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Periodic expansion in determining minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms

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Abstract. We apply the representation of Lefschetz numbers of iterates in the form of periodic expansion to determine the minimal sets of Lefschetz periods of Morse–Smale diffeomorphisms. Applying this approach we present an algorithmic method of finding the family of minimal sets of Lefschetz periods for N_g, a non-orientable compact surfaces without boundary of genus g. We also partially confirm the conjecture of Llibre and Sirvent (J Diff Equ Appl 19(3):402–417, 2013) proving that there are no algebraic obstacles in realizing any set of odd natural numbers as the minimal set of Lefschetz periods on N_g for any g.

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1. Introduction

Let $f\colon M \to M$ be a Morse–Smale diffeomorphism, where M is a compact manifold without boundary. Morse–Smale diffeomorphisms, structurally stable and having relatively simple dynamics, constitute an important subclass of diffeomorphisms that were carefully studied during past decades (cf. [17] and the references therein).

One of the problems studied for Morse–Smale diffeomorphisms is the structure of the set of its minimal periods. The promising results in this direction may be obtained by the comparison (via Lefschetz–Hopf theorem) of the global behavior of f expressed by Lefschetz numbers of iterates $(L(f^n)_n)$ and local properties of f near periodic points represented by local fixed point indices of iterates. Basing on this relation $\mathrm{MPer}_L(f)$ called *minimal set of Lefschetz periods* is considered (cf. Definition 3.3). $\mathrm{MPer}_L(f)$ provides the information about the set of periodic points of f as it is the subset of minimal

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periods of f. The description of $\mathrm{MPer}_L(f)$ was performed (in dependence on the behavior of f_*, a map induced by f on homology groups) for the sphere S^n [21], orientable compact surfaces [29], non-orientable compact surfaces [28], disk with n holes [19], product of l-dimensional spheres [3], n-dimensional torus T^n [23]. Let us mention here that using similar approach the set of periods as well as so-called *minimal set of periods* of the Morse–Smale diffeomorphisms for T^2 was characterized in [22].

The method of determining $\mathrm{MPer}_L(f)$ in the above papers is based on the decomposition of Lefschetz zeta function into all possible factors and taking the exponents which are present in all such decompositions. However, this is sometimes not an easy task, especially in case when the dimensions of homologies are large. The purpose of this paper was to present another approach, which simplifies calculations a lot. Namely, we use the language of so-called periodic expansions, which enables us to represent the sequences of Lefschetz numbers and fixed point indices of iterates in the convenient form of the combination of some basic k-periodic sequences $(\mathrm{reg}_k)_n$ [1,25,30]. Using this apparatus we are able to reformulate the definition of $\mathrm{MPer}_L(f)$ and to express it by decompositions of Lefschetz numbers into sums of $(\mathrm{reg}_k)_n$, each of which represents, for odd k, a periodic orbit of minimal period k. As a consequence, we prove that $\mathrm{MPer}_L(f)$ is equal to the set of all non-zero odd k that appear in such decomposition (Theorem 6.2).

Our method is equivalent to the approach based on Lefschetz zeta functions but is computationally much simpler. We also provide the general form of sequences of Lefschetz numbers of iterates for Morse–Smale diffeomorphisms (Sect. 4.1), which could be a useful device for other investigations. In Sect. 7 we show the advantages of our approach considering self-maps of N_g, a non-orientable compact surface without boundary of genus g, and confirming the algebraic part of the problem of the realization of elements of $\mathrm{MPer}_L(f)$ for self-maps of N_g (Conjecture 7.1) from [28].

In the final part of the paper we also provide an algorithm for computing the minimal sets of Lefschetz periods for self-maps of N_g. We verify by a computer program $\mathrm{MPer}_L(f)$ for $g < 10$ found in [28] correcting one omission and compute $\mathrm{MPer}_L(f)$ for higher values of g.

Finally, let us mention that all the obtained results remain valid not only for Morse–Smale diffeomorphisms, but also for a larger class of maps, namely for maps with finitely many periodic points all of them hyperbolic (cf. Remark 6.3).

2. The class of Morse–Smale diffeomorphisms

In this section we recall some basic definitions related to the class of Morse–Smale diffeomorphisms.

Let $f\colon X \to X$, by $\mathrm{P}(f)$ we will denote the set of periodic points of f. Let us remind that x is a periodic point with minimal period n if $f^n(x) = x$ and $f^m(x) \neq x$ for $m < n$ (we will also call x an n-periodic point for short).

The set of all n-periodic points will be denoted by $\mathrm{P}_n(f)$ and the set of all minimal periods by $\mathrm{MPer}(f)$.

Definition 2.1. Let $f\colon M \to M$ be a C^1 diffeomorphism of a manifold M. A point $x \in M$ will be called non-wandering if for any neighbourhood U of x there exists an integer $n > 1$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points of f will be denoted by $\Omega(f)$.

Definition 2.2. Let $f\colon M \to M$ be a C^1 diffeomorphism of a manifold M and x be an n-periodic point. Then x is called hyperbolic if none of the eigenvalues of the derivative $Df(x)$ of f at x has modulus equal to 1.

Definition 2.3. For a hyperbolic n-periodic point x we define a *stable manifold* $W^s(x)$ as

$$W^s(x) = \{y \in X : f^{nk}(y) \to x \text{ as } k \to \infty\}$$

and an *unstable manifold* $W^u(x)$ as

$$W^u(x) = \{y \in X : f^{nk}(y) \to x \text{ as } k \to -\infty\}.$$

Definition 2.4. Let M be a compact manifold. A diffeomorphism $f\colon M \to M$ is *Morse–Smale* if

(i) $\Omega(f)$ is finite,
(ii) all periodic points are hyperbolic,
(iii) for each $x, y \in \mathrm{P}$, $W^s(x)$ and $W^u(y)$ have transversal intersections.

Remark 2.5. It follows from (i) that the set $\Omega(f)$ consists only of periodic points, i.e. $\Omega(f) = \mathrm{P}(f)$.

3. Lefschetz numbers of iterates and Lefschetz zeta function

First we remind the definition of Lefschetz numbers of iterates; for simplicity we will consider homology with rational coefficients.

