



## 2-Coloring number revisited

Robert Janczewski<sup>a</sup>, Paweł Obszarski<sup>a</sup>, Krzysztof Turowski<sup>b,\*</sup>

<sup>a</sup> Department of Algorithms and System Modeling, Gdańsk University of Technology, Poland

<sup>b</sup> Center for Science of Information, Purdue University, United States of America



### ARTICLE INFO

#### Article history:

Received 26 September 2018

Received in revised form 27 June 2019

Accepted 5 September 2019

Available online 10 September 2019

Communicated by C. Kaklamani

#### Keywords:

Graph chromatic game

2-Coloring number

Subcubic graphs

Thorny graphs

### ABSTRACT

2-Coloring number is a parameter, which is often used in the literature to bound the game chromatic number and other related parameters. However, this parameter has not been precisely studied before.

In this paper we aim to fill this gap. In particular we show that the approximation of the game chromatic number by the 2-coloring number can be very poor for many graphs. Additionally we prove that the 2-coloring number may grow quadratically as a function of the maximum degree of a graph, whereas the game chromatic number is always at most linear.

Moreover, we establish the values of the 2-coloring number for several graph classes, such as complete  $k$ -partite graphs, cacti and thorny graphs, trees and subcubic graphs. It is shown that in all these cases one may compute the exact value in a polynomial time.

© 2019 Elsevier B.V. All rights reserved.

## 1. Introduction

All graphs considered in this paper are finite, undirected and simple, i.e. without loops or multiple edges. We assume the standard graph notation following [3]: a graph  $G$  has a vertex set  $V(G)$  and an edge set  $E(G)$  – with cardinalities  $n(G)$  and  $m(G)$ , respectively. The set of vertices adjacent to a vertex  $v$  in a graph  $G$  is denoted by  $N_G(v)$ , and its cardinality by  $\deg_G(v) := |N_G(v)|$ . Similarly, we denote by  $N_G(v, r)$  the set of vertices, which are exactly at the distance  $r$  from  $v$  in  $G$ , where the distance of two vertices equals the length of any shortest path connecting them. Moreover, we denote by  $\Delta(G)$  and  $\delta(G)$  the maximum and minimum degree of a vertex in the graph, by  $\chi(G)$  the chromatic number of the graph, and by  $g(G)$  the girth of a graph, that is, the length of a shortest cycle in the graph (provided that  $G$  has at least one cycle). We also denote by  $G[A]$  the subgraph of  $G$ , induced by a nonempty set  $A \subseteq V(G)$ .

Note that for brevity, whenever it is obvious from the context to which graph we refer, we simply drop it from the above (and below) notation.

Through the paper we use the notion of a linear ordering of  $V(G)$ . Usually it is understood as a bijection  $L: V(G) \rightarrow \{1, 2, \dots, n\}$ , but sometimes it is more convenient to use a sequence  $(v_i)_{i=1,2,\dots,n(G)}$  of vertices of  $G$  (a permutation of set  $V(G)$ ) where  $v_i := L^{-1}(i)$  for  $i = 1, 2, \dots, n(G)$ . It is obvious that both descriptions are equivalent – and a handful of times we will be using both, writing (with a slight abuse of notation) that  $L = (v_i)_{i=1,2,\dots,n(G)}$ .

For a given linear ordering  $L$  of  $V(G)$  and  $v \in V(G)$  its *back-neighborhood* may be defined as  $N_G^-(v, L) := \{u \in V(G) : uv \in E(G) \wedge L(u) < L(v)\}$  – that is, the set of neighbors of  $v$ , which precede it in  $L$ . Now, we are ready to define the *coloring*

\* Corresponding author.

E-mail addresses: robert@kaims.pl (R. Janczewski), pawel.obszarski@eti.pg.gda.pl (P. Obszarski), krzysztof.szczepan.turowski@gmail.com (K. Turowski).

number  $\text{col}(G)$  of a graph  $G$  as the minimum  $k$  for which there exists a linear ordering  $L$  on  $V(G)$  such that  $|N_G^-(v, L)| \leq k - 1$  for every  $v \in V(G)$ .

