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Sensitivity Analysis of Flexural and Torsional Buckling Loads of Laminated Columns

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Abstract. The paper concerns first order sensitivity analysis of flexural and torsional buckling loads of axially compressed thin-walled columns with bisymmetric or axisymmetric cross-section made of unidirectional fibre-reinforced laminate. The first variation of critical loads versus some variations of the column material properties and cross-sectional dimensions is derived. Numerical examples dealing with simply supported I-columns are presented. The distributions of sensitivity functions of critical loads are presented with respect to variations of the parameters assumed along the column axis are shown. Some differences between sensitivity functions of both kinds of buckling are described and accuracy of sensitivity analysis in assessment of the critical buckling load changes due to some variations of the column parameters is discussed.

INTRODUCTION

Application of the unidirectional fibre-reinforced laminate structures in engineering increases due to their economical and mechanical advantages [1-3]. In structural design, including its optimization, it is necessary to provide re-analysis in order to achieve desired behaviour requirements. Instead of repetitive re-analysis of structures with updated parameters it is better to apply sensitivity analysis to find the relevant location of parameter changes and to establish its necessary variations leading to better designed structural properties [4, 5]. The paper presents application of the sensitivity analysis of the first order to flexural and torsional buckling loads of columns made of a unidirectional fibre-reinforced laminate [6]. Two approaches to column material modelling are applied: homogenization based on the Voigt-Reuss hypothesis and homogenization based on the theory of mixtures and periodicity cells [7]. The first variations of critical loads with respect to variations of the column material properties and the column cross-section dimensions are derived by means of variational calculus [8]. It is assumed that both kinds of buckling are independent, this occurs for bisymmetric or axisymmetric cross-sections. Distribution of the sensitivity functions along the column axis versus the investigated parameter may be stated, so approximation is made of the buckling load change due to point variation of the column parameters along the column axis.

Numerical examples of simply supported I-columns are given. The fibre volume fraction and the flange width are assumed to be the column parameters under consideration. Accuracy sensitivity analysis is discussed while assessing the effect of the parameter variation on the critical loads in comparison with results of the stability reanalysis of the column with updated parameters.

This paper continues our previous research described in [9-12].

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BUCKLING SENSITIVITY ANALYSIS OF COLUMN

Material Description

Let us consider an axially loaded thin-walled column of bisymmetric (or axisymmetric) cross-section made of unidirectional fibre-reinforced laminate presented in Fig. 1. Two models of laminate material behaviour after homogenization procedure are considered here, as follows:

- Model A - based on the theory of mixture and periodicity cell, the relations read

$$
\begin{aligned}
E_l &= E_m\left(1-f\right)+E_f f\\
E_t &= E_m\,\frac{E_m\left(1-\sqrt{f}\right)+E_f\sqrt{f}}{E_m\left[1-\sqrt{f}\left(1-\sqrt{f}\right)\right]+E_f\sqrt{f}\left(1-\sqrt{f}\right)}\\
G &= G_m\,\frac{G_m\sqrt{f}\left(1-\sqrt{f}\right)+G_f\left[1-\sqrt{f}\left(1-\sqrt{f}\right)\right]}{G_m\sqrt{f}+G_f\left(1-\sqrt{f}\right)}\\
\nu &= \nu_m\left(1-\sqrt{f}\right)+\nu_f\sqrt{f}
\end{aligned}
\tag{1}
$$

where E_l, E_t - homogenized Young's moduli in longitudinal and transverse direction, G - homogenized shear modulus, ν - Poisson's ratio in longitudinal direction, E_m, E_f, G_m, G_f and ν_m, ν_f - Young's moduli, shear moduli and Poisson's ratios of matrix and fibre, and f - fibre volume fraction.

