ARCHIVES OF ACOUSTICS
Vol. **44**, No. 2, pp. 321–328 (2019)
DOI: 10.24425/aoa.2019.128495

Impact of Boundary Conditions on Acoustic Excitation of Entropy Perturbations in a Bounded Volume of Newtonian Gas

Anna PERELOMOVA

Faculty of Applied Physics and Mathematics
Gdansk University of Technology
Narutowicza 11/12, 80-233 Gdansk, Poland; e-mail: anna.perelomova@edu.pg.pl

(*received September 30, 2018; accepted February 17, 2019*)

Excitation of the entropy mode in the field of intense sound, that is, acoustic heating, is theoretically considered in this work. The dynamic equation for an excess density which specifies the entropy mode, has been obtained by means of the method of projections. It takes the form of the diffusion equation with an acoustic driving force which is quadratically nonlinear in the leading order. The diffusion coefficient is proportional to the thermal conduction, and the acoustic force is proportional to the total attenuation. Theoretical description of instantaneous heating allows to take into account aperiodic and impulsive sounds. Acoustic heating in a half-space and in a planar resonator is discussed. The aim of this study is to evaluate acoustic heating and determine the contribution of thermal conduction and mechanical viscosity in different boundary problems. The conclusions are drawn for the Dirichlet and Neumann boundary conditions. The instantaneous dynamic equation for variations in temperature, which specifies the entropy mode, is solved analytically for some types of acoustic exciters. The results show variation in temperature as a function of time and distance from the boundary for different boundary conditions.

Keywords: nonlinear acoustics; acoustic heating in resonators; Burgers equation; first and second type boundary conditions; acoustic heating in a half-space.

1. Introduction

The modes of linear flow are represented by acoustic (wave) and non-wave modes. The latter modes are entropy and vortex types of fluid motion. The entropy mode is a potential flow, usually specified by isobaric increase in temperature and, in thermoconducting fluids, by a weak bulk flow. Modes of infinitely small magnitudes do not interact in the course of a flow. However, in the real flows of finite magnitudes with attenuation of any kind, they do interact. This interaction is followed by two phenomena. First, by transfer of macroscopic energy and momentum into chaotic motion of molecules (that is, increase of the background temperature). And second, by excitation of macroscopic bulk flows. Nonlinearity and some kind of attenuation in a fluid are required for this transfer. Usually, nonlinear acoustics focuses on the nonlinear effects that are associated with intense sound. However, it is necessary to describe nonlinear distortion of sound itself along with excitation of non-wave modes in its field. In bounded flows, the geometry of a flow and boundary conditions are of key importance. The nonlinear excitation of the thermodynamic perturbations that belong to the entropy mode in the half-space and in one-dimensional resonator, and satisfy the physically meaningful boundary conditions, is the subject of this study.

The method of projections is used to this end. It was proposed by the author and successfully applied for many problems of fluid dynamics (Perelomova, 2003a; 2012; 2018). Its foundation is the linear projection of the total vector of perturbations onto specific disturbances by means of matrix projectors. These follow from the conservation laws in the differential form in a linear flow. The total field of perturbations in a planar linear flow is split into two acoustic modes and the entropy mode (Chu, 1958; Rudenko, Soluyan, 2005). Suitability for weakly nonlinear flows is the advantage of the projection method (Leble, Perelomova, 2018). By means of projection, the nonlinear system of conservation laws may be decomposed into individual nonlinear dynamic equations that govern perturbations in the corresponding mode. Excitation of the entropy mode in the field of sound (that is, acoustic heating) assumes dominance of sound and weakness of the entropy perturbations. Instantaneous variation of temperature in the entropy mode is described by the diffusion equation with an acoustic driving force that is quadratically nonlinear in the leading order. Acoustic force is proportional to the total attenuation. The fluid velocity in the sound mode satisfies the Burgers dynamic equation (Rudenko, Soluyan, 2005). This is one of the consequences of projecting, where nonlinear term in the Burgers equation can be interpreted as an acoustic force that reflects the self-action of sound. The summary perturbations at the boundaries that consist of parts belonging to entropy and wave modes must satisfy the proper boundary conditions.

The presented method is preferable due to instantaneous description of acoustic heating. The previous approach made use of periodicity as the condition for splitting the energy balance equation into acoustic and non-acoustic parts by means of averaging over the sound period (Rudenko, Soluyan, 2005; Makarov, Ochmann, 1996). Averaging over the sound period is actually some kind of projection. This, however, does not consider aperiodic acoustic disturbances and detailed dynamics of perturbations in the entropy mode. In this study, the instantaneous equation is used for description of heating in the bounded space: the half-space and one-dimensional resonator with the Dirichlet or Neumann boundary conditions that correspond to different physical conditions of a flow. The results may be of especial interest for flows of fluids with noticeable thermal conduction, in particular, for all gases and metallic liquids. If the thermal conduction equals zero, the diffusion coefficient also equals zero, and the theoretical description is simplified to a great extent.