Coloring number can be considered as a measure of sparseness of a graph:  $\text{col}(G) \leq k$  if and only if every subgraph of  $G$  has a vertex of degree at most  $k - 1$ , that is, if every subgraph of  $G$  is sparse. It is known that the optimal linear ordering for  $\text{col}(G)$  can be found in  $O(n + m)$  time using the Smallest Last algorithm [11], that is, by iterative removal from  $G$  vertices of the smallest degree and appending them at the beginning of the sequence.

Kierstead and Yang [10] generalized the definition of the coloring number in the following way: for a given  $r \geq 1$  and a linear ordering  $L$  of  $V(G)$  let

$$N_G^-(v, r, L) := \{u \in V(G) : \exists w_1, w_2, \dots, w_{r-1} u w_1, w_1 w_2, \dots, w_{r-1} v \in E(G) \\ \wedge L(u) < L(v) \wedge L(w_i) > L(v) \text{ for all } 1 \leq i \leq r - 1\}$$

be the  $r$ -back-neighborhood of a vertex  $v$  and

$$\text{deg}_G^*(v, r, L) := \left| \bigcup_{i=1}^r N_G^-(v, i, L) \right|$$

be the  $r$ -extended degree of  $v$ . Then the  $r$ -coloring number  $\text{col}_r(G)$  of a graph  $G$  is defined as follows:

$$\text{col}_r(G) := 1 + \min_L \max_{v \in V(G)} \text{deg}_G^*(v, r, L),$$

where the minimum is taken over all possible linear orderings  $L$ . We would use the name *optimal ordering* for any ordering  $L$ , for which

$$\max_{v \in V(G)} \text{deg}_G^*(v, r, L) = \text{col}_r(G) - 1,$$

given  $r$  and  $G$ . Note that since later on we use  $\text{deg}_G^*(v, r, L)$  only for  $r = 2$ , we may unambiguously simplify the notation to  $\text{deg}_G^*(v, L)$  and use the name *extended degree* as a synonym of 2-extended degree.

It is known (see Kierstead and Kostochka [7]) that the general simple bounds for  $\text{col}_r(G)$  are as follows:

$$\chi(G) \leq \text{col}_r(G) \leq \Delta(G)(\Delta(G) - 1)^{r-1} + 1.$$

The upper bound, as shown by van den Heuvel et al. [5], can be improved for  $G$  excluding  $K_t$  as a minor (that is, it cannot be obtained from  $G$  by contraction and removal of vertices and edges):

$$\text{col}_r(G) \leq \binom{t-1}{2} (2r+1).$$

Moreover, it was proved [6] that there exist simple upper bound using the treewidth of  $G$ :  $\text{col}_r(G) \leq \text{tw}(G) + 1$  for every  $r$ . This entails for example that for cacti and outerplanar graphs it holds that  $\text{col}_r(G) \leq 3$ .

For  $r = 2$  we have many more results, proved in contexts of various problems. For example, it was proved in [4] that  $\text{col}_2(G) \leq 8$  for planar  $G$  (and this result is strict). Originally, it was used by Chen and Schelp [2] in the context of the Erdős-Burr theorem, stating that for any  $p$ -degenerate graph  $G$  on  $n$  vertices, i.e. a graph with  $\text{col}(G) \leq p$ , no monochromatic copy of  $G$  exists in two-edge-colored complete graph on  $c_p n$  vertices for some constant  $c_p$ . They showed that it holds for graphs with bounded  $\text{col}_2(G)$  – and therefore, for example, for all planar graphs.

It was shown in [9] (although using different terminology) that the game chromatic number  $\chi_g(G) \leq 4 \text{col}_2(G) + 1$  for any planar graph  $G$ . It was generalized in slightly weaker form in [1] to the general graphs as  $\chi_g(G) \leq \chi(G)(\text{col}_2(G) + 1)$ .

We just mention further that  $\text{col}_2(G)$  is a parameter used in proofs and bounds of many coloring-related parameters such as acyclic chromatic number [10], oriented game chromatic number [8], generalized coloring number [12] and coloring game number [1].