- Model B - based upon the Voigt-Reuss hypothesis, in which material moduli are

$$
\begin{aligned}
E_l &= E_m\left(1-f\right)+E_f f\\
E_t &= \frac{E_f E_m}{E_f-E_f f+E_m f}\\
G &= \frac{G_f G_m}{G_f-G_f f+G_m f}\\
\nu &= \nu_m\left(1-f\right)+f\nu_f
\end{aligned}
\tag{2}
$$

Taking into account the plane stress field the Young's the modified elastic moduli in longitudinal D_l and transverse directions D_t are assumed in the form

$$
D_l=\frac{E_l}{1-\frac{E_t}{E_l}\nu^2},\qquad D_t=\frac{E_t}{1-\frac{E_t}{E_l}\nu^2}
\tag{3}
$$

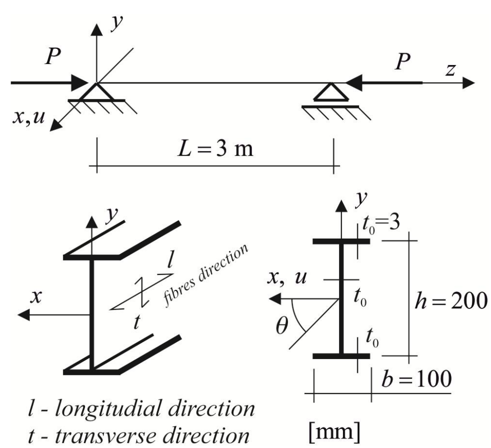

FIGURE 1. Axially compressed column; numerical example – data.

Flexural and torsional buckling patterns of the column are independents, so they be considered separately.

Flexural Buckling – First Variation of Critical Load

Firstly, the flexural buckling about y-axis is examined. The total potential energy V of the column of a length L and subjected to the end axial loads P may be written as

$$V = \frac{1}{2}\int_0^L D_l I_y u''^2 dz - \frac{1}{2}P\int_0^L u'^2 dz \tag{4}$$

where u - displacement in x direction (Fig. 1), I_y - moment of inertia of the cross-section, denotes prime first derivative with respect to axial coordinate z.
The bifurcation point is featured if the total potential energy (4) for the critical load P=P_{cr} vanishes [6]. In order to find the first order variation of the critical load with respect to variations of arbitrary column parameter s the first variation of the both sides of (4) is calculated

$$\delta V = \frac{1}{2}\int_0^L \left[\left(D_l I_y\right),_s u''^2 - \delta P_{cr} u'^2\right]\delta s\,dz - \int_0^L \left(D_l I_y u'' u'',_s - P_{cr} u' u',_s\right)\delta s\,dz = 0 \tag{5}$$

where $(...),_s$ marks the first partial derivative with respect to s is denoted and u stands for the buckling mode. It should be noted that the second part of (5) may be neglected on the basis of the virtual work theorem, as a variation of the critical load is presented

$$\delta P_{cr} = \int_0^L \left[\left(D_l I_y\right),_s u''^2\right]\delta s\ dz \bigg/ \int_0^L u'^2 dz = \int_0^L W_F,_s\,\delta s\ dz \tag{6}$$

where

$$W_F,_s = \left(D_l I_y\right),_s u''^2 \bigg/ \int_0^L u'^2 dz \tag{7}$$

the sensitivity function of critical load of flexural buckling with respect to arbitrary variation of parameter s is denoted.

Torsional Buckling - First Variation of Critical Load

Torsional buckling of the column is the next step. A first the total potential energy of the column and applied end loads following [13, 14] is presented as

$$V = \frac{1}{2}\int_0^L \left(D_l I_\omega \theta''^2 + GI_d \theta'^2\right) dz - \frac{1}{2}P\int_0^L r_0^2 \theta'^2 dz \tag{8}$$

where I_ω - the cross-section warping constant, θ - the cross-section rotation angle (see Fig. 1), G - the shear modulus, I_d - the cross-section free torsion moment of inertia and r_o - the polar radius of inertia.
While the expression in (8) at the bifurcation point is zero, first variation of the total energy with respect to arbitrary column parameter s leads to

$$\begin{aligned}\delta V = \frac{1}{2}\int_0^L \left\{\left[\left(D_l I_\omega\right),_s \theta''^2 + \left(GI_d\right),_s \theta'^2 - P_{cr}\left(r_o^2\right),_s \theta'^2\right] - \delta P_{cr} r_0^2 \theta'^2\right\}\delta s\,dz + ...\\ + \int_0^L \left[D_l I_\omega \theta'' \theta'',_s + GI_d \theta' \theta',_s - P_{cr} r_0^2 \theta' \theta',_s\right]\delta s\,dz = 0\end{aligned} \tag{9}$$

