. Then,

$$|\mathcal{L}| \le \frac{1}{2}\lambda|v|\left[G^\star\left(\frac{\xi}{\lambda}\right) + C\right] + G^\star\left(\frac{\xi}{\lambda}\right) + \beta(t) \le \max\{1, \lambda|v|\}\left[G^\star\left(\frac{\xi}{\lambda}\right) + C + \beta(t)\right], \tag{29}$$

which is an estimate like the right hand side of (28).

Now, we provide an estimate for $|\nabla_v\mathcal{L}|$. Applying the same technique as above, we get

$$|\nabla_v\mathcal{L}| = \frac{1}{2}|J\xi| \le |\xi| \le \lambda\left[G^\star\left(\frac{\xi}{\lambda}\right) + C\right], \tag{30}$$

which is also an estimate of the desired type.

Finally, we deal with $G(\nabla_\xi\mathcal{L}/\lambda_0)$. Since G is a convex, even function, we have

$$G\left(\frac{\nabla_\xi\mathcal{L}}{\lambda_0}\right) = G\left(\frac{-\frac{1}{2}Jv}{\lambda_0} + \frac{\nabla\mathcal{H}^\star(t,\xi)}{\lambda_0}\right) \le \frac{1}{2}G\left(\frac{Jv}{\lambda_0}\right) + \frac{1}{2}G\left(\frac{2\nabla\mathcal{H}^\star(t,\xi)}{\lambda_0}\right).$$

Now, choosing $\lambda_0 = 2/\lambda$ and applying (26), we have

$$\begin{aligned} G\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) &\le \frac{1}{2} G\left(\frac{\lambda J v}{2}\right) + \frac{1}{2} G^\star\left(r\frac{\xi}{\lambda}\right) + \frac{1}{2}\frac{r}{r-1}(\beta+\gamma) \\ &= \frac{1}{2}\max\left\{G\left(\frac{\lambda J v}{2}\right), 1\right\}\left[G^\star\left(r\frac{\xi}{\lambda}\right) + \frac{r}{r-1}(\beta+\gamma)\right], \end{aligned} \tag{31}$$

which again is an estimate of the desired form.

Therefore, from (30),(29) and (31), we see that condition (28) holds for appropriate functions a and b and for $\Lambda_0 = \lambda/r$.

This implies differentiability of χ in a set $\mathbf{W}^1_T\mathbf{L}^{G^\star}([0,T],\mathbb{R}^N)\cap\{v|d(\dot v,\mathbf{L}^\infty) < \Lambda_0\}$. Since r is any number bigger than 1, Λ_0 is arbitrary close to λ. Thus χ is differentiable on $\mathbf{W}^1_T\mathbf{L}^{G^\star}([0,T],\mathbb{R}^N)\cap\{v|d(\dot v,\mathbf{L}^\infty) < \lambda\}$.

Let $v \in \mathbf{W}^1_T\mathbf{L}^{G^\star}([0,T],\mathbb{R}^N)\cap\{d(\dot v,\mathbf{L}^\infty) < \lambda\}$ be a critical point of χ. Then, from [10, Theorem 4.5], we obtain

$$\int_0^T \langle \nabla\mathcal{H}^\star(t,\dot v) - \frac{1}{2}Jv, \dot h\rangle dt = -\int_0^T \frac{1}{2}\langle J\dot v, h\rangle dt.$$

From Proposition 3.4 and (26) we deduce that the functions $\nabla\mathcal{H}^\star(t,\dot v) - \frac{1}{2}Jv$ and $J\dot v$ are in the space $\mathbf{L}^G$. Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^1$, from the Fundamental Lemma (see [1, Chapter 1]) we deduce that $\mathcal{H}^\star(t,\dot v) - \frac{1}{2}Jv$ is absolutely continuous. It follows that v solves $J\dot v = \nabla\mathcal{H}^\star(t,\dot v)$ and therefore by duality we obtain desired result. □

Remark 4.2. *If in addition we assume that $G^\star \in \Delta_2$ then $d(\dot v,\mathbf{L}^\infty) = 0$, since $\mathbf{L}^\infty$ is dense in $\mathbf{E}^{G^\star} = \mathbf{L}^{G^\star}$. In this case χ is continuously differentiable on the whole space $\mathbf{W}^1_T\mathbf{L}^{G^\star}$ (see [10]).*

5. Existence of periodic solutions for Hamiltonian system

The following theorem establishes the existence of minimum for the dual action functional. Our result is a generalization of [1, Theorem 3.1], where the existence was established for $G(u) = |u|^2/2$. Even for the function $|u|^2/2$ our theorem is slightly better than [1, Theorem 3.1]. We obtain existence when under assumption that the functions $\xi \in \mathbf{L}^2$ and $\alpha \in \mathbf{L}^1$ instead of $\mathbf{L}^4$ and $\mathbf{L}^2$ respectively, as it was assumed in [1]. This little improvement is due to the observation in Remark 2.3.