2. Projecting onto acoustic and entropy modes in the linear flow

We start from the set of conservation equations in the planar flow of a thermoconducting Newtonian fluid in the differential form. They are: the momentum equation, the energy balance equation, and the continuity equation:

$$\begin{gathered}\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)+\frac{\partial p}{\partial x}=\frac{4\mu}{3}\frac{\partial^2 v}{\partial x^2},\\ \rho\left(\frac{\partial e}{\partial t}+v\frac{\partial e}{\partial x}\right)+p\frac{\partial v}{\partial x}=\chi\Delta T+\frac{4\mu}{3}\left(\frac{\partial v}{\partial x}\right)^2,\\ \frac{\partial \rho}{\partial t}+\frac{(\partial\rho v)}{\partial x}=0,\end{gathered}\tag{1}$$

where x, t are the spatial co-ordinate and time, and ρ, p, v, T denote density, pressure, velocity of a fluid, and temperature; χ, μ are thermal conductivity and viscosity, both assumed to be constants. The caloric and thermal equations of state of an ideal gas complete the system (1). Its internal energy and temperature are related as

$$e=C_vT=\frac{p}{(\gamma-1)\rho},$$

with C_v denoting the heat capacity under constant volume per unit mass; and $\gamma=C_p/C_v$ being the ratio of specific heats. We consider the constant equilibrium thermodynamic parameters without bulk flows and make use of the excess quantities $p'=p-p_0$, $\rho'=\rho-\rho_0$, where ρ_0 and p_0 are equilibrium values. Equations (1) are readily rearranged as

$$\begin{aligned}&\frac{\partial v}{\partial t}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x}-\delta_1\frac{\partial^2 v}{\partial x^2}=-v\frac{\partial v}{\partial x}+\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x}-\delta_1\frac{\rho'}{\rho_0}\frac{\partial^2 v}{\partial x^2},\\ &\frac{\partial p'}{\partial t}+\rho_0c_0^2\frac{\partial v}{\partial x}-\frac{\delta_2}{(\gamma-1)}\frac{\partial^2(\gamma p'-c_0^2\rho')}{\partial x^2}=-v\frac{\partial p'}{\partial x}\\ &\qquad-\gamma p'\frac{\partial v}{\partial x}+\delta_1(\gamma-1)\rho_0\left(\frac{\partial v}{\partial x}\right)^2\\ &\qquad+\frac{\delta_2}{(\gamma-1)\rho_0}\frac{\partial^2(c_0^2\rho'^2-\gamma\rho'p')}{\partial x^2},\\ &\frac{\partial \rho'}{\partial t}+\rho_0\frac{\partial v}{\partial x}=-v\frac{\partial \rho'}{\partial x}-\rho'\frac{\partial v}{\partial x},\end{aligned}\tag{2}$$

where

$$\delta_1=\frac{4\mu}{3\rho_0},\qquad \delta_2=\frac{\chi}{\rho_0}\left(\frac{1}{C_v}-\frac{1}{C_p}\right),\qquad c_0=\sqrt{\frac{\gamma p_0}{\rho_0}}.$$

All three Eqs (2) are written within the accuracy up to quadratic nonlinear terms, including those proportional to dissipative coefficients. This reflects smallness of magnitudes of perturbations. Hence, the equations are valid if the Mach number of a flow M is much lower than the unity. All conclusions which follow are valid in the leading order.

A linear fluid flow, that is, a flow of infinitely small magnitudes, is represented by the system (2) with zero right-hand side which may be rearranged as

$$\frac{\partial\psi}{\partial t}+L\psi=0,\tag{3}$$

where

$$\psi=\begin{pmatrix}v\\ p'\\ \rho'\end{pmatrix},\qquad L=\begin{pmatrix}-\delta_1\dfrac{\partial^2}{\partial x^2} & \dfrac{1}{\rho_0}\dfrac{\partial}{\partial x} & 0\\ \rho_0c_0^2\dfrac{\partial}{\partial x} & \dfrac{\gamma\delta_2}{\gamma-1}\dfrac{\partial^2}{\partial x^2} & -\dfrac{c_0^2\delta_2}{\gamma-1}\dfrac{\partial^2}{\partial x^2}\\ \rho_0\dfrac{\partial}{\partial x} & 0 & 0\end{pmatrix}.$$

Transfering it into the Fourier space, one arrives at

$$\frac{\partial\widetilde{\psi}}{\partial t}+\widetilde{L}\widetilde{\psi}=0,\tag{4}$$

where

$$\widetilde{\psi} = \begin{pmatrix} \widetilde{v} \\ \widetilde{p} \\ \widetilde{\rho} \end{pmatrix}, \qquad \widetilde{L} = \begin{pmatrix} \delta_1 k^2 & -\dfrac{ik}{\rho_0} & 0 \\ -i\rho_0 c_0^2 k & -\dfrac{\gamma\delta_2}{\gamma-1}k^2 & \dfrac{c_0^2\delta_2}{\gamma-1}k^2 \\ -i\rho_0 k & 0 & 0 \end{pmatrix},$$

and

$$\widetilde{\psi}(k)\exp(i\omega(k)t) = \int \psi(x,t)\exp(ikx)\,\mathrm{d}x.$$

The vectors that follow are solutions to Eq. (4) and represent all possible types of the linear flow (Perelomova, 2003a; Chu, 1958; Perelomova, 2003b):

$$\widetilde{\psi}_1 = \begin{pmatrix} \dfrac{c_0}{\rho_0} + ik\dfrac{\beta}{2\rho_0} \\ c_0^2 + ik\delta_2 c_0 \\ 1 \end{pmatrix}\widetilde{\rho}_1, \qquad \widetilde{\psi}_2 = \begin{pmatrix} -\dfrac{c_0}{\rho_0} + ik\dfrac{\beta}{2\rho_0} \\ c_0^2 - ik\delta_2 c_0 \\ 1 \end{pmatrix}\widetilde{\rho}_2,$$

$$\widetilde{\psi}_3 = \begin{pmatrix} ik\dfrac{\delta_2}{(\gamma-1)\rho_0} \\ 0 \\ 1 \end{pmatrix}\widetilde{\rho}_3.$$