The second term of (9) expresses the virtual work of internal forces and external load P on virtual displacements which should be equal zero, hence it is possible to find the first variation of the critical loads

$$\delta P_{cr} = \int_0^L \left[\left(D_l I_\omega\right),_s \theta''^2 + \left(GI_d\right),_s \theta'^2 - P_{cr}\left(r_o^2\right),_s \theta'^2\right]\delta s\ dz \bigg/ \int_0^L r_0^2 \theta'^2 dz = \int_0^L W_T,_s\,\delta s\ dz \tag{10}$$

where θ is the buckling mode and

$$W_T,_s = \left[\left(D_l I_\omega\right),_s \theta''^2 + \left(GI_d\right),_s \theta'^2 - P_{cr}\left(r_0^2\right),_s \theta'^2\right] \bigg/ \int_0^L r_0^2 \theta'^2 dz \tag{11}$$

denotes the sensitivity function of the torsional buckling load with respect to arbitrary variation of parameter *s*.

NUMERICAL EXAMPLES

Consider a simply supported thin-walled I-column (L=3 m, h=200 mm, b=100 mm, t_0=3 mm) made of unidirectorial fibre-reinforced laminate (see Tab. 1) shown in Fig. 1 is considered. The fibre volume fraction arbitrary variable along the column axis is assumed to be the parameter undergoing variation s=f. Thus the sensitivity function of the critical load of flexural buckling arrives at (see (7))

$$W_F,_s = \left[D_l,_f \, I_y u''^2 \right] \Big/ \int_0^L u'^2 dz \qquad (12)$$

The sensitivity function corresponding to the torsional buckling can be written as (see (11))

$$W_T,_s = \left[D_l,_f \, I_\omega \theta''^2 + G,_f \, I_d \theta'^2 \right] \Big/ \int_0^L r_0^2 \theta'^2 dz \qquad (13)$$

TABLE 1. Material properties for matrix and fibres [15].

Density [kg/m^3]	1380	2450
Young's modulus [GPa]	2.5	420
Kirchhoff's modulus [GPa]	1.2	170
Poisson's ratio [-]	0.33	0.20

The geometric parameters of I cross-section to calculate the sensitivity functions distributions are as follows

$$A = (2b+h)t_0 ,$$

$$I_y = \frac{1}{12}\left(2b^3 t_0 + h t_0^{\ 3}\right) ,$$

$$I_0 = \frac{1}{12}\left(2b^3 + 6bh^2 + h^3\right) t_0 ,$$

$$I_d = \frac{1}{3}\left(2b+h\right) t_0^{\ 3} ,$$

$$I_\omega = \frac{1}{24} b^3 h^2 t_0 ,$$

$$r_0 = \sqrt{\frac{I_0}{A}} = \frac{1}{2}\sqrt{\frac{2b^3+6bh^2+h^3}{6b+3h}} .$$

The functions (7) and (11) for simply supported I column in the case of flexural and torsional buckling are

$$W_F,_s = D_l,_f \, \frac{\pi^2 t_0 (2b^3 + h t_0^{\ 2})}{6L^3} \sin\left(\frac{\pi z}{L}\right)^2 \qquad (14)$$

$$W_T,_s = D_l,_f \, \frac{\pi^2 t_0 (2b+h) b^3 h^2}{(2b^3+6bh^2+h^3)L^3} \sin\left(\frac{\pi z}{L}\right)^2 + G,_f \, \frac{8 t_0^{\ 3} (2b+h)^2}{(2b^3+6bh^2+h^3)L} \cos\left(\frac{\pi z}{L}\right)^2 \qquad (15)$$

The sensitivity functions (14) and (15) distributions along the column axis for both kinds bifurcation under investigation are presented in Figs. 2-6 and 7-11, respectively.

Furthermore, comparative values of the critical buckling loads using FEM (ABAQUS) [16] for uniform distribution of the fibre volume fraction along the column axis are determined. In order to estimate the critical loads, a linear perturbation procedure (LBA) is used. The columns are modelled by shell elements with reduce integration type-S4R. The main element size is 0.01x0.01 m^2, i.e. 40 elements along the cross section. The total amount of finite elements in all cases equals 1200. Material behaviour is modelled by linear elastic orthotropic lamina-type procedure [16]. The value parameters of material types A and B are shown in Tab. 2.