Theorem 5.1. *Suppose that $G\colon \mathbb{R}^{2n} \to [0,\infty)$ is a G-function such that $G \in \Gamma(\mathbb{R}^{2n})$, $G^\star$ is semi-symplectic and $G^\star \in \Delta_2$. Assume $\mathcal{H}\colon [0,T]\times\mathbb{R}^{2n} \to \mathbb{R}$ is C^1 and $\mathcal{H}(t,\cdot) \in \Gamma(\mathbb{R}^{2n})$. Additionally suppose that*

(H_1) *There exists $\xi \in \mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n})$ such that for every $u \in \mathbb{R}^{2n}$ and a.e. $t \in [0,T]$*

$$\mathcal{H}(t,u) \ge \langle \xi(t), u\rangle.$$

(H_2) *There exist* Λ *with* $\Lambda^{-1} > T\max\{1, C_{G^\star}(T)/2\}$ *and* $\alpha \in \mathbf{L}^1([0,T],\mathbb{R})$ *such that, for every* $u \in \mathbb{R}^{2n}$ *and a.e.* $t \in [0,T]$, *we have*

$$\mathcal{H}(t,u) \le G(\Lambda u) + \alpha(t).$$

(H_3)

$$\int_0^T \mathcal{H}(t,u)dt \to +\infty, \quad \textit{when } |u| \to +\infty.$$

Then, there exists $u \in \mathbf{W}_T^1\mathbf{L}^G([0,T],\mathbb{R}^{2n})$ *which is a solution of the problem*

$$\begin{cases} \dot{u} = J\nabla\mathcal{H}(t,u), & \textit{a.e. on } [0,T] \\ u(0) = u(T), \end{cases} \tag{HS}$$

and such that $v = -J\widetilde{u}$ *minimizes the dual action*

$$\chi(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + \mathcal{H}^\star(t,\dot{v})\,dt.$$

Proof. **Step 1:** Suppose that $0 < r < 1$ and $\varepsilon > 0$ are small enough to have

$$\Lambda^{-1} > (1+r)T\max\{1, C_{G^\star}(T)/2\} \quad \text{and} \quad \varepsilon < r\Lambda.$$

We define the perturbed Hamiltonian by

$$\mathcal{H}_\varepsilon(t,u) = \mathcal{H}(t,u) + G(\varepsilon u).$$

By (H_1), inequality (8) and Proposition 2.5, we have

$$\mathcal{H}_\varepsilon(t,u) \ge \langle \xi(t), u\rangle + G(\varepsilon u) \ge -G^\star\left(\frac{1}{r\varepsilon}\xi(t)\right) - G(r\varepsilon u) + G(\varepsilon u) \ge G((1-r)\varepsilon u) - \beta(t), \tag{32}$$

where, since $G^\star \in \Delta_2$, $\beta(t) := G^\star(\frac{1}{r\varepsilon}\xi(t)) \in \mathbf{L}^1$. On the other hand, Proposition 2.5 implies that

$$\mathcal{H}_\varepsilon(t,u) \le G(\Lambda u) + \alpha(t) + G(\varepsilon u) \le G((1+r)\Lambda u) + \alpha(t). \tag{33}$$

From (32), (33) and properties of Fenchel conjugate, we get

$$G^\star\left(\frac{v}{(1+r)\Lambda}\right) - \alpha(t) \le \mathcal{H}_\varepsilon^\star(t,v) \le G^\star\left(\frac{v}{(1-r)\varepsilon}\right) + \beta(t). \tag{34}$$

Define the perturbed dual action $\chi_\varepsilon\colon \mathbf{W}_T^1\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n}) \to \mathbb{R}$ by

$$\chi_\varepsilon(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + \mathcal{H}_\varepsilon^\star(t,\dot{v})\,dt. \tag{35}$$