Their analogues in the (x,t) space,

$$\psi_1 = \begin{pmatrix} \dfrac{c_0}{\rho_0} - \dfrac{\beta}{2\rho_0}\dfrac{\partial}{\partial x} \\ c_0^2 - \delta_2 c_0\dfrac{\partial}{\partial x} \\ 1 \end{pmatrix}\rho_1, \quad \psi_2 = \begin{pmatrix} -\dfrac{c_0}{\rho_0} - \dfrac{\beta}{2\rho_0}\dfrac{\partial}{\partial x} \\ c_0^2 + \delta_2 c_0\dfrac{\partial}{\partial x} \\ 1 \end{pmatrix}\rho_2,$$
$$\psi_3 = \begin{pmatrix} -\dfrac{\delta_2}{(\gamma-1)\rho_0}\dfrac{\partial}{\partial x} \\ 0 \\ 1 \end{pmatrix}\rho_3, \tag{5}$$

are modes with the ordering numbers j ($j = 1, 2, 3$), where ρ_1, ρ_2, ρ_3 are excess densities which specify the corresponding mode, and $\beta = \delta_1 + \delta_2$ denotes the total attenuation. The definition of modes is conditioned by the kinds of dispersion relations (Perelomova, 2003b; Leble, Perelomova, 2018)

$$\omega_1 = c_0 k + i\frac{\beta k^2}{2}, \qquad \omega_2 = -c_0 k + i\frac{\beta k^2}{2},$$
$$\omega_3 = i\frac{\delta_2 k^2}{\gamma - 1}. \tag{6}$$

Three linearly independent vectors reflect three types of links between specific perturbations in a planar flow: the first two are acoustic, rightwards and leftwards progressive, and the third one is the entropy mode (Chu, 1958; Rudenko, Solyan, 2015; Leble, Perelomova, 2018). The specific excess densities ρ_1, ρ_2, ρ_3 determine the total dimensionless perturbations v, p', ρ' in a one-to-one way:

$$\psi = \sum_{j=1}^{3}\begin{pmatrix} v_j & p_j & \rho_j \end{pmatrix}^{\mathrm{T}}.$$

The specific excess densities may be extracted from the vector of total perturbation by means of projecting rows:

$$L_j\psi = \rho_j, \qquad j = 1, 2, 3. \tag{7}$$

These rows take the forms

$$L_1 = \begin{pmatrix} \dfrac{\rho_0}{2c_0} + \dfrac{\delta_2\rho_0}{2c_0^2}\dfrac{\partial}{\partial x} & \dfrac{1}{2c_0^2} + a^* & b^* \end{pmatrix},$$

$$L_2 = \begin{pmatrix} -\dfrac{\rho_0}{2c_0} + \dfrac{\delta_2\rho_0}{2c_0^2}\dfrac{\partial}{\partial x} & \dfrac{1}{2c_0^2} - a^* & -b^* \end{pmatrix},$$

$$L_3 = \begin{pmatrix} -\dfrac{\delta_2\rho_0}{c_0^2}\dfrac{\partial}{\partial x}, & -\dfrac{1}{c_0^2} & 1 \end{pmatrix},$$

where

$$a^* = \left(\frac{\beta}{4c_0^3} - \frac{\delta_2}{2c_0^3(\gamma-1)}\right)\frac{\partial}{\partial x},$$

$$b^* = \frac{\delta_2}{2c_0(\gamma-1)}\frac{\partial}{\partial x}.$$

Obviously,

$$\sum_{j=1}^{3} L_j = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

The important property of projecting rows L_j is to extract the linear dynamic equation for ρ_j when applying at Eqs (3) (Perelomova, 2003b; Leble, Perelomova, 2018). For example,

$$L_1\left(\frac{\partial\psi}{\partial t} + L\psi\right) = \frac{\partial\rho_1}{\partial t} + c_0\frac{\partial\rho_1}{\partial x} - \frac{\beta}{2}\frac{\partial^2\rho_1}{\partial x^2} = 0.$$

3. Projecting in a weakly nonlinear flow and acoustic heating in a bounded space

3.1. The Burgers equation

Going to the flow with finite magnitudes of perturbations, the wave modes need to be corrected in order to hold adiabaticity within accuracy up to terms proportional to M^2. Quasi-adiabaticity plays a key role in description of nonlinear distortions and nonlinear effects of sound which are proportional to M^2 in the leading order. The relations as follows support adiabaticity with the required accuracy. In particular, the relations for the rightwards progressive mode take the form

$$\psi_1=\begin{pmatrix}1\\ c_0\rho_0+\dfrac{(\beta-2\delta_2)\rho_0}{2}\dfrac{\partial}{\partial x}\\ \dfrac{\rho_0}{c_0}+\dfrac{\beta\rho_0}{2c_0^2}\dfrac{\partial}{\partial x}\\ 1\end{pmatrix}v_1+\begin{pmatrix}0\\ \dfrac{(\gamma+1)\rho_0}{4}\\ -\dfrac{(\gamma-3)\rho_0}{4c_0^2}\end{pmatrix}v_1^2. \tag{8}$$