TABLE 2. Material parameters for model types A and B for different fibre volume fraction *f*.

f	Material's homogenization type A	Material's homogenization type B	Material's homogenization type A	Material's homogenization type B	Material's homogenization type A	Material's homogenization type B	Material's homogenization type A	Material's homogenization type B
	E_l [GPa]		E_t [GPa]		G [GPa]		ν_{lt} [-]	
0.85	357.375		29.764	16.123	13.178	7.692	0.210	0.219
0.90	378.250		43.655	23.729	19.685	11.283	0.207	0.213
0.95	399.125		79.944	44.920	36.353	21.162	0.203	0.207

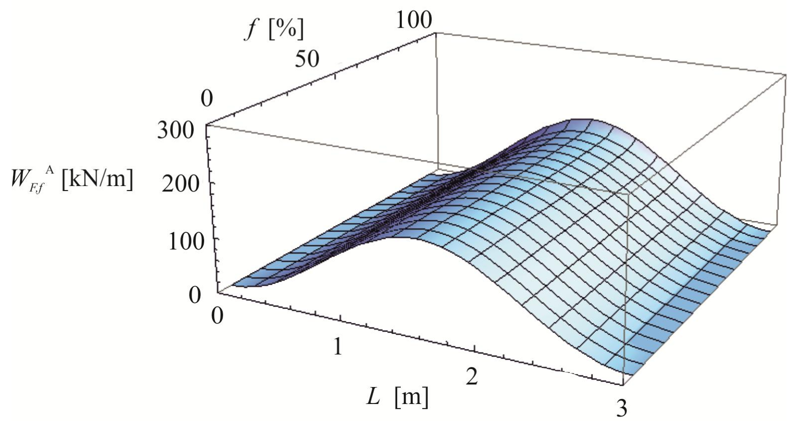

FIGURE 2. The sensitivity functions of flexural buckling critical load depending of the fibre volume fraction parameter *f* (from 5 to 95%) along the I-column axis for material homogenization model type A.

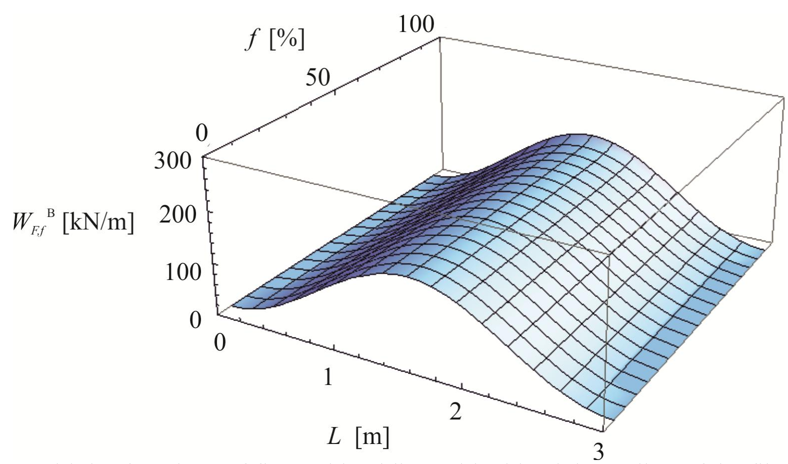

FIGURE 3. The sensitivity functions of flexural buckling critical load depending of the fibre volume fraction parameter *f* (from 5 to 95%) along the I-column axis for material homogenization model type B.

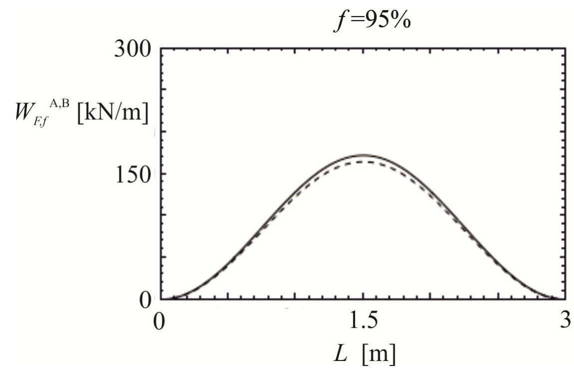

FIGURE 4. The sensitivity functions of flexural buckling critical load for the fibre volume fraction parameter *f*=95% along the I-column axis for two material homogenization model types A (-) (P_{cr}^{FEM}=209.9 kN) and B (--) (P_{cr}^{FEM}=204.5 kN).