Let K be a CW-complex of dimension m with the homology groups $H_i(K;\mathbb{Q})$, where $i = 0, 1, \ldots, m$. In case the homology coefficients are equal to $\mathbb{Q}$ the groups $H_i(K;\mathbb{Q})$ are finite dimensional linear spaces over $\mathbb{Q}$. For a self-map f of K we denote by f_{*i} the linear map induced by f on $H_i(K;\mathbb{Q})$ and by f_* the self-map $\bigoplus_{i=0}^m f_{*i}$ of $\bigoplus_{i=0}^m H_i(K;\mathbb{Q})$. The Lefschetz number $L(f^n)$ of f^n is then equal to

$$L(f^n) = \sum_{i=0}^{m} (-1)^i \operatorname{tr}(f^n)_{*i}, \tag{3.1}$$

where $\operatorname{tr}(f^n)_{*i}$ is the trace of the integer matrix representing $(f^n)_{*i}{:}H_i(K;\mathbb{Q}) \to H_i(K;\mathbb{Q})$. Notice that if A is a matrix of f_{*i}, then A^n is a matrix of $(f^n)_{*i}$, representing the homomorphism induced on $H_i(K;\mathbb{Q})$ by f^n (cf. [25]).

Lefschetz zeta function, Z_f, which is a useful tool in periodic points theory, codes information on the whole sequence of Lefschetz numbers of iterates:

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right). \tag{3.2}$$

We will also consider zeta function associated with a given integer sequence $(a_n)_n$, which will be defined in an analogous way:

$$Z_{a_n}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} t^n\right). \tag{3.3}$$

Remark 3.1. An alternative formula for the Lefschetz zeta function may be given by a use of eigenvalues of f_*, namely if λ_i is an eigenvalue of f_*, taken with algebraic multiplicities k_i, then

$$Z_f(t) = \prod_i (1 - \lambda_i z)^{(-1)^{m_i+1} k_i}, \tag{3.4}$$

where m_i denotes the index of the homology group associated with λ_i [25].

It turns out that the Lefschetz zeta function for Morse–Smale diffeomorphisms has very special form, first described by Franks in [9].

Let M be a smooth manifold and x be a hyperbolic p-periodic point of a map $f\colon M \to M$. Denote by E^{un}_x the subspace of the tangent TM_x spanned by eigenvectors of $Df^p(x)$ which correspond to eigenvalues which are greater than one in absolute value. Let γ be an orbit of x, we define $u = \dim E^{un}_x$ and Δ, the orientation type of γ, as $+1$ if $Df^p(x)$ preserves the orientation and -1 if it reverses the orientation.

By Σ we denote *periodic data*, i.e. a collection of all the triples (p, u, Δ) which corresponds to periodic orbits of f.

Theorem 3.2. [9] *Let M be a closed manifold, and $f\colon M \to M$ be a C^1 map with finitely many periodic points, all of them hyperbolic; then*

$$Z_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}, \tag{3.5}$$

where (p, u, Δ) belongs to periodic data of f.

Definition 3.3. Let $Z_f(t) \neq 1$, the *minimal set of Lefschetz periods* of f is defined as

$$\mathrm{MPer}_L(f) := \bigcap \{r_1, \ldots, r_{N(f)}\},$$

where the intersection is considered over all the possible expressions of $Z_f(t)$ in the following form:

$$Z_f(t) = \prod_{i=1}^{N(f)} (1 + \Delta_i t^{r_i})^{m_i}, \tag{3.6}$$

where $\Delta_i \in \{-1, 1\}$, r_i are positive integer values, m_i are nonzero integers and $N(f)$ is a positive integer that depends on the function f. For $Z_f(t) = 1$ we define $\mathrm{MPer}_L(f) := \emptyset$.

The importance of the minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms results from the following fact, which is a straightforward consequence of Theorem 3.2

$$\mathrm{MPer}_L(f) \subset \mathrm{MPer}(f). \tag{3.7}$$

Now, we introduce the notion of quasi-unipotent maps.

Definition 3.4. A rational linear transformation is called quasi-unipotent if their eigenvalues are roots of unity. We will call a continuous map $f\colon M \to M$ *quasi-unipotent* if the maps f_{*_k} are quasi-unipotent for $0 \le k \le m$, where m is the dimension of the manifold M.

The following fact makes it possible to determine Lefschetz zeta function for a Morse–Smale diffeomorphism in a relatively easy way.

Proposition 3.5 [32]. Let f be a Morse–Smale diffeomorphism of a compact manifold, then f is quasi-unipotent.

Due to Proposition 3.5 and the formula (3.4) Lefschetz zeta function for a Morse–Smale diffeomorphism may be expressed as a rational function with the nominator and denominator being a product of cyclotomic polynomials, whose degrees are bounded by the dimensions of homology spaces. As for a given n there is a finite number of cyclotomic polynomials of degree $\le n$, for a given manifold M there is a finite number of different forms of zeta functions on M. This observation was a base for a strategy of finding minimal set of Lefschetz periods for all Morse–Smale diffeomorphisms on a given manifold M, which may be described in the following steps. For a given Morse–Smale diffeomorphism f find zeta functions $Z_f(t)$ (expressed in terms of cyclotomic polynomials), next determine all their decompositions into the products of elements of the form (3.6), and finally take the common part of coefficients r_i (which are related to minimal periods of f) over all such products. We proposed below the alternative strategy which is based on representing $(L(f^n))_n$ as the sum of basic periodic sequences and decomposing it into sum of sequences related to periodic orbits of the considered map.

4. Periodic expansion of Lefschetz numbers of iterates

Definition 4.1. A sequence of integer numbers $(a_n)_{n=1}^{\infty}$ will be called a *Dold sequence* if the following congruences (called Dold congruences or Dold relations) are fulfilled:

$$\sum_{k|n} \mu(k) a_{\frac{n}{k}} \equiv 0 \pmod{n} \quad \text{for each } n \ge 1, \tag{4.1}$$

where $\mu : \mathbb{N} \to \mathbb{Z}$ is the classical Möbius function, given by the following formula:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ for different primes } p_i; \\ 0 & \text{otherwise.} \end{cases}$$

Dold sequences play important role not only in dynamics but also in number theory (cf. [2,16]). There is a convenient way of writing down a Dold sequence by using so-called periodic expansion, i.e. representing the sequence as a combination of some basic periodic sequences.

