

## Weakly connected Roman domination in graphs

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### Abstract

A *Roman dominating function* on a graph  $G = (V, E)$  is defined to be a function  $f : V \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for which  $f(v) = 2$ . A dominating set  $D \subseteq V$  is a *weakly connected dominating set* of  $G$  if the graph  $(V, E \cap (D \times V))$  is connected. We define a *weakly connected Roman dominating function* on a graph  $G$  to be a Roman dominating function such that the set  $\{u \in V : f(u) \in \{1, 2\}\}$  is a weakly connected dominating set of  $G$ . The weight of a weakly connected Roman dominating function is the value  $f(V) = \sum_{u \in V} f(u)$ . The minimum weight of a weakly connected Roman dominating function on a graph  $G$  is called the *weakly connected Roman domination number* of  $G$  and is denoted by  $\gamma_R^{wc}(G)$ . In this paper, we initiate the study of this parameter.

*Keywords:* Roman domination number, weakly connected set, weakly connected Roman domination number, trees.

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### 1. Introduction

Cockayne et al. in [7] defined a *Roman dominating function* (RDF) on a graph  $G = (V, E)$  to be a function  $f : V \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for

5 which  $f(v) = 2$ . For a real-valued function,  $f: V \rightarrow \mathbb{R}$ , the *weight* of  $f$  is  
 $w(f) = \sum_{v \in V} f(v)$ , and for  $S \subseteq V$  we define  $f(S) = \sum_{v \in S} f(v)$ , so  $w(f) =$   
 $f(V)$ . The *Roman domination number*, denoted  $\gamma_R(G)$ , is the minimum weight  
of an RDF in  $G$ ; that is,  $\gamma_R(G) = \min\{w(f) : f \text{ is an RDF in } G\}$ . An RDF  
of weight  $\gamma_R(G)$  is called a  $\gamma_R(G)$ -function. Roman domination in graphs has  
10 been studied, for example, in [7, 9, 13].

As it is mention in [14], this definition of a Roman dominating function was  
motivated by an article in Scientific American by Ian Stewart entitled "Defend  
the Roman Empire!" [16]. Each vertex in our graph represents a location in the  
Roman Empire. A location (vertex  $v$ ) is considered *unsecured* if no legions are  
15 stationed there (i.e.,  $f(v) = 0$ ) and *secured* otherwise (i.e., if  $f(v) \in \{1, 2\}$ ). An  
unsecured location (vertex  $v$ ) can be secured by sending a legion to  $v$  from an  
adjacent location (an adjacent vertex  $u$ ). In the fourth century A.D. emperor  
Constantine the Great decreed that a legion cannot be sent from a secured loca-  
tion to an unsecured location if doing so leaves that location unsecured. Thus,  
20 two legions must be stationed at a location ( $f(v) = 2$ ) before one of the legions  
can be sent to an adjacent location. In this way, Emperor Constantine the Great  
can defend the Roman Empire. Since it is expensive to maintain a legion at  
a location, the Emperor would like to station as few legions as possible, while  
still defending the Roman Empire. A Roman dominating function of weight  
25  $\gamma_R(G)$  corresponds to such an optimal assignment of legions to locations.

In order to generalize or improve some properties of the Roman domination  
in its standard form, some variants of Roman domination have been introduced  
and studied. Those variants are often related to modifying the conditions in  
which the vertices are dominated, or to adding extra properties to the Roman  
30 domination property itself. For instance we remark here variants like the fol-  
lowing ones: total Roman domination (see [3, 5]), mixed Roman domination  
(see [2]) or strong Roman domination (see [4]).

In this paper we explore the idea of strengthening security of the Roman  
Empire by providing a better communication in emergency between the legions,  
35 while still having substantial costs of maintaining legions as low as possible.

Two legions at different location (vertices  $u$  and  $v$ ) can *contact directly* if there is at most one unsecured location between them and the distance between  $u$  and  $v$  is at most 2. Moreover,  $u$  and  $v$  can *contact undirectly* if there is a sequence of secured vertices ( $u = u_1, u_2, \dots, u_k = v$ ) such that  $u_i$  and  $u_{i+1}$  can contact directly for  $i = 1, 2, \dots, k - 1$ . The Roman Empire is *communicated* if any two  
 40 legions at different locations can contact directly or undirectly.

Let  $G = (V, E)$  be a graph and let  $f: V \rightarrow \{0, 1, 2\}$  be a function. Let  $V_0, V_1$ , and  $V_2$  be the sets of vertices assigned with the values 0, 1, and 2, respectively, under  $f$ . Note that there is a one to one correspondence between the functions  
 45  $f: V \rightarrow \{0, 1, 2\}$  and the ordered triple  $(V_0, V_1, V_2)$  of  $V$ . Thus we will write  $f = (V_0, V_1, V_2)$ .

Denote  $|V(G)| = n(G)$ . The *neighbourhood*  $N_G(v)$  of a vertex  $v \in V(G)$  is the set of all vertices adjacent to  $v$  in  $G$  and the closed neighbourhood is  $N_G[v] = N_G(v) \cup \{v\}$ . The *degree*  $d_G(v)$  of  $v$  is the number of edges incident to  
 50  $v$  in  $G$ ,  $d_G(v) = |N_G(v)|$ . Let  $L(G)$  be the set of all leaves of  $G$ , that is the set of vertices with degree 1, and let  $n_1(G)$  be the cardinality of  $L(G)$ . A vertex  $v$  is called a *support vertex* if  $v$  is a neighbour of a leaf. Denote by  $S(G)$  the set of all support vertices in  $G$  and let  $n_S(G)$  be the cardinality of  $S(G)$ . A *strong support vertex* is a vertex adjacent to at least two leaves. A vertex adjacent to  
 55 exactly one leaf is a *weak support vertex*.

A set  $D \subseteq V(G)$  is a *dominating set* of  $G$  if for every vertex  $v \in V(G) - D$ , there exists a vertex  $u \in D$  such that  $v$  and  $u$  are adjacent. The minimum cardinality of a dominating set in  $G$  is the *domination number* of  $G$  and is denoted by  $\gamma(G)$ . A minimum dominating set of a graph  $G$  is called a  $\gamma(G)$ -set.

From now on,  $G$  will be assumed to be connected. The *subgraph weakly induced by a set*  $D \subseteq V(G)$  is the graph  $\langle D \rangle_w = (N[D], E_w)$ , where  $E_w$  consists of the set of all edges of  $G$  having at least one vertex in  $D$ . A set  $D \subseteq V(G)$  is a *weakly connected dominating set* (WCDS) of  $G$  if  $D$  is dominating and  $\langle D \rangle_w$  is connected. The *weakly connected domination number* of  $G$ , denoted  $\gamma_{wc}(G)$ ,  
 60 is the minimum cardinality of a WCDS. A minimum WCDS of a graph  $G$  is called a  $\gamma_{wc}(G)$ -set. The weakly connected domination number was introduced



in 1997 by Dunbar et al. [10] and studied for example in [8], [15] and [17].