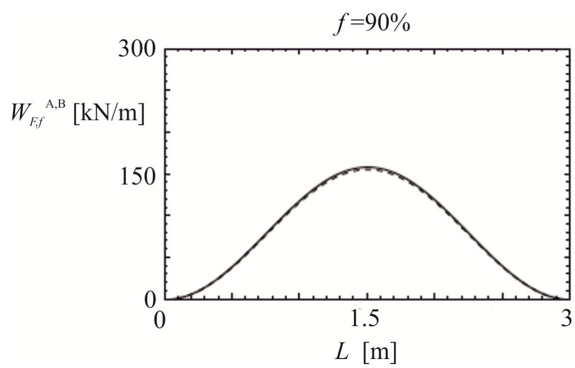

FIGURE 5. The sensitivity functions of flexural buckling critical load for the fibre volume fraction parameter *f*=90% along the I-column axis for two material homogenization model types A (-) (P_{cr}^{FEM}=193.9 kN) and B (--) (P_{cr}^{FEM}=184.4 kN).

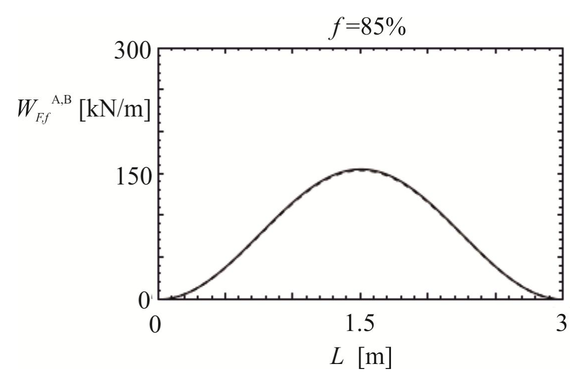

FIGURE 6. The sensitivity functions of flexural buckling critical load for the fibre volume fraction parameter *f*=85% along the I-column axis for two material homogenization model types A (-) (P_{cr}^{FEM}=178.8 kN) and B (--) (P_{cr}^{FEM}=166.5 kN).

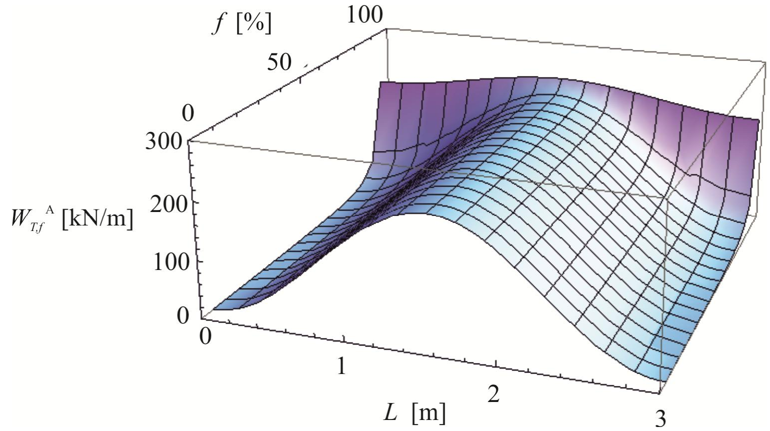

FIGURE 7. The sensitivity functions of torsional buckling critical load depending of the fibre volume fraction parameter *f* (from 5 to 95%) along the I-column axis for material homogenization model type A.

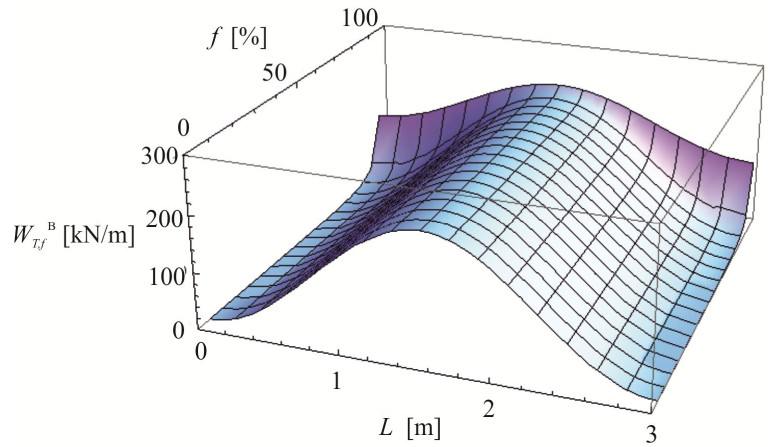

FIGURE 8. The sensitivity functions of torsional buckling critical load depending of the fibre volume fraction parameter *f* (from 5 to 95%) along the I-column axis for material homogenization model type B.