Definition 4.2. Let k be a fixed natural number. We define

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Thus, reg_k is the periodic sequence:

$$(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots),$$

where the non-zero entries appear for indices divisible by k.

Proposition 4.3 (Proposition 2.7 in [30]). *Any arithmetic function $(a_n)_{n=1}^{\infty}$ can be written uniquely in the following form of a periodic expansion:*

$$a_n = \sum_{k=1}^{\infty} b_k \,\mathrm{reg}_k(n), \text{ where } b_n = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right) a_k \in \mathbb{Q}. \tag{4.2}$$

Moreover, $(a_n)_n$ is integral valued and satisfies Dold congruences iff $b_k \in \mathbb{Z}$ for every $k \in \mathbb{N}$.

Remark 4.4. By Proposition 4.3 every Dold sequence is an integral combination (possibly infinite) of basic sequences reg_k (with coefficients b_k which are called Dold coefficients).

There is a deep relation between sequences $(a_n)_n$ and $(b_n)_n$ that appear in Proposition 4.3, which may be expressed in the language of formal power series and formal products.

Theorem 4.5 (Theorem 2 in [6]). *Let $(a_n)_n$, $(b_n)_n$ be some complex valued sequences. Then $(a_n)_n$ is a Dold sequence with Dold coefficients $(b_n)_n$ if and only if*

$$\exp\left(-\sum_{n=1}^{\infty} a_n \frac{x^n}{n}\right) = \prod_{n=1}^{\infty} (1 - x^n)^{b_n}. \tag{4.3}$$

A sequence of fixed point indices of iterates turned out to be a Dold sequence.

Theorem 4.6 [5]. *The sequence of fixed point indices of iterates $(\mathrm{ind}(f^n))_n$ is a Dold sequence (provided it is well-defined). As a consequence it has a periodic expansion of the form (4.2) with integral coefficients.*

The language of periodic expansion is a convenient tool that has been recently used in different contexts such as: study of the fixed point indices of an iterated map [4,24,34] and minimization of the number of periodic points in homotopy class [13,14,30].

In particular the sequence $(L(f^n))_n$ is also Dold sequence (cf. [33] for different proofs of this fact). As a consequence, by Remark 4.4, we get

$$L(f^n) = \sum_{k=1}^{\infty} b_k \,\mathrm{reg}_k(n), \tag{4.4}$$

where b_k are integers.

Remark 4.7. The coefficients b_k in the periodic expansion of Lefschetz numbers in the formula (4.4) are called *Lefschetz numbers for periodic points* (denoted also as $l(f^k)$) and play important role in the periodic points theory (cf. [26]). In particular, the determination of the set $L = \{k \in \mathbb{N} : b_k \neq 0\}$ provides valuable knowledge about the structure of periodic points obtained by confronting the information carried by L with the local information about periodic points (expressed by fixed point indices at orbits). This comparison could be done either straighforwardly (Lefschetz–Hopf theorem) or by zeta function or equivalently by periodic expansion. During the past 30 years the program of determination of the set L for different types of manifolds (in terms of action of induced maps on homology groups) and its application to various classes of maps has been realized. Among other cases this program was accomplished for:

- transversal maps of a compact manifold M such that its rational homology is $H_j(M;\mathbb{Q}) = \mathbb{Q}$ if $j \in J \cup \{0\}$, and $H_j(M;\mathbb{Q}) = \{0\}$ otherwise, where J is a subset of $\mathbb{N}$ with cardinality 1, 2 or 3 [27],
- transversal maps of a compact manifold M such that its rational homology is $H_0(M;\mathbb{Q}) = \mathbb{Q}$, $H_1(M;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ and $H_k(M;\mathbb{Q}) = \{0\}$ for $k \neq 0, 1$ [18],
- C^1 maps of rational exterior spaces and simple rational Hopf spaces [10,12] (see also [11]), and transversal maps of some simple rational Hopf spaces [20],
- holomorphic maps of some complex compact manifolds [8].

4.1. Periodic expansion of Lefschetz numbers of Morse–Smale diffeomorphisms

For the considered class of maps, i.e. Morse–Smale diffeomorphisms periodic expansion of Lefschetz numbers may be expressed by a use of roots of cyclotomic polynomials.

Definition 4.8. The dth cyclotomic polynomial $\omega_d(z)$ is defined by the following formula:

$$\omega_d(z) = \prod_{\varepsilon \in U_d} (z - \varepsilon), \tag{4.5}$$

where U_d denotes the set of all primitive dth roots of unity.

Let us remark that $\omega_d(z)$ is an irreducible polynomial with integer coefficients of degree $\varphi(d)$, where φ is the Euler function (i.e. $\varphi(d)$ is the number of positive integers less than or equal to d that are co-prime to d). For example $\omega_3(z) = \frac{1-z^3}{1-z} = 1 + z + z^2$, $\mathrm{Deg}(\omega_3(z)) = 2 = \varphi(3)$.

Now, our aim is to establish the coefficients b_k for the periodic expansion of Lefschetz numbers of iterated quasi-unipotent maps.

Let $\varepsilon_1, \ldots, \varepsilon_{\varphi(d)}$ be the all dth primitive roots of unity. For a given d we define

$$L_d(n) = \varepsilon_1^n + \cdots + \varepsilon_{\varphi(d)}^n. \tag{4.6}$$

The cyclotomic polynomial $\prod_{i=1}^{\varphi(d)} (z - \varepsilon_i)$ has integer coefficients; thus $L_d(n)$ is equal to $\operatorname{tr} A^n$, for some integer matrix A, having the cyclotomic polynomial as the characteristic polynomial. On the other hand, the sequence $\operatorname{tr} A^n$ for an integer matrix A always satisfies Dold relations (see Theorem 3.1.4 in [25]). As a consequence, by Theorem 1.2 $L_d(n)$ could be uniquely represented as an integral combination of basic sequences reg_k.