We call the function  $f$  a *weakly connected Roman dominating function* in  $G$  (WCRDF) if each vertex  $u \in V_0$  is adjacent to a vertex  $v \in V_2$  and the sub-  
70 graph  $\langle V_1 \cup V_2 \rangle_w$  weakly induced by  $V_1 \cup V_2$  is connected in  $G$ . The weight  $w(f)$  of  $f$  is  $|V_1| + 2|V_2|$ . The *weakly connected Roman domination number*, denoted  $\gamma_R^{wc}(G)$ , is the minimum weight of a WCRDF in  $G$ ; that is,  $\gamma_R^{wc}(G) = \min\{w(f) : f \text{ is a WCRDF in } G\}$ . A WCRDF of weight  $\gamma_R^{wc}(G)$  is called a  $\gamma_R^{wc}(G)$ -function.

75 This definition of a WCRDF is motivated as follows. Using the notation introduced earlier, we define a location of a legion to be *uncommunicated* if there exists another location of a legion such that the legions cannot contact directly nor undirectly. If the locations are uncommunicated, they cannot safely inform the other locations nor ask them for help in case of urgent emergency. When  
80 all locations of legions are communicated, Emperor Constantine the Great can defend the Roman Empire more efficiently: he can supervise whole Empire and send orders to his legions in reasonable time. Such a placement of legions corresponds to a WCRDF and a minimum such placement of legions corresponds to a minimum WCRDF. Hence this concept of weakly connected Roman domi-  
85 nation is an attractive alternative to Emperor Constantines notion of Roman domination.

For a vertex  $v \in V$ , we denote by  $f[v]$  the set  $\{f(u) : u \in N[v]\}$  for notational convenience. For any unexplained terms and symbols see [12].

In [1] Ahangar et al. introduced the concept of outer-independent Roman  
90 domination as follows: a function  $f : V(G) \rightarrow \{0, 1, 2\}$  is an *outer-independent Roman dominating function* (OIRDF) on  $G$  if every vertex  $u \in V$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for which  $f(v) = 2$  and  $\{v : f(v) = 0\}$  is an independent set. The *outer-independent Roman domination number*  $\gamma_{oiR}(G)$  is the minimum weight of an OIRDF on  $G$ .

Clearly, any outer-independent Roman dominating function on a connected

graph  $G$  is an WCRDF of  $G$ , so

$$\gamma_{oiR}(G) \geq \gamma_R^{wc}(G).$$

On the other hand, for any tree  $T$ , it is easy to see that any WCRDF of  $T$  is an OIRDF of  $T$  and this implies that

$$\gamma_{oiR}(T) \leq \gamma_R^{wc}(T).$$

Therefore, for any tree  $T$

$$\gamma_{oiR}(T) = \gamma_R^{wc}(T). \quad (1)$$

## 95 2. Preliminary results

In this section we study basic properties of weakly connected Roman domination number of graphs.

**Proposition 1.** *If  $G$  is a connected graph, then*

$$\gamma_{wc}(G) \leq \gamma_R^{wc}(G) \leq 2\gamma_{wc}(G).$$

PROOF. Let  $f = (V_0, V_1, V_2)$  be  $\gamma_R^{wc}(G)$ -function. Then  $V_1 \cup V_2$  is a WCDS of  $G$ . Hence  $\gamma_{wc}(G) \leq \gamma_R^{wc}(G)$ .

If  $D_w$  is a  $\gamma_{wc}(G)$ -set, then the function

$$f(u) = \begin{cases} 2 & \text{for } u \in D_w \\ 0 & \text{otherwise} \end{cases}$$

100 is a WCRDF in  $G$ . Thus  $\gamma_R^{wc}(G) \leq 2\gamma_{wc}(G)$ .

**Proposition 2.** *For any connected graph  $G$  of order  $n$ ,  $\gamma_{wc}(G) = \gamma_R^{wc}(G)$  if and only if  $G = K_1$ .*

PROOF. It is obvious that if  $G = K_1$ , then  $\gamma_{wc}(G) = \gamma_R^{wc}(G)$ .

Let  $f = (V_0, V_1, V_2)$  be a  $\gamma_R^{wc}(G)$ -function. Then  $\gamma_{wc}(G) \leq |V_1| + |V_2| \leq$   
 105  $|V_1| + 2|V_2| = \gamma_R^{wc}(G)$ . Since  $\gamma_{wc}(G) = \gamma_R^{wc}(G)$ , we obtain  $|V_2| = 0$  and hence  $|V_0| = 0$ . Therefore,  $\gamma_R^{wc}(G) = |V_1| = n$ . This implies that  $\gamma_{wc}(G) = n$ , which, in turn, implies that  $G = K_1$ .



**Proposition 3.** For any connected graph  $G$  of order  $n$ ,

$$\gamma_R^{wc}(G) \leq n.$$

The equality  $\gamma_R^{wc}(G) = n$  holds if and only if  $G \in \{K_1, K_2\}$ .

PROOF. Let  $G = (V, E)$  be a connected graph. Then  $f = (\emptyset, V, \emptyset)$  is a WCRDF  
110 in  $G$  and hence  $\gamma_R^{wc}(G) \leq n$ .

If  $G = K_1$  or  $G = K_2$ , then clearly  $\gamma_R^{wc}(G) = n$ . Thus suppose  $G \notin \{K_1, K_2\}$   
and  $\gamma_R^{wc}(G) = n$ . If  $u \in V$  is a vertex of degree at least 2 and  $x, y \in N(u)$ ,  
then  $f = (\{x, y\}, V - \{u, x, y\}, \{u\})$  is a WCRDF in  $G$  of weight smaller than  
 $\gamma_R^{wc}(G)$ , which is impossible.

115 **Corollary 4.** If  $\gamma_R^{wc}(G) < n$  and  $f = (V_0, V_1, V_2)$  is a  $\gamma_R^{wc}(G)$ -function, then  
 $|V_0| > 0$  and  $|V_2| > 0$ .

### 3. Complexity results

In this section, we show that the problem of computing  $\gamma_R^{wc}(G)$ -function  
is NP-hard. We will state the corresponding decision problem in the standard  
120 form (see [11]) and we indicate the polynomial time reduction used to prove  
that it is NP-complete. Details are omitted.

#### WEAKLY CONNECTED ROMAN DOMINATING FUNCTION (WCRDF)

*Instance:* A connected graph and a positive integer  $k$ .

*Question:* Does  $G$  have a weakly connected Roman dominating function of  
125 weight at most  $k$ ?

A *split graph* is a graph in which the vertex set can be partitioned into a  
clique and an independent set.

**Theorem 5.** WCRDF is NP-complete, even for split graphs and even for bi-  
partite graphs.

130 PROOF. (*Outline*) It is obvious that WCRDF is a member of NP, since we can, in polynomial time, guess at a function  $f: V(G) \rightarrow \{0, 1, 2\}$  and verify that  $f$  has weight at most  $k$  and is a WCRDF.

The reduction is from EXACT COVER BY 3-SETS (X3C). Given an instance  $X = \{x_1, \dots, x_{3q}\}$  and  $\mathcal{C} = \{C_1, \dots, C_m\}$  of X3C, where  $C_j \subseteq X$  and  
 135  $|C_j| = 3$  for  $1 \leq j \leq m$ , construct a split graph  $G$  with vertices for each  $x_i \in X$ , and with edges  $x_i C_j$  for all  $x_i \in C_j$  and edges so that  $\langle \{C_1, \dots, C_m\} \rangle = K_m$ . Let  $k = 2q$ . It is not hard to show that  $\mathcal{C}$  contains an exact cover if and only if  $G$  has a weakly connected Roman dominating function of weight at most  $k$ .