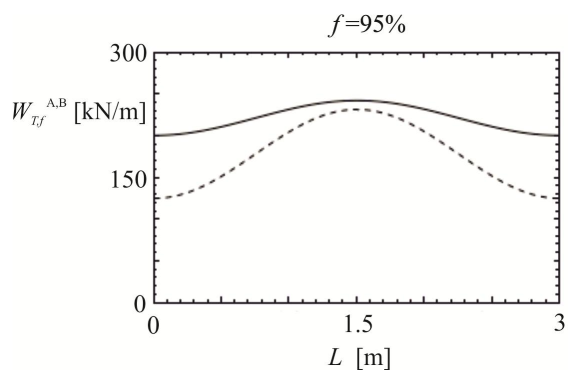

FIGURE 9. The sensitivity functions of torsional buckling critical load for the fibre volume fraction parameter *f*=95% along the I-column axis for two material homogenization model types A (-) (P_{cr}^{FEM}=321.6 kN) and B (--) (P_{cr}^{FEM}=311.3 kN).

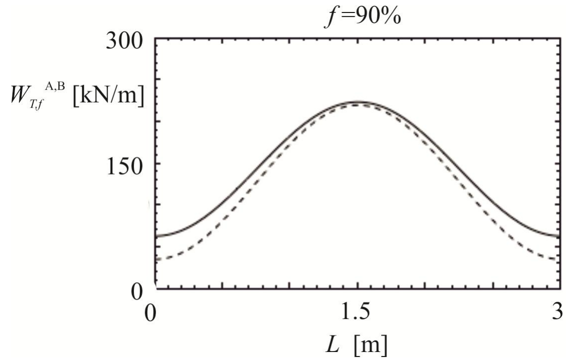

FIGURE 10. The sensitivity functions of torsional buckling critical load for the fibre volume fraction parameter f=90% along the I-column axis for two material homogenization model types A (-) (P_{cr}^{FEM}=294.7 kN) and B (--) (P_{cr}^{FEM}=286.1 kN).

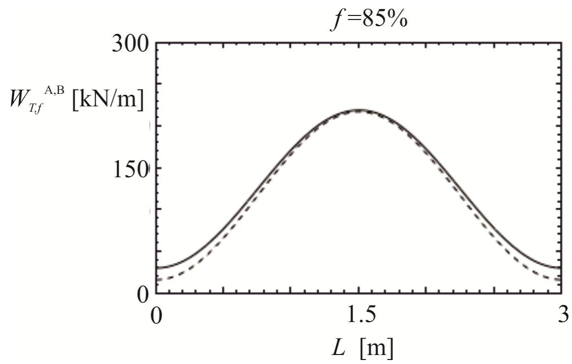

FIGURE 11. The sensitivity functions of torsional buckling critical load for the fibre volume fraction parameter f=85% along the I-column axis for two material homogenization model types A (-) (P_{cr}^{FEM}=273.4 kN) and B (--) (P_{cr}^{FEM}=265.3 kN).

CONCLUSIONS

The paper deals with the first order sensitivity analysis of flexural and torsional buckling loads of thin-walled bisymmetric (or axisymmetric) open cross-section columns subjected to axial compressive loads. The columns are made of unidirectional fibre-reinforced laminates and the fibre volume fraction undergoes variation. Two homogenization types of the column material are taken into account. The closed analytical form of the first order variations of the critical buckling loads are derived for both investigated sorts of buckling. In numerical examples related to the simply supported columns the distributions of the sensitivity functions along the column axis are presented. Some differences in the critical load variation distributions for flexural and torsional buckling and for both types of material are worth noting (see Figs. 2-11). Comparative values of the critical loads determined by means of the analytical formula and in numerical way using the ABAQUS software are shown, as well. The acceptable agreement of the results is achieved. The sensitivity analysis solutions presented in the paper can be a useful tool for finding the variation of fibre volume fraction along the column axis to obtain the assumed critical load change. Moreover, it is may be useful in optimal design of the laminated thin-walled columns.

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