Let us consider a Morse–Smale diffeomorphism f of a closed manifold M of dimension m. Let $e_i(\lambda)$ be the algebraic multiplicity of λ as an eigenvalue of f_{*i}. Define

$$e(\lambda) := \sum_{i=0}^{m} (-1)^i e_i(\lambda).$$

We will call an eigenvalue $\lambda \neq 0$ *essential* provided $e(\lambda) \neq 0$. It is obvious that only essential eigenvalues give the contribution to $\{L(f^n)\}_{n=1}^{\infty}$.

Denote by $\sigma_{es}(f)$ the set of essential eigenvalues of f. We define

$$e(d) = \sum_{\lambda \in U_d \cap \sigma_{es}(f)} e(\lambda).$$

Notice that the essential dth primitive roots of unity appear in groups of $\varphi(d)$ elements, contributing $\frac{e(d)}{\varphi(d)} L_d(n)$ to $L(f^n)$. As a result we get:

$$L(f^n) = \sum_{d} \frac{e(d)}{\varphi(d)} L_d(n). \tag{4.7}$$

As a consequence, to find the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$ it is enough to determine the expansions of each $\{L_d(n)\}_{n=1}^{\infty}$.

We represent $\{L_d(n)\}_{n=1}^{\infty}$ as an integral combination of basic sequences reg_k:

$$L_d(n) = \sum_{k=1}^{\infty} b_k^d \, \mathrm{reg}_k(n), \tag{4.8}$$

where b_k^d are integers, d is fixed.

The following theorem gives the value of b_k^d, and thus allows us to determine the periodic expansion of $\{L_d(n)\}_{n=1}^{\infty}$. It was proved in [15] in an elementary but rather long way. Below we will give much simpler proof that is based on Theorem 4.5.

Theorem 4.9. *The coefficient b_k^d of the periodic expansion of $\{L_d(n)\}_{n=1}^{\infty}$ is equal to:*

$$b_k^d = \begin{cases} \mu(\frac{d}{k}) & \text{if } k \mid d, \\ 0 & \text{if } k \nmid d. \end{cases} \tag{4.9}$$

Proof. If $a_n = \sum_{i=1}^{r} m_i \lambda_i^n$, where m_i are integers and λ_i are some complex numbers, then (cf. [25] (3.1.26)):

$$Z_{a_n}(z) = \prod_{i=1}^{r} (1 - \lambda_i z)^{-m_i}.$$

On the other hand, in our case each $\lambda_i \in U(d)$, where $U(d)$ denotes the set of all primitive roots of unity of degree d and $r = \varphi(d)$. As $\varphi(d)$ is always even for $d > 1$, $U(d)$ is closed for taking the inverses and in our case each $m_i = 1$, we get:

$$Z_{L_d(n)}(z) = \prod_{i=1}^{\varphi(d)} (\lambda_i z - 1)^{-1} = \prod_{i=1}^{\varphi(d)} \left(z - \frac{1}{\lambda_i} \right)^{-1} = \prod_{i=1}^{\varphi(d)} (z - \lambda_i)^{-1} = \omega_d(z)^{-1}.$$

On the other hand, the following well-known fact holds (cf. for example [7]):

$$\omega_d(z)^{-1} = \prod_{k|d} (1 - z^k)^{-\mu(\frac{d}{k})}. \tag{4.10}$$

Applying the formula (4.3) for $a_n = L_d(n)$ we get that

$$Z_{L_d(n)}(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-b_k^d}. \tag{4.11}$$

Comparing the formulas (4.10) and (4.11) we get the equality (4.9). □

5. Periodic expansion of indices of iterates at periodic points for Morse–Smale diffeomorphisms

Let us denote for short the derivative of f at $x_0 \in \mathrm{Fix}(f)$ by $D = Df(x_0)$ and by $\sigma(D)$ its spectrum. By σ_+ we denote the number of real eigenvalues of D greater than 1 and by σ_- the number of real eigenvalues of D less than -1, in both cases counting with multiplicity. We consider hyperbolic maps (i.e. maps having only hyperbolic periodic points). For a fixed point x_0 we get [4]:

$$\mathrm{ind}(f^n, x_0) = \operatorname{sgn} \det(Id - D^n) = \begin{cases} (-1)^{\sigma_+} & \text{if } n \text{ is odd,} \\ (-1)^{\sigma_+ + \sigma_-} & \text{if } n \text{ is even.} \end{cases} \tag{5.1}$$

In dependence of parity of the values σ_+, σ_- and n, we obtain four possibilities:

$$\mathrm{ind}(f^n, x_0) = \begin{cases} \mathrm{reg}_1(n), \\ -\mathrm{reg}_1(n), \\ \mathrm{reg}_1(n) - \mathrm{reg}_2(n), \\ -\mathrm{reg}_1(n) + \mathrm{reg}_2(n). \end{cases} \tag{5.2}$$

Let us consider a point x_0 with minimal period k and its orbit $O_{x_0} = \{x_0, f(x_0), \ldots, f^{k-1}(x_0)\}$. Then by (5.2) we obtain that there are only four possible forms of indices for the orbit:

$$\mathrm{ind}(f^n, O_{x_0}) = \begin{cases} \mathrm{reg}_k(n), \\ -\mathrm{reg}_k(n), \\ \mathrm{reg}_k(n) - \mathrm{reg}_{2k}(n), \\ -\mathrm{reg}_k(n) + \mathrm{reg}_{2k}(n). \end{cases} \tag{5.3}$$

There is a one-to-one correspondence between the sequences in (5.3) and the forms of the Lefschetz zeta functions, namely the following relations hold:

Theorem 5.1.

$$(A)\ \text{ for } \ a_n = l\,\mathrm{reg}_k(n), \ \ Z_{a_n}(t) = \frac{1}{(1-t^k)^l}.$$

$$(B)\ \text{ for } \ a_n = l(\mathrm{reg}_k(n) - \mathrm{reg}_{2k}(n)), \ \ Z_{a_n}(t) = (1+t^k)^l.$$

Proof. We will consider each case separately:

(A) Let $a_n = l\,\mathrm{reg}_k(n)$; by Theorem 4.5 we have the following relation between sequences a_n and b_n:

$$Z_{a_n}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{l\,\mathrm{reg}_k(n)}{n} t^n\right) = \prod_{n=1}^{\infty} (1-t^n)^{-l\cdot b_n},$$

where $b_n = \frac{1}{n}\sum_{s|n} \mu(s) a_{\frac{n}{s}}$ and we get by the definition of $(\mathrm{reg}_k)_n$ that b_n is equal to 1 for $n = k$, and zero otherwise, which gives us the desired form of $Z_{a_n}(t)$.