Similarly, construct a bipartite graph in the same way, except that rather  
 140 than adding all the edges between vertices of  $\mathcal{C}$ , add four new vertices,  $y_0, y_1, y_2, y_3$  and edges  $y_0 y_1, y_0 y_2, y_0 y_3$  and  $y_0 C_j$  for all  $j$ . Set  $k = 2q + 2$ .

#### 4. Lower bound on the weakly connected Roman domination number of a tree without strong support vertices

In this section we prove a lower bound for the weakly connected Roman  
 145 domination number of a tree without strong support vertices in terms of the order of a graph. We start with a result for general graphs.

**Lemma 6.** *Let  $G$  be a graph and let  $P = (v_1, v_2, v_3, v_4)$  be an induced path in  $G$  such that  $d(v_1) = 1$ ,  $d(v_2) = d(v_3) = d(v_4) = 2$ . Denote  $G' = G - P$ . Then*

$$\gamma_R^{wc}(G) = \gamma_R^{wc}(G') + 3. \quad (2)$$

PROOF. Let  $f' = (V_0, V_1, V_2)$  be a  $\gamma_R^{wc}(G')$ -function. Then  $(V_0 \cup \{v_1, v_3\}, V_1 \cup \{v_4\}, V_2 \cup \{v_2\})$  is a WCRDF of  $G$ . Hence,  $\gamma_R^{wc}(G) \leq \gamma_R^{wc}(G') + 3$ .

On the other hand, let  $f = (V_0, V_1, V_2)$  be a  $\gamma_R^{wc}(G)$ -function. Let  $v_5 \neq v_3$   
 150 be a neighbour of  $v_4$ . If  $f(v_5) \in \{1, 2\}$ , we may assume that  $f(v_2) = f(v_4) = 0$ ,  $f(v_1) = 1$  and  $f(v_3) = 2$ . Then  $(V_0 - \{v_2, v_4\}, V_1 - \{v_1\}, V_2 - \{v_3\})$  is a WCRDF of  $G'$ . If  $f(v_5) = 0$  and  $f(v_4) = 2$ , we may assume that  $f(v_1) = f(v_3) = 0$ ,  $f(v_2) = 2$  and then  $(V_0 - \{v_1, v_3, v_5\}, V_1 \cup \{v_5\}, V_2 - \{v_2, v_4\})$  is a WCRDF of  $G'$ . If  $f(v_5) = 0$  and  $f(v_4) = 1$ , we may assume that  $f(v_1) = f(v_3) = 0$ ,  $f(v_2) = 2$

155 and then  $(V_0 - \{v_1, v_3\}, V_1 - \{v_4\}, V_2 - \{v_2\})$  is a WCRDF of  $G'$ . Notice that the situation when  $f(v_4) = f(v_5) = 0$  is impossible. In all situations we obtain a WCRDF of  $G'$  of weight smaller than the weight of  $f$  by three. Therefore,  $\gamma_R^{wc}(G') \leq \gamma_R^{wc}(G) - 3$ . Hence the equality (2) follows.

160 Let  $T$  be a tree and let  $f = (V_0, V_1, V_2)$  be a  $\gamma_R^{wc}(T)$ -function. If  $v \in V(T)$  is a strong support vertex, then without loss of generality we may assume that  $v \in V_2$  and each leaf neighbour of  $v$  belongs to  $V_0$ . If  $v \in V(T)$  is a weak support vertex and  $x$  is the leaf adjacent to  $v$ , then without loss of generality we may assume that either  $v \in V_2$  and  $x \in V_0$  or  $v \in V_0$  and  $x \in V_1$ .

165 Let  $\mathcal{T}_1, \mathcal{T}_2$  and  $\mathcal{T}_3$  be the following three operations defined on a tree  $T$ . Let  $f$  be a  $\gamma_R^{wc}(T)$ -function and let  $v \in V(T)$ .

**Operation  $\mathcal{T}_1$ .** If  $f(v) = 0$  and  $v$  is not a support vertex, then add a vertex  $x$  and the edge  $vx$ .

**Operation  $\mathcal{T}_2$ .** If  $f(v) = 2$ , add a path  $(x, y)$  and the edge  $vx$ .

**Operation  $\mathcal{T}_3$ .** If  $f(v) \in \{1, 2\}$ , add a path  $(x, y, z)$  and the edge  $vx$ .

170 Let  $\mathcal{T}$  be the minimum family of trees obtained from the path  $P_2$  by a finite sequence of Operations  $\mathcal{T}_2$  and at most one either Operation  $\mathcal{T}_1$  or  $\mathcal{T}_3$ .

**Theorem 7.** *Let  $T$  be a tree of order  $n$  without a strong support vertex. Then*

$$\gamma_R^{wc}(T) \geq \left\lceil \frac{n}{2} \right\rceil + 1, \quad (3)$$

*with equality if and only if  $T$  belongs to the family  $\mathcal{T}$ .*

PROOF. First we prove that if  $T$  is a tree without a strong support vertex, then equation (8) is true and if equality in (8) holds, then  $T$  belongs to the family  $\mathcal{T}$ .  
175 If  $\text{diam}(T) = 1$ , then  $T = P_2$  and the statement is clearly true. If  $\text{diam}(T) = 2$ , then  $T$  is a star and the central vertex is a strong support vertex, which is impossible. If  $\text{diam}(T) = 3$ , then, since  $T$  is a tree without a strong support vertex,  $T = P_4$  and the statement holds, since  $P_4$  can be obtained from  $P_2$  by Operation  $\mathcal{T}_2$ .



180 Hence assume  $\text{diam}(T) \geq 4$ . We proceed by induction on  $n$ . Assume for each tree  $T'$  without a strong support vertex and with  $n(T') < n$  the inequality (8) holds for  $T'$  and in case of equality in (8),  $T' \in \mathcal{T}$ . Let  $(v_1, v_2, \dots, v_k)$  be a longest path in  $T$ . Then  $d(v_2) = 2$ . We consider a few cases depending on the structure of  $T$ .

*Case 1:*  $d(v_3) > 2$ . Then without loss of generality we let  $f$  be a minimum WCRDF of  $T$  such that  $f(v_3) = 2$ , the weight assigned to every neighbour of  $v_3$ , except possibly  $v_4$ , is 0, and the weight assigned to every leaf vertex at distance 2 from  $v_3$  is 1. Let  $T' = T - \{v_1, v_2\}$ . Since  $T$  is without strong support vertices and  $d(v_3) > 2$ ,  $T'$  is also a tree without strong support vertices and hence equation (8) holds for  $T'$ . Moreover, the function  $f$  restricted on  $T'$  is a WCRDF of  $T'$ . Hence,

$$\gamma_R^{wc}(T) \geq 1 + \gamma_R^{wc}(T') \geq 1 + \left\lceil \frac{n-2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil + 1. \quad (4)$$

185 Hence the inequality (8) holds for  $T$ .

If  $\gamma_R^{wc}(T) = \left\lceil \frac{n}{2} \right\rceil + 1$ , then we have equalities throughout the inequality chain (4). Particulary,  $\gamma_R^{wc}(T') = \left\lceil \frac{n(T')}{2} \right\rceil + 1$ . By the induction,  $T' \in \mathcal{T}$  and  $f$  restricted on  $T'$  is a  $\gamma_R^{wc}(T')$ -function. Hence, for some minimum WCRDF  $f'$  of  $T'$  is  $f'(v_3) = 2$ . Therefore  $T$  may be obtained from  $T'$  by  
190 Operation  $\mathcal{T}_2$  and we conclude that  $T \in \mathcal{T}$ .