Equation (8) recall the links which specify the Riemann wave. Additionally, they take into account attenuation of a wave. We make use of relations (8) in order to study the weakly nonlinear flow in the case of a dominant first sound mode. They have impact on the coupling of modes and dynamic equations which take into account nonlinear interaction of modes. An excess density which specifies the first acoustic mode, satisfies the equation

$$\frac{\partial\rho_1}{\partial t}+c_0\frac{\partial\rho_1}{\partial x}+\frac{(\gamma+1)c_0}{2\rho_0}\rho_1\frac{\partial\rho_1}{\partial x}-\frac{\beta}{2}\frac{\partial^2\rho_1}{\partial x^2}=0.$$

It may be obtained by applying L_1 on the system (2) if ψ_1 is taken alone due to its dominance. The nonlinear term may be interpreted as a result of nonlinear self-action of the dominant sound mode. In view of links (5), the velocity of a fluid specifying the first mode, satisfies the Burgers equation

$$\frac{\partial v_1}{\partial t}+c_0\frac{\partial v_1}{\partial x}+\frac{(\gamma+1)}{2}v_1\frac{\partial v_1}{\partial x}-\frac{\beta}{2}\frac{\partial^2 v_1}{\partial x^2}=0. \tag{9}$$

We consider the dominant first wave mode. This condition determines some temporal and spatial domains. A solution to Eq. (9) should take into account proper boundary conditions.

3.2. Acoustic heating in a half-space

Applying L_3 in Eqs (2) and making use of ψ_1 from (8), yields a leading-order dynamic equation:

$$\frac{\partial\rho_3}{\partial t}-\frac{\delta_2}{\gamma-1}\frac{\partial^2\rho_3}{\partial x^2}=\frac{\beta(\gamma-1)\rho_0}{2c_0^2}\left(v_1\frac{\partial^2 v_1}{\partial x^2}-\left(\frac{\partial v_1}{\partial x}\right)^2\right)$$
$$=F_3(x,t), \tag{10}$$

where F_3 represents an "acoustic force" which excites the entropy perturbations. Let us consider the processes in the half space $x\geq 0$ at $t\geq 0$. The solution to (10) which satisfies the second-type (Neumann) boundary condition is as follows:

$$\frac{\partial\rho_3}{\partial x}(t,x=0)=0, \tag{11}$$

$$\rho_3(x,t)=\sqrt{\gamma-1}\int_0^t\int_0^\infty\frac{e^{-\frac{(\gamma-1)(x-\xi)^2}{4\delta_2(t-\tau)}}+e^{-\frac{(\gamma-1)(x+\xi)^2}{4\delta_2(t-\tau)}}}{2\sqrt{\delta_2\pi(t-\tau)}}$$
$$\cdot F_3(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{12}$$

The boundary condition (11) corresponds to a zero flow of energy through the boundary which associates with the entropy mode, since $T_3=-\frac{T_0}{\rho_0}\rho_3$ and hence

$$\frac{\partial T_3}{\partial x}(t,x=0)=0.$$

This is valid for zero total energy flux at the boundary, at least, on average, if excited by the periodic sound, since the spatial derivatives of acoustic perturbations are zero on average. The first-type (Dirichlet) boundary condition for an excess density

$$\rho_3(t,x=0)=0 \tag{13}$$

corresponds to constant temperature at the boundary which associates with the entropy mode,

$$T_3(t,x=0)=0.$$

The total excess temperature at the boundary is also zero, at least, on average. This condition requires an inflow of external energy. The solution to (10) with the boundary condition (13), takes the form

$$\rho_3(x,t)=\sqrt{\gamma-1}\int_0^t\int_0^\infty\frac{e^{-\frac{(\gamma-1)(x-\xi)^2}{4\delta_2(t-\tau)}}-e^{-\frac{(\gamma-1)(x+\xi)^2}{4\delta_2(t-\tau)}}}{2\sqrt{\delta_2\pi(t-\tau)}}$$
$$\cdot F_3(\xi,\tau)\,\mathrm{d}\xi\tau. \tag{14}$$

The harmonic fluid velocity specifying the first mode, that is, solution to the linear wave equation without taking into account attenuation,

$$v_1=V_0\sin(\omega(t-x/c_0)) \tag{15}$$

leads to a uniform solution

$$-\frac{\rho_3}{\rho_0}=\frac{T_3}{T_0}=\frac{\beta(\gamma-1)\omega^2V_0^2t}{2c_0^4}=\frac{(\gamma-1)M^2\theta}{2} \tag{16}$$

in the case of the second-type boundary condition (zero flux of energy at the boundary) and to the solution

$$-\frac{\rho_3}{\rho_0}=\frac{T_3}{T_0}=\frac{\beta(\gamma-1)\omega^2V_0^2}{2c_0^4}\int_0^t Erf\left(\frac{x}{2a\sqrt{(t-\tau)}}\right)\mathrm{d}\tau$$
$$=\frac{(\gamma-1)M^2}{2}\int_0^\theta Erf\left(\frac{X}{\sqrt{\theta-\xi}}\right)\mathrm{d}\xi \tag{17}$$

in the case of the first-type condition (constant temperature at the boundary), where

$$X=\frac{\omega\sqrt{\beta}}{2ac_0}x,\qquad \theta=\frac{\beta\omega^2}{c_0^2}t,$$

$$a=\sqrt{\frac{\delta_2}{\gamma-1}},\qquad M=\frac{V_0}{c_0}.$$

Since $Erf(a) < 1$ for any $a \geq 0$, the absolute values of perturbations which associate with the Dirichlet boundary condition, are always smaller than that corresponding to the Neumann boundary condition. This reflects reduction of energy in the volume of the resonator due to a flux of energy through the boundary in the case of the second-type boundary condition. Table 1 represents the limits to which a dimensionless excess temperature tends if X is much smaller or much larger than the unity for the Dirichlet boundary condition (13) in the case of periodic excitation (15).