From (34) and (16), we have

$$\chi_\varepsilon(v) \ge -\frac{C_{G^\star}(T)}{2}\int_0^T G^\star(T\dot{v})\,dt + \int_0^T G^\star\left(\frac{\dot{v}}{(1+r)\Lambda}\right)dt - \int_0^T \alpha(t)\,dt - C_2.$$

Thus, as $T(1+r)\Lambda < 1$ we obtain

$$\begin{aligned}\chi_\varepsilon(v) &\geq -\frac{C_{G^\star}(T)}{2}\int_0^T G^\star(T\dot{v})\,dt + \frac{1}{T(1+r)\Lambda}\int_0^T G^\star\left(T\dot{v}\right)\,dt - \int_0^T \alpha(t)\,dt - C_2 \\ &> \left(\frac{1}{T(1+r)\Lambda} - \frac{C_{G^\star}(T)}{2}\right)\int_0^T G^\star(T\dot{v})\,dt - \int_0^T \alpha(t)\,dt - C_2 \\ &=: C_\chi \int_0^T G^\star(T\dot{v})\,dt - B_\chi.\end{aligned} \tag{36}$$

By the definition of Λ and our choice of r we have that $C_\chi > 0$. Since $\chi_\varepsilon(v) = \chi_\varepsilon(v+c)$ with $c \in \mathbb{R}^{2n}$, it is sufficient to minimize χ_ε on $\widetilde{\mathbf{W}}^1_T\,\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n})$.

The perturbed dual action is coercive on this space. To see this let $\{v_n\} \subset \widetilde{\mathbf{W}}^1_T\,\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n})$ and suppose that $\|v_n\|_{\mathbf{W}^1\,\mathbf{L}^{G^\star}} \to \infty$. Then $\|\dot{v}_n\|_{\mathbf{L}^{G^\star}} \to \infty$ or $|\overline{v}_n| \to \infty$. Since $\overline{v}_n = 0$, $\|\dot{v}_n\|_{\mathbf{L}^{G^\star}} \to \infty$. Hence from (10) we obtain that $\int_0^T G^\star(T\dot{v}_n)\,dt \to \infty$ and consequently $\chi_\varepsilon(\dot{v}_n) \to \infty$, by (36).

It follows that if $\{v_n\} \subset \widetilde{\mathbf{W}}^1_T\,\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n})$ is a minimizing sequence for χ_ε then $\dot{v}_n$ is a bounded sequence in $\mathbf{L}^{G^\star} = \left(\mathbf{E}^G\right)^\star$. Following a standard argument (see [10, Theorem 3.2]), we obtain a function $v_\varepsilon \in \widetilde{\mathbf{W}}^1_T\,\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n})$ which is a minimum of χ_ε.

As $G^\star \in \Delta_2$ then $\mathbf{L}^\infty$ is dense in $\mathbf{L}^{G^\star}$ and consequently $d(\dot{v}_\varepsilon, \mathbf{L}^\infty) = 0$. Theorem 4.1 implies that

$$u_\varepsilon(t) = \nabla\mathcal{H}^\star_\varepsilon(t,\dot{v}_\varepsilon) \in \mathbf{W}^1_T\,\mathbf{L}^G([0,T],\mathbb{R}^{2n})$$

is a solution to

$$\begin{cases}\dot{u} = J\nabla\mathcal{H}_\varepsilon(t,z) = \varepsilon J\nabla G\left(\varepsilon u\right) + J\nabla\mathcal{H}(t,u) \\ u(0) = u(T)\end{cases} \tag{37}$$

and the relation $\dot{u}_\varepsilon = J\dot{v}_\varepsilon$ holds.

Step 2: Now, we provide a posteriori estimates on $u_\varepsilon = \nabla\mathcal{H}^\star_\varepsilon(t,v_\varepsilon)$. It is easy to verify (see [1, page 47]) that there exists $\overline{u} \in \mathbb{R}^{2n}$ such that

$$\int_0^T \nabla\mathcal{H}(t,\overline{u})\,dt = 0.$$

We define

$$w(t) = \int_0^t \nabla\mathcal{H}(s,\overline{u})\,ds + c,$$

where c is chosen in order to $\int_0^T w\,dt = 0$. The function w is absolutely continuous, we show that $w \in \mathbf{W}^1\,\mathbf{L}^G$. From (H_1), (H_2) and inequality (8), it follows that for any $t \in [0,T]$ and $u \in \mathbb{R}^{2n}$

$$-G^\star\left(\frac{\xi(t)}{\Lambda}\right) \leq \mathcal{H}(t,u) + G(\Lambda u) \leq 2G(\Lambda u) + \alpha(t).$$