*Case 2:*  $d(v_3) = 2$  and  $f(v_1) = 1$  for some minimum WCRDF  $f$  of  $T$ . Then  $f(v_2) = 0$  and  $f(v_3) = 2$ . Consider  $T' = T - v_1$ . Since  $n(T') < n$  and  $T'$  is without a strong support vertex, we apply the induction hypothesis to  $T'$ . Moreover, the function  $f$  restricted on  $T'$  is a WCRDF of  $T'$ . Therefore,

$$\gamma_R^{wc}(T) \geq 1 + \gamma_R^{wc}(T') \geq \left\lceil \frac{n(T')}{2} \right\rceil + 2 = \left\lceil \frac{n+1}{2} \right\rceil + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1. \quad (5)$$

Hence the inequality (8) holds for  $T$ .

If  $\gamma_R^{wc}(T) = \left\lceil \frac{n}{2} \right\rceil + 1$ , then we have equalities throughout the inequality chain (5) and  $n(T')$  is even. Particulary,  $\gamma_R^{wc}(T') = \left\lceil \frac{n(T')}{2} \right\rceil + 1$ . By the

induction,  $T' \in \mathcal{T}$  and  $f$  restricted on  $T'$  is a minimum WCRDF of  $T'$ . Hence, for some  $\gamma_R^{wc}(T')$ -function  $f'$  is  $f'(v_2) = 0$ . Therefore  $T$  may be obtained from  $T'$  by Operation  $\mathcal{T}_1$  and we conclude that  $T \in \mathcal{T}$ .

*Case 3:*  $d(v_3) = 2$  and  $f(v_1) = 0$  for each minimum WCRDF of  $T$ . Let  $f$  be a minimum WCRDF of  $T$ . Then  $f(v_1) = 0$ ,  $f(v_2) = 2$  and without loss of generality we assume  $f(v_3) = 0$  and  $f(v_4) \in \{1, 2\}$ . Assume additionally  $d(v_4) > 2$  or  $v_5$  is not a support vertex. Let  $T' = T - \{v_1, v_2, v_3\}$ . Then  $T'$  is a tree without a strong support vertex and with less vertices than  $T$ . Moreover,  $f$  restricted on  $T'$  is a WCRDF of  $T'$ . Therefore by the induction, the inequality (8) is true for  $T'$ . Hence,

$$\gamma_R^{wc}(T) \geq 2 + \gamma_R^{wc}(T') \geq \left\lceil \frac{n(T')}{2} \right\rceil + 3 = \left\lceil \frac{n+1}{2} \right\rceil + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1. \quad (6)$$

Hence in this situation the inequality (8) holds for  $T$ .

If  $\gamma_R^{wc}(T) = \left\lceil \frac{n}{2} \right\rceil + 1$ , then we have equalities throughout the inequality chain (6). Particular,  $\gamma_R^{wc}(T') = \left\lceil \frac{n(T')}{2} \right\rceil + 1$ . By the induction,  $T' \in \mathcal{T}$  and  $f$  restricted on  $T'$  is a minimum WCRDF function. Hence, for some minimum WCRDF  $f'$  of  $T'$  is  $f'(v_4) \in \{1, 2\}$ . Therefore  $T$  may be obtained from  $T'$  by Operation  $\mathcal{T}_3$  and we conclude that  $T \in \mathcal{T}$ .

Assume now  $d(v_4) = 2$  and  $v_5$  is a support vertex. Without loss of generality we may assume  $f(v_4) = 1$ . Let  $T' = T - \{v_1, v_2, v_3, v_4\}$ . Then  $T'$  is a tree without a strong support vertex and with less vertices than  $T$ . Therefore by the induction, the inequality (8) is true for  $T'$ . Moreover,  $f$  restricted on  $T'$  is a WCRDF of  $T'$ . Hence,

$$\gamma_R^{wc}(T) \geq 3 + \gamma_R^{wc}(T') \geq \left\lceil \frac{n(T')}{2} \right\rceil + 4 = \left\lceil \frac{n}{2} \right\rceil + 2 > \left\lceil \frac{n}{2} \right\rceil + 1. \quad (7)$$

Hence in this situation the inequality (8) holds for  $T$ .

If  $\gamma_R^{wc}(T) = \left\lceil \frac{n}{2} \right\rceil + 1$ , then we can not have equalities in the inequality chain (7), so this case is impossible.

Notice that Operations  $\mathcal{T}_1$  and  $\mathcal{T}_3$  may be performed on a tree  $T \in \mathcal{T}$  only when

$n(T)$  is even and both of these operations change the parity of the number of vertices of a tree. Therefore these operations may be performed at most once.

This is the end of the proof for inequality (8) and for the case of equality  
 210 in (8).

Now we prove that if  $T \in \mathcal{T}$ , then  $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$ . We proceed by induction on the number  $s(T)$  of operations required to construct the tree  $T$ . If  $s(T) = 0$ , then  $T = P_2$  and clearly  $\gamma_R^{wc}(P_2) = 2 = \lceil \frac{n}{2} \rceil + 1$ .

Assume now that  $T \in \mathcal{T}$  is a tree with  $s(T) = k$  for some positive integer  
 215  $k > 1$  and for each tree  $T' \in \mathcal{T}$  with  $s(T') < k$  is equality in (8). Then  $T$  can be obtained from a tree  $T'$  belonging to  $\mathcal{T}$  by operation  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  or  $\mathcal{T}_3$ . We now consider three possibilities depending on whether  $T$  is obtained from  $T'$  by operation  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  or  $\mathcal{T}_3$ .

*Case 1.*  $T$  is obtained from  $T' \in \mathcal{T}$  by Operation  $\mathcal{T}_1$ . Let  $f'$  be a minimum WCRDF in  $T'$ . Suppose  $T$  is obtained from  $T'$  by adding a vertex  $x$  and the edge  $xv$ , where  $v \in V(T')$  is not a support vertex and  $f'(v) = 0$ . Since the Operation  $\mathcal{T}_1$  is performed,  $T'$  is obtained by applying only Operations  $\mathcal{T}_2$  and hence  $|V(T')|$  is even and  $n = |V(T')| + 1$ . We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 1 to  $x$ . For this reason,

$$\gamma_R^{wc}(T) \leq |f'| + 1 = \frac{|V(T')|}{2} + 2 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Since the inequality (8) is true for  $T$ , we conclude that  $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$ .