(B) Let $a_n = l(\mathrm{reg}_k(n) - \mathrm{reg}_{2k}(n))$; using the result from the previous case we have

$$\begin{aligned} Z_{a_n}(t) &= \exp\left(l\sum_{n=1}^{\infty} \frac{\mathrm{reg}_k(n)}{n} t^n\right) \exp\left(-l\sum_{n=1}^{\infty} \frac{\mathrm{reg}_{2k}(n)}{n} t^n\right) = \frac{(1-t^{2k})^l}{(1-t^k)^l} \\ &= (1+t^k)^l. \end{aligned}$$

□

6. The minimal set of Lefschetz periods expressed by periodic expansions

By Lefschetz–Hopf formula we may represent the sequence of Lefschetz numbers in the following form:

$$L(f^n) = \sum_{x\in \mathrm{P}(f)} \mathrm{ind}(f^n, x) = \sum_k \sum_{O\in \mathrm{Orb}_k(f)} \mathrm{ind}(f^n, O), \tag{6.1}$$

where $\mathrm{Orb}_k(f)$ denotes the set of k-orbits of f and each O has the form (5.3).

For $j \in \{1, 2, 3, 4\}$ let us denote by $c^j_{r_i}$ an integer sequence that has one of the forms of fixed point indices of an r_i-orbit in (5.3), i.e.

$$c^j_{r_i}(n) = \begin{cases} \mathrm{reg}_{r_i}(n), & j = 1, \\ -\mathrm{reg}_{r_i}(n), & j = 2, \\ \mathrm{reg}_{r_i}(n) - \mathrm{reg}_{2r_i}(n), & j = 3, \\ -\mathrm{reg}_{r_i}(n) + \mathrm{reg}_{2r_i}(n), & j = 4. \end{cases} \tag{6.2}$$

Proposition 6.1. *The following formula holds:*

$$\mathrm{MPer}_L(f) = \bigcap \{r_1, r_2, \ldots, r_{N(f)}\}, \tag{6.3}$$

where the intersection is taken over all possible decompositions of $(L(f^n))_n$ *given by the formula* (6.1) *into the sum of sequences of the form* (6.2).

Proof. Each representation of $Z_f(t)$ in the form (3.6) (with the factors $(1+\Delta_i t^{r_i})^{m_i}$) is equivalent by Theorem 5.1 to the unique representation of $(L(f^n))_n$ as the sum of the sequences of the form (6.2) (with $k = r_i$ representing the minimal period of an orbit). □

Theorem 6.2. *The following formula holds:*

$$\mathrm{MPer}_L(f) = \{k : b_k \neq 0 \text{ in the periodic expansion (4.4) of } L(f^n))_n\} \cap Odd, \tag{6.4}$$

where Odd denotes the set of odd natural numbers.

Proof. It is obvious that for odd k for which $b_k \neq 0$ in (4.4) we get that $k \in \mathrm{MPer}(f)$. We show that there are no even numbers in $\mathrm{MPer}(f)$. Assume, contrary to our claim, that there is an even $r_i = 2u$ in $\mathrm{MPer}_L(f)$. Then for every decomposition of $L(f^n)$ into (6.2) there must be the term $c^j_{2u}(n)$ for some j. However, for $j = 1$ we have

$$c^1_{2u}(n) = \mathrm{reg}_{2u}(n) = (\mathrm{reg}_{2u}(n) - \mathrm{reg}_u(n)) + \mathrm{reg}_u(n) = c^4_u(n) + c^2_u(n).$$

For $j = 3$ there is:

$$\begin{aligned} c^3_{2u}(n) &= \mathrm{reg}_{2u}(n) - \mathrm{reg}_{4u}(n) = (\mathrm{reg}_{2u}(n) - \mathrm{reg}_u(n)) + \mathrm{reg}_u(n) - \mathrm{reg}_{4u}(n) = \\ &= c^3_u(n) + c^1_u(n) + c^1_{4u}(n). \end{aligned}$$

Analogously, for $j = 2, 4$ we obtain another decompositions of $c^j_{2u}(n)$, which contradicts our assumption. □

Remark 6.3. Theorem 6.2 holds in fact for a larger class of maps, namely for maps having finitely many periodic points, all of them hyperbolic. For a map f in this class the sequence $(L(f^n))_n$ is bounded (cf. [4]) and thus the only non-zero eigenvalues of f_* which give the contribution to $(L(f^n))_n$ (i.e. have different multiplicity in odd and even homology) are roots of unity [1]. As a consequence, the analysis of such maps reduces to quasi-unipotent case.

7. Applications: the minimal sets of Lefschetz periods on N_g, a non-orientable compact surface without boundary of genus g

Let us remind that N_g is homeomorphic with the connected sum of g real projective planes and its homology groups are the following: $H_0(M, \mathbb{Q}) = \mathbb{Q}$, $H_2(M, \mathbb{Q}) = 0$ and

$$H_1(M, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{g-1}.$$

7.1. Realization of the minimal set of Lefschetz periods on N_g

Conjecture 7.1. *Llibre and Sirvent [28] formulated the following conjecture: can any finite set of odd positive integers be the minimal set of Lefschetz periods for a C^1 Morse–Smale diffeomorphism on some non-orientable compact surface without boundary with a convenient genus?*

We will show that there are no algebraic obstacles on the homology which would prevent the validity of Conjecture 7.1.

Let d be a degree of some primitive root of unity; we take maximal m such that $2m-1 \leq d$ and denote by c^d the vector $[b_1^d, b_3^d, \dots, b_{2m-1}^d, 0, \dots]$, where b_k^d is given by the formula (4.9), i.e. $b_k^d = \begin{cases} \mu(\frac{d}{k}) & \text{if } k \mid d, \\ 0 & \text{if } k \nmid d. \end{cases}$.