Table 1. Limit to which an excess dimensionless temperature which associates with the entropy mode tends if $X = \frac{\omega\sqrt{\beta}}{2ac_0}x$ tends to zero or infinity. The Dirichlet boundary condition (13) and the periodic sound (15).

X	$X \ll 1$	$X \gg 1$
$\dfrac{T_3}{T_0}$	$\dfrac{(\gamma-1)M^2\sqrt{\theta}X}{2}$	$\dfrac{(\gamma-1)M^2}{2}\theta$

The quasi-periodic approximate solution to the Burgers equation (9) takes the form (Rudenko, Soluyan, 2005):

$$v_1 = V_0 \exp(-\beta\omega^2 t/c_0^2)\sin(\omega(t - x/c_0)). \tag{18}$$

The case of the second-type boundary condition yields the expression

$$-\frac{\rho_3}{\rho_0} = \frac{T_3}{T_0} = -\left(1 - \exp\left(-\frac{\beta\omega^2 t}{c_0^2}\right)\right)\frac{(\gamma-1)V_0^2}{2c_0^2}, \tag{19}$$

and the case of the first-type boundary condition leads to the formula

$$\begin{aligned} -\frac{\rho_3}{\rho_0} = \frac{T_3}{T_0} &= \frac{\beta(\gamma-1)\omega^2 V_0^2}{2c_0^4} \\ &\cdot \int_0^t \exp\left(-\frac{\beta\omega^2\tau}{c_0^2}\right) Erf\left(\frac{x}{2a\sqrt{t-\tau}}\right) \mathrm{d}\tau \\ &= \frac{(\gamma-1)M^2}{2}\int_0^\theta \exp(-\xi) Erf\left(\frac{X}{\sqrt{\theta-\xi}}\right)\mathrm{d}\xi. \end{aligned} \tag{20}$$

Boundary conditions of the first and second type for excess density in the entropy mode do not disturb the total velocity at the boundary in the leading order. The perturbation of density develops due to the acoustic force of heating which is proportional to β, and the corresponding velocity v_3 is a small quantity of order β^2 in accordance to the links (5). Hence, there is no bulk flow associating with the entropy mode. Figure 1 shows the excess dimensional temperature in the entropy mode as a function of X and θ for zero temperature perturbations at the boundary in the cases of periodic and quasi-periodic excitation at the boundary.

a)

b)

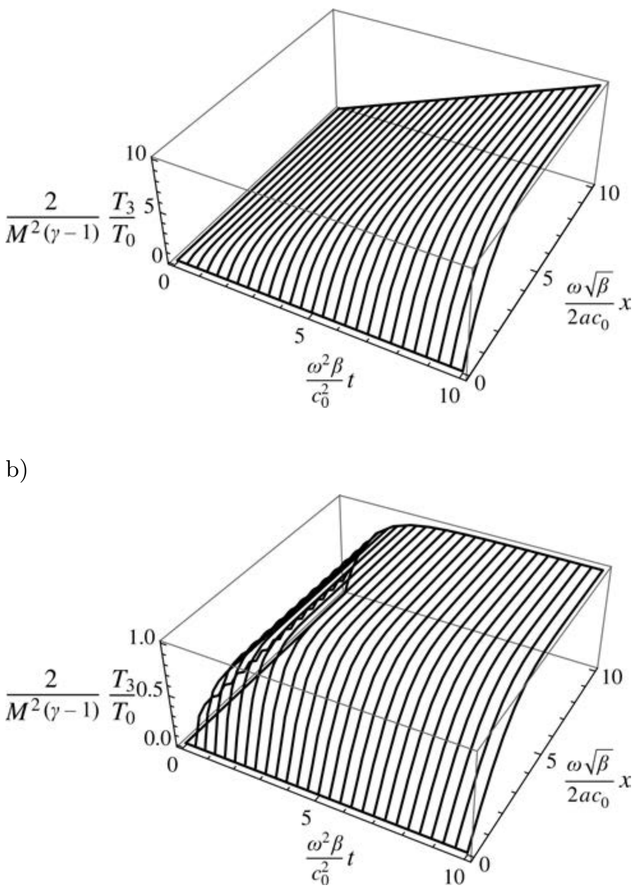

Fig. 1. Excess temperature associated with the entropy mode as a function of $X = \frac{\omega\sqrt{\beta}}{2ac_0}x$ and $\theta = \frac{\omega^2\beta}{c_0^2}t$. The case of constant temperature at the boundary and harmonic excitation (15) (a) and quasi-harmonic (b) excitation at the boundary (18).