*Case 2.*  $T$  is obtained from  $T' \in \mathcal{T}$  by Operation  $\mathcal{T}_2$ . Let  $f'$  be a minimum WCRDF in  $T'$ . Suppose  $T$  is obtained from  $T'$  by adding a path  $(x, y)$  and the edge  $xv$ , where  $v \in V(T')$  and  $f'(v) = 2$ . We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 1 to  $y$  and the weight 0 to  $x$ . For this reason,

$$\gamma_R^{wc}(T) \leq |f'| + 1 = \left\lceil \frac{|V(T')|}{2} \right\rceil + 2 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

220 Since  $\gamma_R^{wc}(T) > \gamma_R^{wc}(T')$ , we conclude that  $\gamma_R^{wc}(T) = \lceil \frac{n}{2} \rceil + 1$ .

*Case 3.*  $T$  is obtained from  $T' \in \mathcal{T}$  by Operation  $\mathcal{T}_3$ . Let  $f'$  be a minimum WCRDF in  $T'$ . Suppose  $T$  is obtained from  $T'$  by adding a path  $(x, y, z)$  and the edge  $xv$ , where  $v \in V(T')$  and  $f'(v) \in \{1, 2\}$ . Since the Operation  $\mathcal{T}_3$  is performed,  $T'$  is obtained by applying only Operations  $\mathcal{T}_2$  and hence  $|V(T')|$  is even. We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 2 to  $y$  and the weight 0 to  $x$  and  $z$ . For this reason,

$$\gamma_R^{wc}(T) \leq |f'| + 2 = \frac{|V(T')|}{2} + 3 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Since the inequality (8) is true for  $T$ , we conclude that  $\gamma_R^{wc}(T) = \left\lceil \frac{n}{2} \right\rceil + 1$ .

Thus if  $T \in \mathcal{T}$ , then  $\gamma_R^{wc}(T) = \left\lceil \frac{n}{2} \right\rceil + 1$ .

The proof is complete.

Since the weakly connected Roman domination number and the outer-independent Roman domination number are equal for trees, we have the following

**Corollary 8.** *Let  $T$  be a tree of order  $n$  without a strong support vertex. Then*

$$\gamma_{oiR}(T) \geq \left\lceil \frac{n}{2} \right\rceil + 1, \tag{8}$$

*with equality if and only if  $T$  belongs to the family  $\mathcal{T}$ .*

## 5. Upper bound on the weakly connected Roman number of a tree

In this section we present an upper bound for the weakly connected Roman domination number of a tree in terms of the order of a tree  $T$ .

Let  $\mathcal{F}$  be a family of all trees  $T$  whose vertex set can be partitioned into sets, each set inducing a path  $P_6$ , such that the subgraph induced by the two central vertices of these  $P_6$ 's is connected. We call the subtree induced by these central vertices the *underlying subtree* of the resulting tree  $T$ , and is called each such path  $P_6$  a *base path* of the tree  $T$ .

A graph  $G$  is a  $\gamma_{wc}$ -*excellent graph* if each vertex of  $G$  is contained in some  $\gamma_{wc}(G)$ -set.



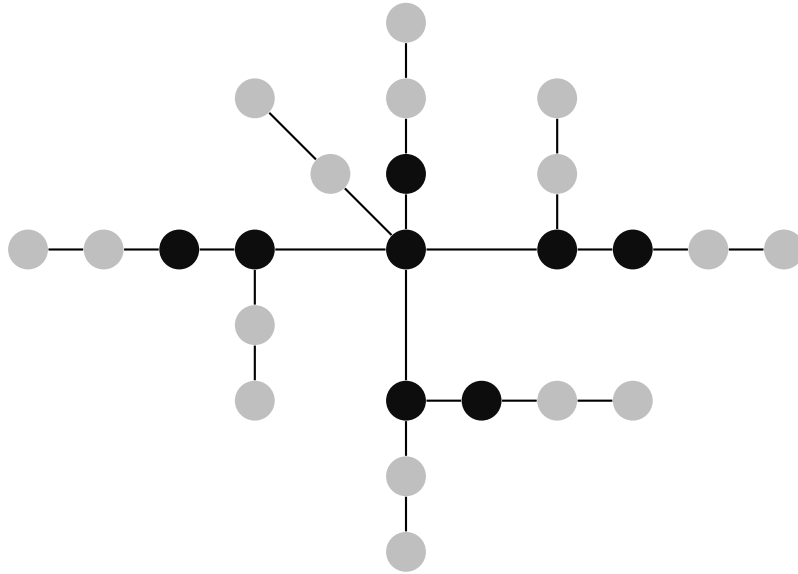


Figure 1: A tree in  $\mathcal{F}$  with underlying tree denoted black

Domke et al. [8] have defined the class  $\mathcal{E}$  to be the class of trees obtained from  $P_2$  by a finite sequence of the following operation: attach to any vertex a  $P_2$ . They have proved the following result.

240 **Theorem 9 (Domke et al. [8]).** *A nontrivial tree  $T$  is  $\gamma_{wc}$ -excellent if and only if  $T$  belongs to the family  $\mathcal{E}$ .*

A set  $S$  of vertices of  $G = (V, E)$  is an *independent set* if no two vertices of  $S$  are adjacent. The *independence number* of  $G$ , denoted  $\beta(G)$ , is the maximum cardinality among all independent sets of vertices of  $G$ .

**Theorem 10 (Domke et al. [8]).** *A nontrivial tree  $T$  of order  $n$  is  $\gamma_{wc}$ -excellent if and only if*

$$\beta(T) = \frac{n}{2}.$$

245 The following result appears in [10].

**Theorem 11 (Dunbar et al. [10]).** *If  $T$  is a nontrivial tree of order  $n$ , then*

$$\gamma_{wc}(T) = n - \beta(T).$$

Therefore, if a tree  $T$  of order  $n$  belongs to the family  $\mathcal{E}$ , then  $\gamma_{wc}(T) = \frac{n}{2}$ .

Our next lemma gives some properties of trees in  $\mathcal{F}$ .

**Lemma 12.** *If  $T$  is a tree of order  $n$  that belongs to the family  $\mathcal{F}$ , then*

$$\gamma_R^{wc}(T) = \frac{5}{6}n. \quad (9)$$

*Additionally,*

1. *if  $v \in V$  is a support vertex, then there exists a  $\gamma_R^{wc}(T)$ -function that assigns to  $v$  value 2;*
2. *if  $v \in V$  is a leaf, then there exists a  $\gamma_R^{wc}(T)$ -function that assigns to  $v$  value 2.*

PROOF. Let  $T \in \mathcal{F}$  have order  $n$  and let the underlying subtree of  $T$  have order  $k$ . Then  $n = 3k$  where  $k \geq 1$ . Let  $f$  be a  $\gamma_R^{wc}(T)$ -function and let  $P = (v_1, v_2, \dots, v_6)$  be an arbitrary base path in  $T$ . Hence  $d(v_1) = d(v_6) = 1$  and  $d(v_2) = d(v_5) = 2$ . Vertices  $v_3$  and  $v_4$  belong to the underlying subtree of  $T$ . The sum of weights given to  $v_1$  and  $v_2$  by  $f$  must be at least 2, unless the weight assigned by  $f$  to  $v_3$  is 2, to  $v_1$  is 1 and to  $v_2$  is 0. Moreover, if the sum of weights given to  $v_1$  and  $v_2$  is 2 and the sum of weights given to  $v_5$  and  $v_6$  is also 2, then the sum of weights given to  $v_3$  and  $v_4$  is at least 1 to ensure  $f$  is a WCRDF of  $T$ . This implies that the sum of the weights assigned by  $f$  to the vertices of the base path  $P$  is at least 5. Since there are at least  $k$  vertex disjoint base paths in  $T$ , each of which receives a total weight at least 5, the weight of  $f$  is  $w(f) \geq 5k$ . Since  $f$  is an arbitrary  $\gamma_R^{wc}(T)$ -function, this implies that  $\gamma_R^{wc}(T) \geq \frac{5}{6}n$ .