Observe that for odd k there is $b_k^d = -b_k^{2d}$ and as a consequence,

$$c^d = -c^{2d}. \tag{7.1}$$

Theorem 7.2. *The set* $\mathbb{A}_m = \{c^1, c^3, c^5, \dots, c^{(2m-1)}\}$ *of* m *vectors is a basis of* $\mathbb{Z}^m$.

Proof. We identify c^d with $[b_1^d, b_3^d \dots, b_{2m-1}^d]$. Consider the matrix A composed of vectors c^i in rows:

$$\begin{array}{l} \\ c^1 \leftrightarrow \mathrm{reg}_1 \\ c^3 \leftrightarrow -\mathrm{reg}_1 + \mathrm{reg}_3 \\ c^5 \leftrightarrow -\mathrm{reg}_1 + \mathrm{reg}_5 \\ \vdots \\ c^{2m-1} \leftrightarrow \dots \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_3 & b_5 & \dots & b_{2m-1} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ . & . & . & . & 1 \end{bmatrix} \end{array} = A.$$

Note that $b_d^d = \mu(\frac{d}{d}) = 1$ and for $i > d$ we have $b_i^d = 0$. As a consequence, the matrix A is lower triangular and $\det A = 1$, so the set $\mathbb{A}_m$ is a basis of $\mathbb{Z}^m$. □

Theorem 7.3. *For any finite set* $\mathcal{G}_m = \{a_1, a_3, \dots, a_{2m-1}\}$ *of* m *integers there exists quasi-unipotent map* $f_* : H_*(N_g) \to H_*(N_g)$ *such that*

(a) $L(f_*^n) = \mathrm{tr}(f_{*_0}^n) - \mathrm{tr}(f_{*_1}^n) = -\sum_{i=1}^{2(2m-1)} a_i \,\mathrm{reg}_i(n)$,
(b) $\mathrm{MPer}_L(f_*) = \{i : a_i \neq 0 \wedge a_i \in \mathcal{G}_m\}$,

Proof. (a) Remind that by the formula (4.9) Lefschetz numbers of self-map of N_g have the form

$$L(f^n) = \mathrm{tr}(f_{*_0}^n) - \mathrm{tr}(f_{*_1}^n) = \mathrm{reg}_1 - \sum_i \sum_{k=1}^{\infty} b_k^{d_i} \,\mathrm{reg}_k(n).$$

We are searching for a homomorphism f_* satisfying for each odd i the equality:

$$L(f_*^n) = -\sum_{i=1}^{2m-1} a_i \,\mathrm{reg}_i(n) = \underbrace{\mathrm{reg}_1(n)}_{\mathrm{tr}(f_{*_0}^n)} - \underbrace{\left((a_1+1)\,\mathrm{reg}_1(n) + \sum_{i=2}^{2m-1} a_i \,\mathrm{reg}_i(n) \right)}_{\mathrm{tr}(f_{*_1}^n)}.$$

Consider the vector $[a_1+1, a_3, \dots, a_{2m-1}]$ of odd indices of coefficients coming from $tr(f_{*_1})$. By Theorem 7.2 there exists unique representation of this vector in the basis $\{c^1, c^3, \dots c^{2m-1}\}$ such that:

$$[a_1+1, a_3, \dots, a_{2m-1}] = \sum_{i=1}^{m} m_i c^{2i-1} = \sum_{i=1}^{m} |m_i| c^{s(i)}$$

for $s(i) = \begin{cases} 2i-1 & \text{if } m_i > 0, \\ 2(2i-1) & \text{if } m_i < 0, \end{cases}$ where in the last equality we used the relation (7.1). Now we define the map f_{*_1} represented by a diagonal matrix of dimension

$$D = \sum_{i=1}^{m} |m_i|\varphi(2i-1), \tag{7.2}$$

which have on the diagonal: $|m_i|$ copy of all primitive $(2i-1)$-roots of unity for each $m_i \geqslant 0$, and $|m_i|$ copies $2(2i-1)$-roots of unity for each $m_i < 0$, where $i = 1, \ldots, m$.

(b) follows directly from (a) and Theorem 6.2.

□

Remark 7.4. Let us notice that Conjecture 7.1 coincides with the item (b) of Theorem 7.3, while item (a) is more general. What is more, it provides the bound for the dimension of the realization (7.2) as well as the bound for the highest degree of roots of unity needed in the realization $(2(2m-1))$.

Corollary 7.5. *Lefschetz zeta function of the realization f_* has the following form:*

$$Z_{f_*}(t) = \frac{1}{t-1} \prod_i \left(\prod_{q|s(i)} (1-z^q)^{-\mu\left(\frac{s(i)}{q}\right)} \right)^{|m_i|}. \tag{7.3}$$

Indeed, observe that $Z_{f_*}(t) = Z_{l_n}(t)$, where $l_n = 1 - \sum_i |m_i| L_{s(i)}(n)$. Thus

$$Z_{f_*}(t) = \frac{1}{t-1} \prod_i \left(Z_{L_{s(i)}}(t)\right)^{|m_i|},$$

and by Theorem 4.11 we get the formula (7.3).

Remark 7.6. There is a "topological" part of Conjecture 7.1 which is still unsolved. Namely, it remains an open question whether the quasi-unipotent map f_* found in Theorem 7.3 could be realized as a map that is induced by some Morse–Smale diffeomorphism on N_g.

7.2. Algorithmic approach to determine the minimal sets of Lefschetz periods for self-maps of N_g

In this section we describe a simple algorithm that enables us to determine the minimal sets of Lefschetz periods for self-maps of N_g. As an application we use a computer program based on this algorithm to verify $\mathrm{MPer}_L(f)$ for $g < 10$ found in [28] and to compute $\mathrm{MPer}_L(f)$ for $g \geq 10$.

We will sketch below the main scheme of the algorithm, while the detailed description is placed in Appendix. The program for computation of $\mathrm{MPer}_L(f)$ based on the algorithm (in Mathematica) is available on the webpage of Myszkowski.[1]

We assume that $g > 1$; otherwise the matrix f_{*_1} has the dimension 0 and the only possibility is $\mathrm{MPer}_L(f) = \{1\}$. The dimension of the matrix f_{*_1}

[1] http://www.mif.pg.gda.pl/homepages/amyszkowski/.

is $g-1$ and all eigenvalues are primitive d_i-roots of unity grouped in $\varphi(d_i)$ elements. Consequently $g-1=\sum_i \varphi(d_i)$.