3.3. Acoustic heating in a resonator

The counterpropagating waves do not nonlinearly interact in the leading order. This was discovered by Kaner and co-authors (Kaner *et al.*, 1977). Recently, a similar result was obtained by Ruderman with the help of the spectral method (Ruderman, 2013). The counter-propagating waves have individual influence on acoustic heating, if they are periodic. The cross acoustic terms in the acoustic force of heating are zero on average. The author has proved that in (Perelomova, 2008). The leading-order equation

$$\begin{aligned} \frac{\partial\rho_3}{\partial t} - \frac{\delta_2}{\gamma-1}\frac{\partial^2\rho_3}{\partial x^2} &= \frac{\beta(\gamma-1)\rho_0}{2c_0^2} \\ &\cdot\left(v_1\frac{\partial^2 v_1}{\partial x^2} - \left(\frac{\partial v_1}{\partial x}\right)^2 + v_2\frac{\partial^2 v_2}{\partial x^2} - \left(\frac{\partial v_2}{\partial x}\right)^2\right) = F_3 \end{aligned} \tag{21}$$

describes evolution of the entropy perturbations. The velocities v_1, v_2 satisfy the Burgers equations for progressive perturbations and the summary conditions at the boundaries of a resonator,

$$v_1(x=0,t)+v_2(x=0,t)=v_1(x=L,t)+v_2(x=L,t)=0.$$

The solution is valid for the small Reynolds numbers, that is, for large enough attenuation, it is a sum of two components:

$$\begin{aligned} v_1 &= V_0\exp(-\beta\omega^2 t/c_0^2)\sin(\omega(t-x/c_0)),\\ v_2 &= -V_0\exp(-\beta\omega^2 t/c_0^2)\sin(\omega(t+x/c_0)), \end{aligned} \tag{22}$$

where the length of a resonator L includes the natural number of wave lengths,

$$\frac{\omega L}{c_0}=N\pi,$$

and N is any natural number ensuring smallness of the Reynolds number which is proportional to $N^{-1}L$. The resulting acoustic force of heating equals to

$$F_3=-\frac{\beta V_0^2(\gamma-1)\omega^2\rho_0}{c_0^4}\exp\left(-\frac{\beta\omega^2 t}{c_0^2}\right). \tag{23}$$

The acoustic force of heating takes the same form for the boundary conditions of the second type at both boundaries or for the first-type condition at one of the boundaries and the second-type condition at the other one. The series

$$\rho_3(x,t)=\sum_{n=1}^{\infty}\left(\int_0^t e^{-\frac{a^2\pi^2n^2}{L^2}(t-\tau)}F_{3,n}(\tau)\,\mathrm{d}\tau\right)\sin\left(\frac{n\pi x}{L}\right), \tag{24}$$

where

$$F_{3,n}(\tau)=\frac{2}{L}\int_0^L F_3(\tau)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$
$$=\frac{2\exp\left(-\frac{\beta\omega^2\tau}{c_0^2}\right)(1-\cos(\pi n))}{n\pi},$$

is a solution satisfying the first-type boundary conditions and ensuring constant temperature at the boundaries,

$$\rho_3(x=0,t)=\rho_3(x=L,t)=0,$$
$$T_3(x=0,t)=T_3(x=L,t)=0.$$

The total fluid velocity $v_1+v_2+v_3$ at both boundaries remains zero. An excess density which associates with the entropy mode, takes the form

$$\frac{\rho_3(x,t)}{\rho_0}=\frac{2\beta(\gamma-1)L^2\omega^2V_0^2}{\pi c_0^2}\cdot\sum_{n=1}^{\infty}\frac{c^*}{\beta L^2\omega^2 n-\pi^2a^2c_0^2n^3}\sin\left(\frac{n\pi x}{L}\right), \tag{25}$$

where

$$c^*=(1-\cos(\pi n))\left(\exp\left(-\frac{\beta\omega^2 t}{c_0^2}\right)-\exp\left(-\frac{a^2n^2\pi^2 t}{L^2}\right)\right).$$

If $\beta L^2\omega^2-\pi^2a^2c_0^2n^2=0$ for some n, the corresponding constituent in the sum,

$$\frac{\left(\exp\left(-\frac{\beta\omega^2 t}{c_0^2}\right)-\exp\left(-\frac{a^2n^2\pi^2 t}{L^2}\right)\right)}{\beta L^2\omega^2 n-\pi^2a^2c_0^2n^3}$$

needs to be substituted by $-t\exp\left(-\frac{\beta\omega^2 t}{c_0^2}\right)$. Making use of the dimensionless variables

$$t'=\frac{\pi^2\beta}{L^2}t,\qquad \nu=\frac{a^2}{\beta},\qquad N=\frac{\omega L}{\pi c_0},$$

Eq. (25) may be readily rearranged as

$$\frac{\rho_3}{\rho_0}=-\frac{T_3}{T_0}=\frac{2(\gamma-1)N^2V_0^2}{\pi c_0^2}\cdot\sum_{n=1}^{\infty}\frac{d^*}{n(N^2-n^2\nu)}\sin\left(\frac{n\pi x}{L}\right), \tag{26}$$

where

$$d^*=(1-\cos(\pi n))(\exp(-N^2t')-\exp(-n^2\nu t')).$$

For any parameters, the sum is negative and an excess temperature associated with the entropy mode is positive. $\nu=\frac{\delta_2}{(\delta_1+\delta_2)(\gamma-1)}$ varies from 0 (zero thermal conduction) untill $(\gamma-1)^{-1}$ (this does not exceed 3, this is the case of $\delta_1=0$). The absolute quantities of perturbations in the entropy mode achieve maximum in the middle of a resonator, that is, at $x=\frac{L}{2}$. They achieve maximum at some time which is difficult to establish due to an infinite amount of constituents. It may be evaluated approximately in view of the fact that the first constituent has the biggest contribution in the series of partial sums. It corresponds to $n=1$ and achieves maximum at $t'=\frac{\ln(N^2)-\ln(\nu)}{N^2-\nu}$. The case $N^2=\nu$ corresponds to the maximum at $t'=N^{-2}$. Examples of numerical evaluations of

$$\frac{1}{(\gamma-1)M^2}\frac{T_3}{T_0},$$

where $M=\frac{V_0}{c_0}$, are shown in Fig. 2.