Therefore, the function $\mathcal{H}(t,u)+G(\Lambda u)$ and the G-function $2G(\Lambda u)$ satisfy hypothesis of Proposition 2.2. Consequently, taking $r=2$

$$G^\star\left(\frac{\nabla\mathcal{H}(t,u)+\Lambda\nabla G(\Lambda u)}{2\Lambda}\right)\le G(2\Lambda u)+2G^\star\left(\frac{\xi(t)}{\Lambda}\right)+2\alpha(t).$$

This inequality and the fact that $\dot{w}=\nabla\mathcal{H}(t,\overline{u})$ imply that

$$\begin{aligned}G^\star\left(\frac{\dot{w}}{4\Lambda}\right)&=G^\star\left(\frac{\nabla\mathcal{H}(t,\overline{u})}{4\Lambda}\right)\\ &\le\frac{1}{2}G^\star\left(\frac{\nabla\mathcal{H}(t,\overline{u})+\Lambda\nabla G(\Lambda\overline{u})}{2\Lambda}\right)+\frac{1}{2}G^\star\left(\nabla G\left(\frac{\Lambda\overline{u}}{2}\right)\right)\\ &\le 2G(2\Lambda\overline{u})+2G^\star\left(\frac{\xi(t)}{\Lambda}\right)+2\alpha(t)\in\mathbf{L}^1\,.\end{aligned}\tag{38}$$

Thus $\dot{w}\in\mathbf{L}^{G^\star}$. Moreover, $\mathcal{H}^\star(t,\dot{w})=\langle\dot{w},\overline{u}\rangle-\mathcal{H}(t,\overline{u})$ so that $\mathcal{H}^\star(\cdot,\dot{w}(\cdot))\in\mathbf{L}^1([0,T],\mathbb{R})$.

From inequality $\mathcal{H}(t,u)\le\mathcal{H}_\varepsilon(t,u)$, we deduce that $\mathcal{H}^\star_\varepsilon(t,v)\le\mathcal{H}^\star(t,v)$. By inequality (36) and (38), we have

$$C_\chi\int_0^T G^\star(T\dot{v}_\varepsilon)\,dt-B_\chi\le\chi_\varepsilon(v_\varepsilon)\le\chi_\varepsilon(w)\le\int_0^T\frac{1}{2}\langle J\dot{w},w\rangle+\mathcal{H}^\star(t,\dot{w})\,dt=:c_1<\infty.$$

Since $G^\star$ is semi-symplectic, there exist $C,k>0$ with $G(Ju)\le G^\star(ku)+C$. Moreover, since $\dot{u}_\varepsilon=J\dot{v}_\varepsilon$, we have

$$\int_0^T G\left(\frac{T}{k}\dot{u}_\varepsilon\right)\,dt=\int_0^T G\left(\frac{T}{k}J\dot{v}_\varepsilon\right)\,dt\le C+\int_0^T G^\star\left(T\dot{v}_\varepsilon\right)\,dt\le c_2,$$

and

$$Jv_\varepsilon=u_\varepsilon-\overline{u}_\varepsilon.\tag{39}$$

It follows from (10) that $\dot{u}_\varepsilon$ is uniformly bounded in $\mathbf{L}^G$. Now, from inequality (12) we deduce that $\widetilde{u}_\varepsilon$ is uniformly bounded in $\mathbf{L}^\infty$. Therefore, there exists c_3 such that

$$\int_0^T G\left(\Lambda\widetilde{u}_\varepsilon\right)dt\le c_3.$$

Thus, using (39) and Theorem 3.5, we have

$$\int_0^T\langle J\dot{u}_\varepsilon,u_\varepsilon\rangle\,dt=\int_0^T\langle-\dot{v}_\varepsilon,Jv_\varepsilon+\overline{u}_\varepsilon\rangle\,dt\ge-C_{G^\star}\int_0^T G^\star\left(T\dot{v}_\varepsilon\right)-C_1\ge-c_4.\tag{40}$$