We define a finite family P of partitions P_j of the number $g-1$ into possible values of φ, i.e.

$$P=\left\{P_j=\bigcup_i \{d_i\}:\ g-1=\sum_i \varphi(d_i),\ d_i\in\mathbb{N}\right\}.$$

The next lemma allows us to find the bound for the number of elements in the family P.

Lemma 7.7 (cf. [31]). *Let φ be the Euler function. For all $n\in\mathbb{N}$, with the exception of $n=2,4,6,10,12,18,30$ we have*

$$\varphi(n)\geqslant n^{\log_3 2}.$$

Corollary 7.8. *Let φ be the Euler function. For all $n\in\mathbb{N}$ we have*

$$\varphi(n)\geqslant \left(\frac{n}{2}\right)^{\log_3 2}. \tag{7.4}$$

By application of the inequality (7.4) we get that:

$$P=\left\{\bigcup_i \{d_i\}:\ g-1=\sum_i \varphi(d_i),\ d_i\leqslant 2(g-1)^{\log_2 3},\ d_i\in\mathbb{N}\right\}.$$

We can associate with each partition P_j (i.e. each set of degrees) the corresponding sequence of Lefschetz numbers in the form of a periodic expansion:

$$L(f^n)=1-\sum_{d_i\in P_j} L_{d_i}(n)=\mathrm{reg}_1(n)-\sum_{d_i\in P_j}\sum_k b_k^{d_i}\,\mathrm{reg}_k(n),$$

where $b_k^{d_i}$ are defined by the formula (4.9) of Theorem 4.9. By Theorem 6.2 the set of minimal periods M_{P_j} for a map corresponding to a group of degrees P_j is

$$M_{P_j}=\left\{k:\sum_{i=1}^{|P_j|} a_k^{d_i}\neq 0,\ 2\nmid k,\ d_i\in P_j\right\},\quad \text{where}\quad a_1^{d_i}=b_1^{d_i}-1,$$
$$a_k^{d_i}=b_k^{d_i}\ \text{for}\ k>1.$$

Finally, the family of the minimal sets of Lefschetz periods $\mathrm{MPer}_L(f)$ over all f being Morse–Smale diffeomorphisms of N_g is given as follows:

$$\mathrm{MPer}_L=\bigcup_{j=1}^{|P|}\{M_{P_j}\}.$$

Table 1. The family of all minimal sets of Lefschetz periods for Morse–Smale diffeomorphism of N_4

P_j	$L(f^n)$	$M_{P_j} = \mathrm{MPer}_L(f)$
$\{1,3\}$	$\mathrm{reg}_1(n) - \mathrm{reg}_3(n)$	$\{1,3\}$
$\{1,4\}$	$\mathrm{reg}_2(n) - \mathrm{reg}_4(n)$	$\emptyset$
$\{1,6\}$	$-\mathrm{reg}_1(n) + \mathrm{reg}_2(n) + \mathrm{reg}_3(n) - \mathrm{reg}_6(n)$	$\{1,3\}$
$\{2,3\}$	$3\,\mathrm{reg}_1(n) - \mathrm{reg}_2(n) - \mathrm{reg}_3(n)$	$\{1\}$
$\{2,4\}$	$2\,\mathrm{reg}_1(n) - \mathrm{reg}_4(n)$	$\{1\}$
$\{2,6\}$	$\mathrm{reg}_1(n) + \mathrm{reg}_3(n) - \mathrm{reg}_6(n)$	$\{1,3\}$
$\{1,1,1\}$	$-2\,\mathrm{reg}_1(n)$	$\{1\}$
$\{1,1,2\}$	$-\mathrm{reg}_2(n)$	$\emptyset$
$\{1,2,2\}$	$2\,\mathrm{reg}_1(n) - 2\,\mathrm{reg}_2(n)$	$\{1\}$
MPer_L		$\{\emptyset, \{1\}, \{3\}, \{1,3\}\}$

Table 2. Selected calculations of the family MPer_L of the minimal sets of Lefschetz periods for Morse–Smale diffeomorphism of N_g computed on CPU: Phenom II x4 965

Genus	MPer_L	Time of execution (s)
9	{{1}, {3}, {5}, {7}, {9}, {1, 3}, {1, 5}, {1, 7}, {1, 9}, {3, 5}, {3, 9}, {1, 3, 5}, {1, 3, 7}, {1, 3, 9}, {3, 5, 15}, {1, 3, 5, 15}}	0.00572536
10	{{}, {1}, {3}, {5}, {1, 3}, {1, 5}, {1, 7}, {1, 9}, {3, 5}, {3, 7}, {3, 9}, {1, 3, 5}, {1, 3, 7}, {1, 3, 9}, {1, 3, 5, 15}}	0.00690033
30	Number of lists: 363	0.986523
40	Number of lists: 1230	10.7953
50	Number of lists: 3568	117.685

Example 7.9. Let $g = 4$; then $\dim f_{*_1} = 3$. There are 9 possible families P_j of degrees of primitive roots of unity (each corresponding to some induced map f_*). In Table 1 the set $M_{P_j} = \mathrm{MPer}_L(f)$ is described for each P_j and MPer_L is determined.

Comparing the results of Theorem 8 in [28] for $g = 9$ with the computations based on our algorithm (see Table 2) one can observe that in [28] the case $\{3, 5\}$ is omitted. We will show straightforwardly the realization of this set in two ways.

Consider f_* that corresponds to the set $\{5, 6, 6\}$ of degrees of primitive roots of unity (satisfying the equation $g-1=8=\varphi(5)+\varphi(6)+\varphi(6)$). We have in the terms of periodic expansion

$$L(f^n) = \mathrm{reg}_1(n) - (L_5(n) + L_6(n) + L_6(n)) = -2\,\mathrm{reg}_2(n) - 2\,\mathrm{reg}_3(n) + \mathrm{reg}_5(n) - 2\,\mathrm{reg}_6(n)$$

and thus by Theorem 6.2 $\mathrm{MPer}_L(f) = \{3, 5\}$.