The next case represents the boundary conditions which correspond to zero temperature flux at the boundaries, that is, thermally isolated boundaries:

$$\frac{\partial\rho_3}{\partial x}(x=0,t)=\frac{\partial\rho_3}{\partial x}(x=L,t)=0,$$
$$\frac{\partial T_3}{\partial x}(x=0,t)=\frac{\partial T_3}{\partial x}(x=L,t)=0.$$

The total fluid velocity $v_1+v_2+v_3$ at both boundaries remains zero. The solution to Eq. (21) with acoustic force (23) is uniform:

$$\frac{\rho_3}{\rho_0}=-\frac{T_3}{T_0}=-M^2(\gamma-1)\left(1-\exp\left(-\frac{\beta\omega^2 t}{c_0^2}\right)\right). \tag{27}$$

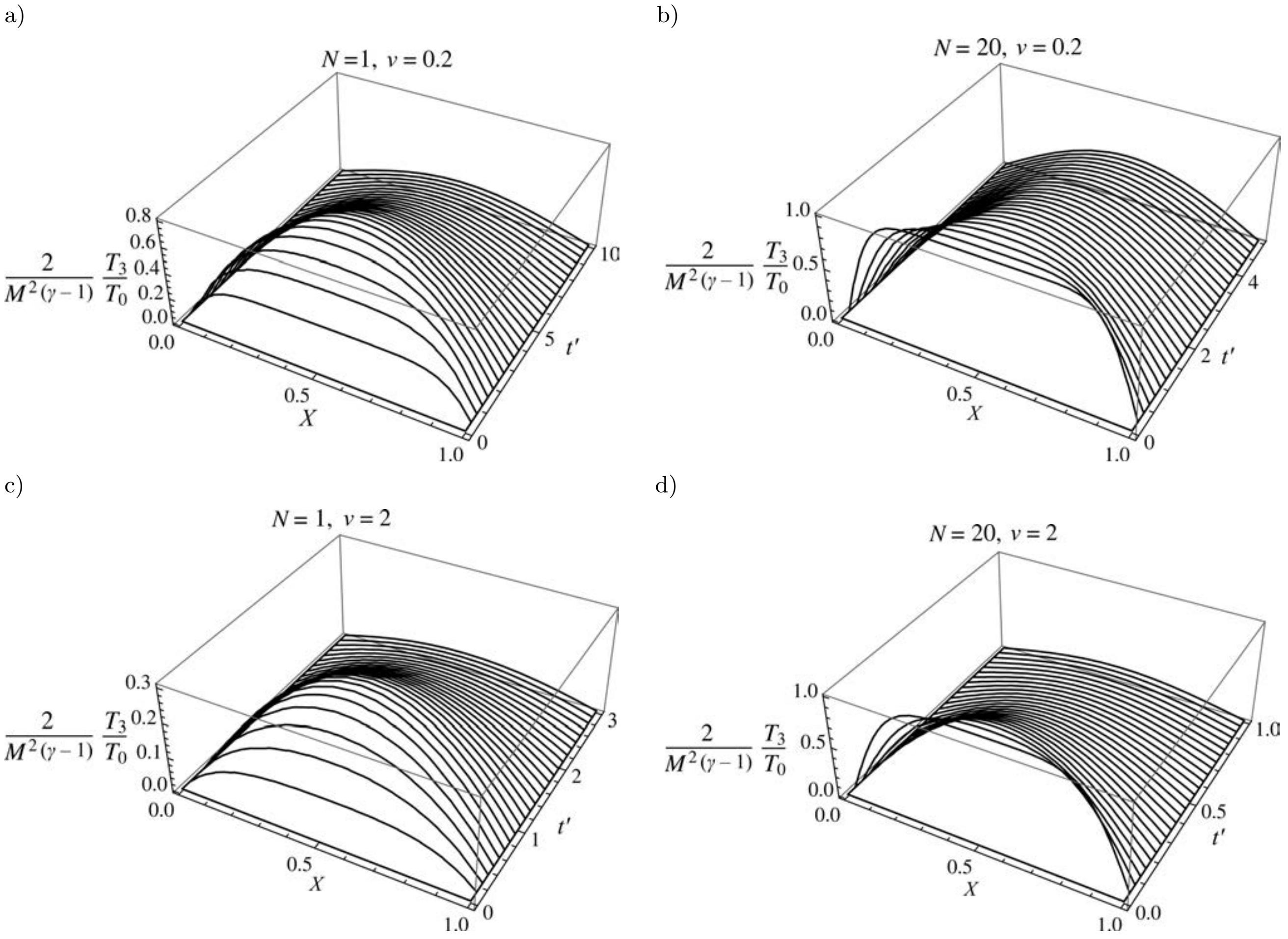

Fig. 2. Dimensionless excess temperature associated with the entropy mode, $\frac{1}{(\gamma-1)M^2}\frac{T_3}{T_0}$ as a function of $t' = \frac{\pi^2\beta}{L^2}t$ and $X = \frac{x}{L}$. Cases of different N and ν.