The convexity of $\mathcal{H}(t,\cdot)$, inequality (3), (H_2), (37) and the fact that $\langle u,\nabla G(u)\rangle\ge 0$ for any $u\in\mathbb{R}^{2n}$, imply

$$\begin{aligned}2\mathcal{H}\left(t,\frac{\overline{u}_\varepsilon(t)}{2}\right)&\le\mathcal{H}(t,u_\varepsilon)+\mathcal{H}(t,-\widetilde{u}_\varepsilon)\\ &\le\langle\nabla\mathcal{H}(t,u_\varepsilon(t)),u_\varepsilon\rangle+\mathcal{H}(t,0)+G(\Lambda\widetilde{u}_\varepsilon)+\alpha(t)\\ &=\langle-J\dot{u}_\varepsilon-\varepsilon\nabla G(\varepsilon u_\varepsilon),u_\varepsilon\rangle+\mathcal{H}(t,0)+G(\Lambda\widetilde{u}_\varepsilon)+\alpha(t)\\ &\le\langle-J\dot{u}_\varepsilon,u_\varepsilon\rangle+\mathcal{H}(t,0)+G(\Lambda\widetilde{u}_\varepsilon)+\alpha(t).\end{aligned}$$

Integrating the previous inequality and using (40), it follows that

$$\int_0^T \mathcal{H}\left(t, \frac{\overline{u}_\varepsilon}{2}\right) dt \leq c_5.$$

Now, by (H_3) we have that $\overline{u}_\varepsilon$ is uniformly bounded. Thus, we have that u_ε is uniformly bounded in $\mathbf{W}^1\,\mathbf{L}^G([0,T],\mathbb{R}^{2n})$.

Step 3: By a standard argument (see [10]), we can suppose that there exists a sequence ε_n such that $u_n := u_{\varepsilon_n}$ converges uniformly to a continuous function $u \in \mathbf{W}^1\,\mathbf{L}^G([0,T],\mathbb{R}^N)$ and that $\dot{u}_n$ converges to $\dot{u}$ in the weak* topology of $\mathbf{L}^G([0,T],\mathbb{R}^N)$. From (37) in integrated form

$$Ju_n(t) - Ju_n(0) = -\int_0^t \varepsilon_n \nabla G(\varepsilon_n u_n) + \nabla \mathcal{H}(t, u_n)\, dt,$$

we deduce u is a solution of the original problem.

It remains to prove that v minimizes the dual action integral. Since $\dot{v}_n = \nabla\mathcal{H}(t,u_n)$, we have

$$\begin{aligned}
\chi_{\varepsilon_n}(v_{\varepsilon_n}) &= \int_0^T \left[\frac{1}{2}\langle J\dot{v}_n, v_n\rangle + \langle u_n, \dot{v}_n\rangle - \mathcal{H}_{\varepsilon_n}(t,u_n)\right] \\
&= \int_0^T \left[\frac{1}{2}\langle J\dot{v}_n, v_n\rangle + \langle u_n, \dot{v}_n\rangle - \mathcal{H}(t,u_n) - G(\varepsilon_n u_n)\right] dt.
\end{aligned}$$

Taking into account that $\dot{v}_n \overset{\star}{\rightharpoonup} \dot{v}$ in $\mathbf{L}^{G^\star}$ and $u_n \to u$ uniformly, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \chi_{\varepsilon_n}(v_{\varepsilon_n}) &= \lim_{n\to\infty} \int_0^T \left[\frac{1}{2}\langle J\dot{v}_n, v_n\rangle + \langle u_n, \dot{v}_n\rangle - \mathcal{H}(t,u_n) - G(\varepsilon_n u_n)\right] dt \\
&= \int_0^T \left[\frac{1}{2}\langle J\dot{v}, v\rangle + \langle u, \dot{v}\rangle - \mathcal{H}(t,u)\right] dt.
\end{aligned}$$

Now, (39) implies that $v = -J(u - \overline{u})$. Thus, using (HS) we get $\dot{v} = -J\dot{u} = \nabla\mathcal{H}(t,u)$. Consequently,

$$\lim_{n\to\infty} \chi_{\varepsilon_n}(v_{\varepsilon_n}) = \int_0^T \left[\frac{1}{2}\langle J\dot{v}, v\rangle + \mathcal{H}^\star(t,\dot{v})\right] dt = \chi(v).$$