Conversely, it is no problem to observe, that the underlying tree of  $T$  belongs to the family  $\mathcal{E}$ . Hence, by Theorems 9, 10 and 11, the weakly connected domination number of underlying tree of  $T$  is equal to  $\frac{k}{2}$ . Hence the function  $f$  that assigns the weight 2 to every support vertex of  $T$ , the weight 0 to every leaf and the weight 1 to each vertex of a minimum weakly connected dominating set of the underlying subtree of  $T$  is a WCRDF of  $T$  of weight  $\frac{5}{2}k$ , which

proves statement 1. Therefore,  $\gamma_R^{wc}(T) \leq w(f) = \frac{5}{2}k = \frac{5}{6}n$ , which proves the equality (9).

Let  $v_1$  be a leaf of  $T \in \mathcal{F}$  and let  $(v_1, v_2, v_3, v_4, v_5, v_6)$  be a base path of  $T$ . In  
 275 what follows we construct a  $\gamma_R^{wc}(T)$ -function which assigns to  $v$  value 2. Since  
 the underlying tree of  $T$  belongs to the family  $\mathcal{E}$ , it is  $\gamma_{wc}$ -excellent. Thus there  
 exists a  $\gamma_{wc}$ -set of the underlying tree of  $T$  containing  $v_3$ . Let  $f$  be the function  
 that assigns the weight 2 to  $v_1$  and to every support vertex of  $T$  except of  $v_2$ ,  
 the weight 0 to  $v_2$  and every leaf except of  $v_1$  and the weight 1 to each vertex  
 280 of a minimum weakly connected dominating set of the underlying subtree of  
 $T$  that contains  $v_3$ . Then  $f$  is a WCRDF of  $T$  of weight  $\gamma_R^{wc}(T) = \frac{5}{6}n$ , which  
 proves statement 2.

**Theorem 13.** *If  $T$  is a tree of order  $n \geq 3$ , then*

$$\gamma_R^{wc}(T) \leq \frac{5}{6}n,$$

*with equality if and only if  $T \in \mathcal{F}$ .*

PROOF. We proceed by induction on the order  $n \geq 3$  of a tree  $T$ . If  $n = 3$ , then  
 285  $T = P_3$  and  $\gamma_R^{wc}(T) = 2 < \frac{5}{6}n$ . This establishes the base case.

Let  $n \geq 4$  and assume that if  $T'$  is a tree of order  $n'$ , where  $3 \leq n' < n$ , then  
 $\gamma_R^{wc}(T') \leq \frac{5}{6}n'$  with equality if and only if  $T' \in \mathcal{F}$ .

If  $T$  is a star, then the function that assigns the weight 2 to the central  
 vertex and the weight 0 to every leaf of the star is a WCRDF of  $T$  of weight 2,  
 290 and so  $\gamma_R^{wc}(T) = 2 < \frac{5}{6}n$ . Hence we may assume that  $\text{diam}(T) \geq 3$ .

If  $T = P_4$ , then  $\gamma_R^{wc}(T) = 3 < \frac{5}{6}n$ . If  $T$  is a double star which is not  $P_4$ ,  
 then the function that assigns the weight 2 to the two central vertices and the  
 weight 0 to every leaf of the double star is a WCRDF of  $T$  of weight 4, and so  
 $\gamma_R^{wc}(T) = 4 < \frac{5}{6}n$ . Hence we may assume that  $\text{diam}(T) \geq 4$ .

Let  $v_1$  and  $r$  be two vertices at maximum distance apart in  $T$ . Necessarily,  
 295  $v_1$  and  $r$  are leaves and  $d(v_1, r) = \text{diam}(T)$ . We now root the tree  $T$  at the  
 vertex  $r$ . Let  $v_2$  be the parent of  $v_1$ ,  $v_3$  parent of  $v_2$ ,  $v_4$  parent of  $v_3$  and  $v_5$   
 parent of  $v_4$ . We note that if  $\text{diam}(T) = 4$ , then  $r = v_5$ .



Suppose that  $d_T(v_2) \geq 3$ . Let  $T'$  be the tree obtained from  $T$  by deleting  
 300  $v_2$  and its children. Let  $T'$  have order  $n'$ , and so  $n' \leq n - 3$ . Since  $\text{diam}(T) \geq$   
 $4$ , we note that  $n' \geq 3$ . Applying the inductive hypothesis to the tree  $T'$ ,  
 $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n - 3)$ . Let  $f'$  be a  $\gamma_R^{wc}(T')$ -function. We can extend  $f'$   
 to the WCRDF of  $T$  by assigning the weight 2 to  $v_2$  and the weight 0 to the  
 children of  $v_2$ . The resulting function  $f$  has weight  $w(f) = w(f') + 2$ . Hence,  
 305  $\gamma_R^{wc}(T) \leq w(f) = w(f') + 2 \leq \frac{5}{6}(n - 3) + 2 < \frac{5}{6}n$ .

Therefore we may assume that every child of  $v_3$  in  $T$  is a leaf or has degree 2,  
 for otherwise the desired result follows. By symmetry, we assume that every  
 support vertex on a longest path of  $T$  is of degree 2.

If  $\text{diam}(T) = 4$ , then  $T$  is a spider graph, that is a tree with  $\text{diam}(T) = 4$ ,  
 310  $d_T(v_3) \geq 3$ ,  $d_T(v_2) = d_T(v_4) = 2$  and all other vertices with degree at most 2.  
 Denote by  $k_2$  the number of neighbours of  $v_3$  of degree 2. Note that  $k_2 \geq 2$  and  
 $n \geq 2k_2 + 1$ . Then the function that assigns the weight 2 to  $v_3$ , the weight 1  
 to each leaf at distance 2 from  $v_3$  and the weight 0 to every other vertex of the  
 spider is a WCRDF of  $T$  of weight  $2 + k_2$ , and so  $\gamma_R^{wc}(T) = 2 + k_2 < \frac{13}{6} + k_2 \leq$   
 315  $\frac{5}{6} + \frac{4}{6}k_2 + k_2 = \frac{5}{6}(1 + 2k_2) \leq \frac{5}{6}n$ . Hence we may assume that  $\text{diam}(T) \geq 5$ .