4. Conclusions

The main result of this study is a theoretical description of the non-wave perturbations of density and temperature, where boundary conditions and thermal conductivity of a fluid are taken into account. These contribute to the equilibrium thermodynamic state of a fluid. In turn, variations in equilibrium temperature affect the wave processes. The entropy perturbations are nonlinearly excited by sound that may be aperiodic or impulsive. This is the advantage of the current methodology as compared to previously proposed methods that used averaging over the sound period (hence referred only to the periodic sound) and considered heating in unbounded volumes of a fluid (Rudenko, Soluyan, 2005; Makarov, Ochmann, 1996; Hamilton, Blackstock, 1998). The theoretical results of this study are supplemented by analytical examples of dynamics of an excess temperature that is associated with the entropy mode. These examples consider periodic and quasi-periodic acoustic disturbances and boundary conditions of the first and second types. The description may be useful in technical and medical applications of ultrasound, where accurate evaluation of variations in temperature is of great importance (Izadifar *et al.*, 2017). Molevich (2001) studied acoustic heating excited by the periodic sound in acoustically active unbounded media. Recent studies in plasma physics pay special attention to magnetoacoustic heating of coronal plasma which requires proper accounting for the boundary conditions and magnetosound source (Sakurai, 2017; Murawski *et al.*, 2011). Acoustic heating in plasmas is special due to presence of fast and slow magnetosound perturbations.

Evolution of density perturbations belonging to the entropy mode is governed by the diffusion equation with coefficient proportional to the thermal conduction (Eq. (10)). It is instantaneous, and includes an acoustic source in its right-hand side, which is nonlinearly quadratic in the leading order, and proportional to the total attenuation. The fluid velocity specifying the first dominant mode satisfies the Burgers equation (9). The boundary conditions play a key role in dynamics not only of wave perturbations, but also in dynamics of the secondary non-wave motions that get stronger in the wave field. The perturbations at boundaries may be conditioned by excitation of a transducer, and by properties of boundaries themselves to reflect physics, for example, requirement of zero total velocity at the boundary. Usually, these conditions may be satisfied by the discrete wave spectrum, in contrast to the flows in unbounded volumes. It is worth noting that the second-type boundary condition for an excess temperature (the Neumann condition, that is, zero energy flux through the boundary) in a problem that relates to the half-space, yields perturbations in the entropy mode that do not depend on thermal conductivity at all. Solutions (16), (19) overlap with solutions to (10) with zero diffusion coefficient. The same conclusion may be formulated for the entropy mode perturbations in a resonator in the case of the Neumann conditions for an excess temperature at the boundaries. This is related to periodic and quasi-periodic acoustic disturbances that excite acoustic heating. The reason for that is the uniform acoustic force of heating, which in turn creates uniform entropy perturbations.

In this study, Eq. (10) is solved analytically for physically meaningful examples of acoustic perturbations. In media without heat conduction, excess density in the entropy mode may be established by simple integration of the acoustic force over time. Consideration of thermal conduction requires to solve the diffusion equation with proper total boundary conditions. Thus, there is a pronounced difference between analytical description of acoustic heating in thermoconducting fluids and those without thermal conduction.

References

1. Chu B.-T., Kovasznay L.S.G. (1958), *Nonlinear interactions in a viscous heat-conducting compressible gas*, Journ. Fluid. Mech., **3**, 494–514.
2. Hamilton M.F., Blackstock D.T. [Eds.] (1998), *Nonlinear acoustics: theory and applications*, Academic Press.
3. Izadifar Z., Babyn P., Chapman D. (2017), *Mechanical and biological effects of ultrasound: a review of present knowledge*, Ultrasound in Med. and Biol., **43**, 6, 1085–1104.
4. Kaner V.V., Rudenko O.V., Khokhlov R.V. (1977), *Theory of nonlinear oscillations in acoustic resonators*, Sov. Phys. Acoust., **23**, 432–437.
5. Leble S., Perelomova A. (2018), *The dynamical projectors method: hydro and electrodynamics*, CRC Press.
6. Makarov S., Ochmann M. (1996), *Nonlinear and thermoviscous phenomena in acoustics, Part I*, Acustica, **82**, 579–606.
7. Molevich N.E. (2001), *Amplification of vortex and temperature waves in the process of induced scattering of sound in thermodynamically nonequilibrium media*, High Temperature, **39**, 6, 884–888.
8. Murawski K., Zaqarashvili T.V., Nakariakov V.M. (2011), *Entropy mode at a magnetic null point as a possible tool for indirect observation of nanoflares in the solar corona*, Astronomy & Astrophysics, A18.
9. Perelomova A. (2003a), *Interaction of modes in nonlinear acoustics: theory and applications to pulse dynamics*, Acta Acustica, **89**, 86–94.
10. Perelomova A. (2003b), *Heating caused by a nonperiodic ultrasound. Theory and calculations on pulse and stationary sources*, Archives of Acoustics, **28**, 2, 127–138.
11. Perelomova A. (2012), *Acoustic heating produced in resonators filled by a newtonian fluid*, Canadian Journal of Physics, **90**, 7, 693–699.
12. Perelomova A. (2018), *Magnetoacoustic heating in a quasi-isentropic magnetic gas*, Physics of Plasmas, **25**, 042116.
13. Rudenko O.V., Soluyan S.I. (2005), *Theoretical foundations of nonlinear acoustics*, Consultants Bureau, New York.
14. Ruderman M.S. (2013), *Nonlinear damped standing slow waves in hot coronal magnetic loops*, Astronomy and Astrophysics, **553**, A23.
15. Sakurai T. (2017), *Heating mechanisms of the solar corona*, Proc. Jpn Acad. Ser B Phys Biol Sci., **93**, 2, 87–97.