On the other hand, from $\mathcal{H}_\varepsilon^\star \leq \mathcal{H}^\star$ we have that for any $w \in \mathbf{W}_T^1\,\mathbf{L}^{G^\star}([0,T],\mathbb{R}^{2n})$, $\chi_{\varepsilon_n}(v_{\varepsilon_n}) \leq \chi_{\varepsilon_n}(w) \leq \chi(w)$. Therefore, v is a minimum of χ. □

In the case where $G(u) = |u|^2/2$, in [1, Theorem 3.1] it is assumed that constant $\Lambda < \sqrt{2\pi/T}$. Meanwhile, in (H_2) we are assuming that $\Lambda < \min\{1/T, 2\pi\}$, i.e. when $G(u) = |u|^2/2$ our constant Λ is not as good as constant in [1, Theorem 3.1]. Assuming additional hypothesis on the G-function G, we are able to obtain better estimates for the constant Λ.

First, we recall some definitions from [20, Chapter 11]. In that monograph, it were considered a G-function such that $G\colon \mathbb{R} \to [0,+\infty)$. However, all definitions and results remains true in the anisotropic setting.

We denote by α_G and β_G the so-called *Matuszewska-Orlicz indices* of the function G, which are defined by

$$\alpha_G := \lim_{t\to 0^+} \frac{\log\left(\sup\limits_{u\neq 0} \frac{G(tu)}{G(u)}\right)}{\log(t)}, \quad \beta_G := \lim_{t\to +\infty} \frac{\log\left(\sup\limits_{u\neq 0} \frac{G(tu)}{G(u)}\right)}{\log(t)}. \tag{41}$$

We have that $0 \leq \alpha_G \leq \beta_G \leq +\infty$. The relation $\beta_G < \infty$ holds if and only if G satisfies the Δ_2-condition globally. On the other hand, $\alpha_G > 1$ if and only if $G^\star$ satisfies the Δ_2-condition globally.

In the case that G and $G^\star$ satisfy the Δ_2-condition globally, for every $\epsilon > 0$ there exists a constant $K = K(G, \epsilon)$ such that, for every $t, u \geq 0$,

$$K_{G,\varepsilon}^{-1} \min\left\{t^{\beta_G+\epsilon}, t^{\alpha_G-\epsilon}\right\} G(u) \leq G(tu) \leq K_{G,\varepsilon} \max\left\{t^{\beta_G+\epsilon}, t^{\alpha_G-\epsilon}\right\} G(u). \tag{42}$$

Proposition 5.2. *The conclusions of Theorem 5.1 continue to be true if we suppose that G and $G^\star$ satisfy the Δ_2-condition globally and instead of inequality $\Lambda^{-1} > T\max\{1, C_{G^\star}(T)/2\}$ in* (H_2)*, we assume that*

$$K_{G,\varepsilon}^{-1} \min\left\{(T\Lambda)^{-\beta_G-\epsilon}, (T\Lambda)^{-\alpha_G+\epsilon}\right\} \geq \frac{C_{G^\star}(T)}{2}, \tag{43}$$

where the constant $K_{G,\varepsilon}, \alpha_G, \beta_G$ satisfy (42).

Proof. The only change that must be made in the proof of Theorem 5.1 is choosing $0 < r < 1$ such that

$$K_{G,\varepsilon}^{-1} \min\left\{[(1+r)T\Lambda]^{-\beta_G-\epsilon}, [(1+r)T\Lambda]^{-\alpha_G+\epsilon}\right\} \geq \frac{C_{G^\star}(T)}{2}.$$

Now, we use (43) to produce the next inequality

$$\begin{aligned} &\int_0^T G^\star\left(\frac{\dot{v}}{(1+r)\Lambda}\right) dt \geq \\ &\qquad \geq K_{G,\varepsilon}^{-1} \min\left\{[(1+r)T\Lambda]^{-\beta_G-\epsilon}, [(1+r)T\Lambda]^{-\alpha_G+\epsilon}\right\} \int_0^T G^\star\left(T\dot{v}\right) dt. \end{aligned}$$

From here, the proof continues like in Theorem 5.1. □

Remark 5.3. *If $G_2(u) = |u|^2/2$ then inequality* (42) *holds with $\epsilon = 0$, $K_{G_2,\varepsilon} = 1$, $\alpha_{G_2} = \beta_{G_2} = 2$. Since $G_2^\star = G_2$, from* (15) *we have that $C_{G_2^\star} = 1/T\pi$. Thus inequality* (43) *is equivalent to $\Lambda \leq \sqrt{2\pi/T}$, which is the same constant as in [1, Theorem 3.1]. On the other hand, in Theorem 5.1 we assume that $\xi \in \mathbf{L}^2$ and $\alpha \in \mathbf{L}^1$. Meanwhile in order to apply [1, Theorem 3.1] we need $\xi \in \mathbf{L}^4$ and $\alpha \in \mathbf{L}^2$.*

Remark 5.4. *Let us discuss the relation between Proposition 5.2 and the result obtained in [2]. Recall that they consider Hamiltonian $\mathcal{H}$ given by* (2) *and satisfying* (A1) *and* (A2).