Let  $t_1$  be the number of children of  $v_3$  of degree 1 and let  $t_2$  be the number of  
 children of  $v_3$  of degree 2. Then  $t_2 \geq 1$ . Suppose that  $t_1 + t_2 \geq 2$ . Let  $T'$  be the  
 tree obtained from  $T$  by deleting  $v_3$  and its descendants. Let  $T'$  have order  $n'$ ,  
 and so  $n' = n - 2t_2 - t_1 - 1$ . Since  $\text{diam}(T) \geq 5$ , we note that  $n' \geq 3$ . Applying  
 320 the inductive hypothesis to the tree  $T'$ ,  $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n - 2t_2 - t_1 - 1)$ . Let  
 $f'$  be a  $\gamma_R^{wc}(T')$ -function. We can extend  $f'$  to the WCRDF of  $T$  by assigning  
 the weight 2 to  $v_3$ , the weight 1 to each descendant at distance 2 from  $v_3$   
 and the weight 0 to the children of  $v_3$ . The resulting function  $f$  has weight  
 $w(f) = w(f') + 2 + t_2$ . Hence,  $\gamma_R^{wc}(T) \leq w(f) = w(f') + 2 + t_2 \leq \frac{5}{6}(n - 2t_2 -$   
 325  $t_1 - 1) + 2 + t_2 = \frac{1}{6}(5n - 5t_1 - 4t_2 + 7)$  and since we supposed  $t_1 + t_2 \geq 2$ , we  
 obtain  $\gamma_R^{wc}(T) < \frac{5}{6}n$ .

Therefore we may assume that  $t_1 + t_2 = 1$ . Since  $v_3$  is on a longest path  
 of  $T$ , we conclude that  $t_1 = 0$  and  $t_2 = 1$ , which implies that  $d_T(v_3) = 2$ , for  
 otherwise the desired result follows.



330 Suppose now that  $d_T(v_4) = 2$ . Let  $T'$  be the tree obtained from  $T$  by deleting  $v_4$  and its descendants, that is  $v_1, v_2, v_3$  and  $v_4$ . Let  $T'$  have order  $n'$ , and so  $n' = n - 4$ . If  $n' \geq 3$ , then applying the inductive hypothesis to the tree  $T'$ ,  $\gamma_R^{wc}(T') \leq \frac{5}{6}n' = \frac{5}{6}(n - 4)$ . Moreover, Lemma 6 implies that  $\gamma_R^{wc}(T) = \gamma_R^{wc}(T') + 3$ . Hence,  $\gamma_R^{wc}(T) \leq \frac{5}{6}(n - 4) + 3 < \frac{5}{6}n$ . If  $n' \leq 2$ , then since  
 335  $\text{diam}(T) \geq 5$ ,  $n' = 2$  and thus  $T = P_6$ . In this case  $\gamma_R^{wc}(T) = \frac{5}{6}n$  and clearly  $P_6 \in \mathcal{F}$ .

Therefore in what follows we may assume that  $d_T(v_4) \geq 3$ .

Suppose that a child of  $v_4$ , say  $x$ , is a strong support vertex. Let  $T'$  be the tree obtained from  $T$  by deleting  $x$  and the children of  $x$ . Let  $T'$  have order  $n'$ , and  
 340 so  $n' \leq n - 3$ . Since  $\text{diam}(T) \geq 5$ , we note that  $n' \geq 3$ . Applying the inductive hypothesis to the tree  $T'$ ,  $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n - 3)$ . Let  $f'$  be a  $\gamma_R^{wc}(T')$ -function. We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 2 to  $x$  and the weight 0 to the children of  $x$ . The resulting function  $f$  has weight  $w(f) = w(f') + 2$ . Hence,  $\gamma_R^{wc}(T) \leq w(f) = w(f') + 2 \leq \frac{5}{6}(n - 3) + 2 < \frac{5}{6}n$ .

345 Therefore we may assume that every child of  $v_4$  is of degree 1 or 2, for otherwise the desired result follows.

Let  $t_1$  be the number of children of  $v_4$  of degree 1, let  $t_2$  be the number of children of  $v_4$  which are support vertices and let  $t_3$  be the number of children of  $v_4$  which are not support vertices. Then  $d_T(v_4) = t_1 + t_2 + t_3 + 1$ ,  $t_3 \geq 1$  and  
 350  $t_1 + t_2 + t_3 \geq 2$ .

Suppose  $t_2 = 0$ . Then  $t_1 + t_3 \geq 2$ . Let  $T'$  be the tree obtained from  $T$  by deleting  $v_4$  and its descendants. Let  $T'$  have order  $n'$ , and so  $n' = n - (1 + t_1 + 3t_3)$ . Since  $\text{diam}(T) \geq 5$ , we note that  $n' \geq 2$ .

If  $n' = 2$ , then  $n = 3 + t_1 + 3t_3$  and  $V(T') = \{v_5, r\}$ . Let  $f$  be a WCRDF  
 355 of  $T$  which assigns the weight 2 to  $v_4$  and to all support descendants of  $v_4$ , the weight 0 to the remaining descendants of  $v_4$  and to  $v_5$ , and the weight 1 to  $r$ . The resulting function  $f$  has weight  $w(f) = 3 + 2t_3$ . Hence,  $\gamma_R^{wc}(T) \leq w(f) = 3 + 2t_3$ . If  $t_1 \geq 1$ , then  $\gamma_R^{wc}(T) \leq 2 + \frac{1}{2}t_1 + \frac{5}{2}t_3 < \frac{5}{6}(3 + t_1 + 3t_3) = \frac{5}{6}n$ . If  $t_1 = 0$ , then  $t_3 \geq 2$  and  $\gamma_R^{wc}(T) \leq 2 + \frac{5}{2}t_3 < \frac{5}{6}(3 + 3t_3) = \frac{5}{6}n$ .

360 If  $n' \geq 3$ , then by applying the inductive hypothesis to the tree  $T'$ ,  $\gamma_R^{wc}(T') \leq$

$\frac{5}{6}n' \leq \frac{5}{6}(n - 1 - t_1 - 3t_3)$ . Let  $f'$  be a  $\gamma_R^{wc}(T')$ -function. If  $t_1 = 0$ , then  $t_3 \geq 2$  and we can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 1 to  $v_4$ , the weight 2 to all support descendants of  $v_4$ , and the weight 0 to the remaining descendants of  $v_4$ . The resulting function  $f$  has weight  $w(f) = w(f') + 1 + 2t_3$ .  
 365 Hence,  $\gamma_R^{wc}(T) \leq w(f) = w(f') + 1 + 2t_3 \leq \frac{5}{6}(n - 1 - 3t_3) + 1 + 2t_3 < \frac{5}{6}n$ .  
 If  $t_1 \geq 1$ , then we can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 2 to  $v_4$  and to all support descendants of  $v_4$ , and the weight 0 to the remaining descendants of  $v_4$ . The resulting function  $f$  has weight  $w(f) = w(f') + 2 + 2t_3$ .  
 Hence,  $\gamma_R^{wc}(T) \leq w(f) = w(f') + 2 + 2t_3 \leq \frac{5}{6}(n - 1 - t_1 - 3t_3) + 2 + 2t_3 < \frac{5}{6}n$ .