Alternatively, using the language of Lefschetz zeta function, we obtain

$$c_5(t) = \frac{1-t^5}{1-t} \quad c_6(t) = \frac{1+t^3}{1+t}.$$

Thus. the zeta Lefschetz function for f has the possible forms

$$Z_f(t) = \frac{(1+t^3)^2(1-t^5)}{(1-t^2)^2} = \frac{(1+t^3)^2(1-t^5)}{(1-t)^2(1+t)^2} = \frac{(1+t^3)^2(1-t^5)}{(1-t)(1+t)(1-t^2)}.$$

Finally, according to Definition 3.3 we get that $\mathrm{MPer}_L(f) = \{2, 3, 5\} \cap \{1, 3, 5\} \cap \{1, 2, 3, 5\} = \{3, 5\}$.

8. Appendix: The algorithm for the determination of MPer_L

For the sake of clarity we divided our algorithm into some steps (Algorithms 1-4 below). In Algorithm 1 we find the set $\varphi(\varphi^{-1}[1, g-1])$ by using the function $PhiValues$.

Algorithm 1: Algorithm for finding set $\varphi(\mathbb{N}) \cap [1, m] = \varphi(\varphi^{-1}[1, m])$.

function PhiValues (m);
Input : Integer $m \geqslant 1$.
Output: List $\varphi(\varphi^{-1}[1, m])$.
List $phiValues$;
for $i = 1$ **to** $\lceil 2m^{\frac{\log 3}{log 2}} \rceil$ **do**
  if $\varphi(i) \leqslant m$ **then**
    add element $\varphi(i)$ to list $phiValues$;
  end
end
return $phiValues$;

Example 8.1. Input: $m = 4$; Output: $PhiValues(4) = \{1, 2, 4\}$.

Algorithm 2 groups degrees of primitive roots of unity by their Euler function value.

Algorithm 2: Algorithm returns the list containing on nth position the list of degrees of roots of unity for which the Euler function takes value n.

function GroupedDegrees (m);
Input : Integer $m \geqslant 1$.
Output: List containing on the nth position the list of degrees of all roots of unity for which the Euler function takes value $n \leq m$.
List $groupedDegrees, listOfDegrees$;
$phiValues = PhiValues(m)$;
for $n = 1$ **to** $Max(phiValues)$ **do**
    for $i = 1$ **to** $\lceil 2m^{\frac{\log 3}{\log 2}} \rceil$ **do**
        if $\varphi(i) == n$ AND $i \nmid 4$ **then**
            add element i to list $listOfDegrees$;
        end
    end
    add list $listOfDegrees$ to $groupedDegrees$;
    clear $listOfDegrees$;
end
return $groupedDegrees$;

Example 8.2.

$$\text{Input: } m = 4; \text{ Output: } \{\{\underbrace{1,2}_{\substack{\varphi(d)=1\\ d \nmid 4}}\},\{\underbrace{3,6}_{\substack{\varphi(d)=2\\ d \nmid 4}}\},\underbrace{\{\}}_{\substack{\varphi(d)=3\\ d \nmid 4}},\{\underbrace{5,10}_{\substack{\varphi(d)=4\\ d \nmid 4}}\}\}.$$

Algorithm 3 determines a list of odd coefficients of b_k^d for every integer d from $GroupedDegrees(m)$.

Note that $\varphi(2k) = \varphi(2)\varphi(k) = \varphi(k)$ for odd number k. Hence we can write them in the list of length $max\{PhiValues(m)\}/2$

Algorithm 3: Algorithm returns the list of all possible odd indices coefficients.

function GroupedCoef (m);
Input : Integer m.
Output: List containing on nth position the lists of lists of coefficients with odd indices.
List $groupedCoefficients$;
$groupedDegrees = GroupedDegrees(m)$;
for $i = 1$ **to** $Max(PhiValues(m))/2$ **do**
    add to $groupedCoefficients$ a list of lists of coefficients computed for every degree from i'th position of $groupedDegrees$;
end
return $groupedCoefficients$

Example 8.3. Input: $m = 4$; Output: {{{1, 0, 0}, {− 1, 0, 0}}, {{− 1, 1, 0}, {1, − 1, 0}}, {}, {{− 1, 0, 1}, {1, 0, − 1}}}.

Definition 8.4. Let $A, B \subset \mathbb{R}^n$, the Minkowski sum of A and B is the set

$$\{a + b : \ a \in A, \ b \in B\}.$$

In the final Algorithm 4 we will use some well-known functions introduced below:

- $MinkowskiSum(A, B)$ which calculates Minkowski sum of the lists A, B. (for example $MinkowskiSum(\{\{1, 2\}, \{3, 4\}\}, \{\{5, 6\}\}) = \{\{6, 8\}, \{8, 10\}\}$),
- $IntegerPartitions(m, A)$ which gives the list of all possible partitions of an integer m into the sum of integers from the list A, (for example $IntegerPartitions(4, \{1, 2, 4\}) = \{\{4\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}\}$),
- $ChangeToOdd$ which changes the number n to $2n - 1$ (for example $ChangeToOdd(3) = 5$).

Algorithm 4: Algorithm returns the family of the minimal sets of Lefschetz periods for self-map f of N_g.

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function MPer (g);
Input : Integer $g > 1$.
Output: List of all possible minimal sets of Lefschetz periods for self-map f of N_g.
$m = g - 1$;
$P = IntegerPartitions(m, PhiValues(m))$;
$groupedCoefficients = GroupedCoefficients(m)$;
List $LefschetzCoef, mSum, MPer_L, M_{P_i}, nonZeroIndices$;
for $i = 1$ **to** $Length(P)$ **do**
  for $j = 1$ **to** $Length(P[i])$ **do**
    $mSum = MinkowskiSum(mSum, groupedCoefficients\big[P[i][j]\big])$;
  end
  $LefschetzCoef = \{\{1, 0, \ldots, 0\}\} - mSum$;
  $nonZeroIndices$ is list of lists of non-zero indices from $LefschetzCoef$;
  $M_{P_i} = ChangeToOdd(nonZeroIndices)$;
  add to $MPer_L$ list M_{P_i};
end
return $MPer_L$;

Example 8.5. Input: $g = 5$; Output: {{1}, {3}, {5}, {1, 3}, {1, 5}}.

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