If $\xi(t) = (l_1(t), l_2(t))$ is a function satisfying

$$\langle \xi(t), u\rangle \leq \frac{1}{a}F(t,u_1) + \frac{a^{q-1}}{q}|u_2|^q,$$

for any $u \in \mathbb{R}^{2n}$, then taking $u = (0, u_2)$ we have that $\langle l_2(t), u_2\rangle \leq \frac{a^{q-1}}{q}|u_2|^q$ and this inequality is true only for $l_2 \equiv 0$. Consequently (H_1) implies

$$\langle l_1(t), u_1\rangle \leq \frac{1}{a}F(t,u_1),$$

and $l_1 \in \mathbf{L}^q$ (recall that $G_p^\star(l_1, l_2) = |l_1|^q/q + |l_2|^p/p$). Therefore our condition (H_1) differs slightly from (A1).

The condition (A2) *for $\mathcal{H}$ implies*

$$\mathcal{H}(t,u_1,u_2) \leq \frac{a}{p}|u_1|^p + \frac{a^{q-1}}{q}|u_2|^q + \frac{\gamma(t)}{a} = G_p(\Lambda u_1, \Lambda u_2) + \alpha(t),$$

where $G_p(u_1,u_2) = |u_1|^p/p + |u_2|^q/q$, $\Lambda = a^{1/p}$ and $\alpha(t) = \gamma(t)/a$. The inequality $0 < a < \min\{T^{-\frac{p}{q}}, T^{-1}\}$ shows that condition (A2) *in [2] implies*

$$\Lambda < \min\{T^{-\frac{1}{q}}, T^{-\frac{1}{p}}\}.$$

On the other hand, it is easy to see that inequality (42) *holds with $K_{G_p,\varepsilon} = 1$, $\alpha_G = \min\{p,q\}$, $\beta_G = \max\{p,q\}$ and $\varepsilon = 0$. Therefore, the fact that $C_{G^\star}(T) = C_{G^\star}(1)/T = C_{G^\star}/T$ implies that Λ satisfies inequality* (43) *if and only if*

$$\Lambda < \min\left\{\left(\frac{2}{C_{G^\star}}\right)^{1/p} T^{-1/q}, \left(\frac{2}{C_{G^\star}}\right)^{1/q} T^{-1/p}\right\}.$$

Recalling that for G symplectic we have $C_{G^\star} \leq 2$, we obtain that the condition (A2) *implies our condition (H_2). We suspect that the estimate $C_{G^\star} \leq 2$ is not the best possible (it is evident when $n = 1$).*

Remark 5.5. *To finish this section let us give a condition that contains conditions* (A1) *and (H_1) as particular cases. Concretely, Theorem 5.1 remains true if we replace (H_1) by the following condition.*

$(H1')$ *There exist $b \in \mathbf{L}^1([0,T],\mathbb{R})$, a G-function $G_0 : \mathbb{R}^{2n} \to [0,\infty)$, $\xi \in \mathbf{E}^{G_0^\star}$ and a map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

$$H(t,u) \geq \langle \xi(t), f(u)\rangle + b(t), \quad u \in \mathbb{R}^{2n}, t \in [0,T]$$

and f satisfies that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$G_0(\delta f(u)) \leq G(\epsilon u), \quad u \in \mathbb{R}^{2n}$$

Note that condition (A1) *for Hamiltonian* (2) *is obtained choosing $f(u_1,u_2) = (|u_1|^{\frac{p-2}{2}}u_1, 0)$, $\xi(t) = (l(t), 0)$, $b \equiv 0$ and $G_0(u) = |u|^2$ in condition $(H1')$ and finally taking $u_2 = 0$. Hence we obtain existence when $l \in \mathbf{L}^2$, while in [2] it is assumed $l \in \mathbf{L}^{2\max\{p,q-1\}}$.*