370 Therefore we may assume that  $t_2 \geq 1$ , for otherwise the desired result follows.

Suppose  $t_1 \geq 1$ . Then  $t_2 \geq 1$  and  $t_3 \geq 1$ . Let  $T'$  be the tree obtained from  $T$  by deleting  $v_4$  and its descendants. Let  $T'$  have order  $n'$ , and so  $n' = n - (1 + t_1 + 2t_2 + 3t_3)$ . Since  $\text{diam}(T) \geq 5$ , we note that  $n' \geq 2$ .

If  $n' = 2$ , then  $n = 3 + t_1 + 2t_2 + 3t_3$  and  $V(T') = \{v_5, r\}$ . Let  $f$  be a  
 375 WCRDF of  $T$  which assigns the weight 2 to  $v_4$  and to all support descendants of  $v_4$  at distance 2 from  $v_4$ , the weight 1 to  $r$  and the leaf descendants of  $v_4$  at distance 2 from  $v_4$ , and the weight 0 to the remaining descendants of  $v_4$  and to  $v_5$ . The resulting function  $f$  has weight  $w(f) = 3 + t_2 + 2t_3$ . Hence,  
 $\gamma_R^{wc}(T) \leq w(f) = 3 + t_2 + 2t_3 \leq 2 + \frac{3}{2}t_2 + \frac{5}{2}t_3 < \frac{5}{6}(3 + t_1 + 2t_2 + 3t_3) = \frac{5}{6}n$ .

If  $n' \geq 3$ , then by applying the inductive hypothesis to the tree  $T'$ ,  $\gamma_R^{wc}(T') \leq \frac{5}{6}n' \leq \frac{5}{6}(n - 1 - t_1 - 2t_2 - 3t_3)$ . Let  $f'$  be a  $\gamma_R^{wc}(T')$ -function. We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 2 to  $v_4$  and to all support descendants of  $v_4$  at distance 2 from  $v_4$ , the weight 1 to the leaf descendants of  $v_4$  at distance 2 from  $v_4$ , and the weight 0 to the remaining descendants of  $v_4$ . The resulting function  $f$  has weight  $w(f) = w(f') + 2 + t_2 + 2t_3$ . Hence,

$$\begin{aligned}
 \gamma_R^{wc}(T) &\leq w(f) = w(f') + 2 + 2t_3 \\
 &\leq \frac{5}{6}(n - 1 - t_1 - 2t_2 - 3t_3) + 2 + t_2 + 2t_3 \quad (10) \\
 &= \frac{1}{6}(5n + 7 - 5t_1 - 4t_2 - 3t_3).
 \end{aligned}$$

380 Since  $t_1 \geq 1$ ,  $t_2 \geq 1$ , and  $t_3 \geq 1$ , equation (10) implies that  $\gamma_R^{wc}(T) < \frac{5}{6}n$ .  
 Therefore we may assume that  $t_1 = 0$ , for otherwise the desired result follows.

Similarly, if  $t_2 \geq 2$  or  $t_3 \geq 2$ , equation (10) again implies that  $\gamma_R^{wc}(T) < \frac{5}{6}n$ . Therefore we may assume that  $t_2 = t_3 = 1$ , for otherwise the desired result follows.

385 Denote by  $x$  the child of  $v_4$  which is a support vertex different from  $v_5$  and let  $y$  be the child of  $x$ . Then  $(v_1, v_2, v_3, v_4, x, y)$  induce a path  $P_6$  in  $T$ . Let  $T'$  be the tree obtained from  $T$  by deleting  $v_4$  and its descendants. Let  $T'$  have order  $n'$ , and so  $n' = n - 6$ . Since  $\text{diam}(T) \geq 5$ , we note that  $n' \geq 2$ . If  $n' = 2$ , then  $n = 8$  and  $V(T') = \{v_5, r\}$ . Let  $f$  be a WCRDF of  $T$  which assigns  
390 the weight 2 to  $v_4$  and  $v_2$ , the weight 1 to  $r$  and  $y$ , and the weight 0 to the remaining vertices of  $T$ . The resulting function  $f$  has weight  $w(f) = 6$ . Hence,  $\gamma_R^{wc}(T) \leq w(f) = 6 < \frac{5}{6}n$ .

Hence  $n' \geq 3$ . By (10), if  $w(f') < \frac{5}{6}n'$ , then  $\gamma_R^{wc}(T) < \frac{5}{6}n$  and the result follows. Hence assume  $\gamma_R^{wc}(T') = \frac{5}{6}n'$ . Then by the induction hypothesis,  
395  $T' \in \mathcal{F}$ . Now it suffices to show, that  $v_5$  belongs to the underlying subtree of  $T'$ . Suppose to the contrary, that  $v_5$  is a support vertex or a leaf in  $T'$ .

Consider first the situation when  $v_5$  is a support vertex. Denote by  $z_1$  the leaf neighbour of  $v_5$  and by  $z_2$  the neighbour of  $v_5$  belonging to the underlying subtree of  $T'$ . Since the underlying subtree of  $T'$  belongs to the family  $\mathcal{E}$ ,  
400 Theorem 9 and Lemma 12 imply that there exists a  $\gamma_R^{wc}(T')$ -function  $f'$  such that the weight assigned to  $z_2$  is 1, the weight assigned to  $v_5$  is 2 *tu poprawi*  $v_5$  and the weight assigned to  $z_1$  is 0. We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 2 to  $v_2$  and  $v_4$ , the weight 1 to  $y$  and the weight 0 to  $v_1, v_3$  and  $x$ . Additionally, we change the weight of  $v_5$  to 0 and the weight of  
405  $z_1$  to 1. The resulting function  $f$  is a WCRDF of  $T$  and has weight  $w(f) = w(f') + 5 - 1 = \frac{5}{6}n' + 4 = \frac{5}{6}(n - 6) + 4 < \frac{5}{6}n$ . Therefore  $v_5$  is not a support vertex.

Suppose now  $v_5$  is a leaf. Denote by  $z$  the neighbour of  $v_5$  in  $T'$ . Then Lemma 12 implies that there exists a  $\gamma_R^{wc}(T')$ -function  $f'$  such that the weight  
410 assigned to  $v_5$  is 2 and hence we can assume that the weight assigned to  $z$  is 0. We can extend  $f'$  to a WCRDF of  $T$  by assigning the weight 2 to  $v_2$  and  $v_4$ , the weight 1 to  $y$  and the weight 0 to  $v_1, v_3$  and  $x$ . Additionally, we change the

weight of  $v_5$  to 0 and the weight of  $z$  to 1. The resulting function  $f$  is a WCRDF of  $T$  and has weight  $w(f) = w(f') + 5 - 1 = \frac{5}{6}n' + 4 = \frac{5}{6}(n - 6) + 4 < \frac{5}{6}n$ .

415 Therefore  $v_5$  is not a leaf.

We conclude that  $v_5$  belongs to the underlying subtree of  $T'$ . For this reason,  $T$  belongs to the family  $\mathcal{F}$ , which completes the proof.

Since the weakly connected Roman domination number and the outer-independent Roman domination number are equal for trees, we have the following

**Corollary 14.** *If  $T$  is a tree of order  $n \geq 3$ , then*

$$\gamma_{oiR}(T) \leq \frac{5}{6}n,$$

420 *with equality if and only if  $T \in \mathcal{F}$ .*

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