6. Application to the existence of solutions of second order systems

The purpose of this section is to apply the previous results to get existence of solutions of the second-order system

$$\begin{cases} \frac{d}{dt}\nabla\Phi(\dot{q}) + \nabla V(t,q) = 0 \quad \text{for a.e. } t\in[0,T] \\ q(0)=q(T), \quad \dot{q}(0)=\dot{q}(T), \end{cases} \tag{EL}$$

where $\Phi\colon \mathbb{R}^N \to \mathbb{R}$ is a G-function in $\Gamma(\mathbb{R}^N)$ such that Φ and $\Phi^\star$ satisfy Δ_2 condition and $V\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}$, $(t,q)\mapsto V(t,q)$ is a Carathéodory function continuously differentiable and convex in q.

Theorem 6.1. *Assume that the following conditions are satisfied:*

(V_1) there exists $l\in \mathbf{L}^\Phi([0,T],\mathbb{R}^N)$ such that for all $q\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, such that

$$\langle l(t), q\rangle \le V(t,q);$$

(V_2) for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$ one has

$$V(t,q)\le \Phi\left(\Lambda^2 q\right)+\gamma(t);$$

where $\Lambda^{-1} > T\max\{1, C_G/2\}$.

(V_3)

$$\lim_{x\to\infty}\int_0^T V(t,x)\,dt=\infty.$$

where $C_G = C_G(T)$ denotes constant corresponding to the G-function $G(q,p) = \Phi(q)+\Phi^\star(p)$. Then the problem (EL) *has at least one solution.*

Our theorem is a generalization of the classical result [1, Theorem 3.5] where the authors proved that under a quadratic growth condition on V, there exists a periodic solution to the problem $\ddot{u}=\nabla V(t,u)$. This result was further extended by Tian and Ge (see [2, Theorem 2.1]) to p-Laplacian setting. They assumed that V has a p-power growth.

Proof. System (EL) is a system of Lagrange equations for the Lagrangian function $L(t,q,p)=\Phi(p)-V(t,q)$. Alternatively, we can use the Lagrangian function $L(t,q,p)=\Phi(p/\Lambda)-V(t,q/\Lambda)$. Clearly, periodic solutions of one system correspond to periodic solutions of the other one. The associated Hamiltonian $\mathcal{H}\colon [0,T]\times\mathbb{R}^{2n}\to\mathbb{R}$ is given by

$$\mathcal{H}(t,z)=\Phi^\star(\Lambda z_2)+V\left(t,\frac{z_1}{\Lambda}\right),$$

where $z=(z_1,z_2)$.

For a.e. $t\in[0,T]$, the function $\mathcal{H}(t,\cdot)$ is convex and C^1. For every $z\in\mathbb{R}^{2n}$ and a.e. $t\in[0,T]$,

$$\mathcal{H}(t,z)\ge \Phi^\star(\Lambda z_2)+\frac{1}{\Lambda}\,\langle l(t),z_1\rangle_{\mathbb{R}^N}\ge\frac{1}{\Lambda}\,\langle (l(t),0),z\rangle_{\mathbb{R}^{2n}}$$

and

$$\mathcal{H}(t,z) \le \Phi^\star(\Lambda z_2) + \Phi(\Lambda z_1) + \gamma(t) = G(\Lambda z) + \gamma(t).$$

Moreover,

$$\int_0^T \mathcal{H}(t,z)\,dt = \Phi^\star(\Lambda z_2)\,T + \int_0^T V\left(t, \frac{z_1}{\Lambda}\right)\,dt \to \infty, \quad z \to \infty$$

Hamiltonian $\mathcal{H}$ satisfies assumptions of Theorem 5.1. Hence, the corresponding Hamiltonian system with periodic boundary conditions has a solution $z \in \mathbf{W}^1_T\,\mathbf{L}^G([0,T],\mathbb{R}^{2n})$. Consequently, $u = z_1/\Lambda$ is a solution of (EL). Since $\dot{u} = \nabla\Phi^\star(z_2/\alpha)$ and $z_2 \in \mathbf{L}^{\Phi^\star}([0,T],\mathbb{R}^N)$, then $u \in \mathbf{W}^1_T\,\mathbf{L}^\Phi([0,T],\mathbb{R}^N)$. This finishes the proof.

□

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