In this paper we analyze the properties of  $\text{col}_2(G)$ . First, in Section 2 we show that  $\text{col}_2(G) \leq \frac{1}{2} \Delta(G)(\Delta(G) - 1) + 1$ , therefore improving previous bound by a factor of 2, but also that for all regular,  $\{C_3, C_4\}$ -free graphs it is true that  $\text{col}_2(G) \geq \frac{1}{8} \Delta(G)^2 + \frac{1}{4} \Delta(G) + 1$ , therefore the parameter  $\text{col}_2(G)$  grows quadratically with  $\Delta(G)$ . This finding leads us to a conclusion that  $\text{col}_2(G)$  cannot be a good bound for  $\chi_g(G)$  for these graphs, as it is known that  $\chi_g(G) \leq \Delta(G) + 1$  – so  $\text{col}_2(G) = \Omega(\chi_g^2(G))$ .

Finally, in Section 3 we use some of the above results to provide a complete classification of subcubic graphs in terms of their  $\text{col}_2(G)$  values and present a polynomial algorithm for computing  $\text{col}_2(G)$  for subcubic graphs.

Note that the existence of isolated vertices do not influence the value of  $\text{col}_2(G)$ . Therefore, from now on without loss of generality we may assume that we consider only graphs without isolated vertices, that is, with  $\delta(G) \geq 1$ .

## 2. General bounds on 2-coloring number

**Theorem 2.1.** *If  $G$  has  $\text{col}(G) \geq 3$ , then  $\text{col}_2(G) \leq \Delta(G)(\text{col}(G) - 2) + 1$ .*

**Proof.** Let  $L$  be an ordering of vertices of  $G$  such that  $|N_G^-(v, L)| \leq \text{col}(G) - 1$  for all  $v \in V(G)$ .

Now, consider any  $v \in V(G)$ . It has  $k = |N_G^-(v, L)| \leq \text{col}(G) - 1$  and  $|N_G(v) \setminus N_G^-(v, L)| = \text{deg}_G(v) - k$ . Any vertex  $u \in N_G(v) \setminus N_G^-(v, L)$  may have at most  $\text{col}(G) - 2$  neighbors in  $N_G^-(v, L)$ , therefore

$$\begin{aligned} \text{deg}_G^*(v, L) &\leq k + (\text{deg}_G(v) - k)(\text{col}(G) - 2) \\ &= \text{deg}_G(v)(\text{col}(G) - 2) + k(3 - \text{col}(G)) \leq \Delta(G)(\text{col}(G) - 2), \end{aligned}$$

which completes the proof.  $\square$

**Theorem 2.2.** *For every graph  $G$  it holds that  $\text{col}_2(G) \leq \frac{1}{2} \Delta(G)(\Delta(G) - 1) + 2$ .*

**Proof.** Let  $L = (v_i)_{i=1,2,\dots,n(G)}$  be a Smallest Last ordering of vertices of  $G$ . Let us look at vertex  $v_i$  and its forward neighbors  $v_{i_1}, v_{i_2}, \dots, v_{i_k}$  such that  $i < i_1 < i_2 < \dots < i_k$ . Let also  $G_{i_j} := G[\{v_1, v_2, \dots, v_{i_j}\}]$ .

Now we have:

$$\forall_{1 \leq j \leq k} \text{deg}_{G_{i_j}}(v_{i_j}) = \min_{1 \leq l \leq i_j} \text{deg}_{G_{i_j}}(v_l) \leq \text{deg}_{G_{i_j}}(v_i) = \text{deg}_{G_i}(v_i) + j.$$

Moreover

$$\begin{aligned} \text{deg}_G^*(v_i, L) &\leq (\text{deg}_G(v_i) - k) + \sum_{j=1}^k (\text{deg}_{G_{i_j}}(v_{i_j}) - 1) \\ &\leq (\text{deg}_G(v_i) - k) + \sum_{j=1}^k (\text{deg}_{G_i}(v_i) + j - 1). \end{aligned}$$

But now observe that  $\text{deg}_{G_i}(v_i) = \text{deg}_G(v_i) - k$ , therefore:

$$\begin{aligned} \text{deg}_G^*(v_i, L) &\leq (k + 1)(\text{deg}_G(v_i) - k) + \frac{1}{2}k(k - 1) \\ &= -\frac{1}{2}k^2 + k \left( \text{deg}_G(v_i) - \frac{3}{2} \right) + \text{deg}_G(v_i). \end{aligned}$$

This upper bound is a quadratic function of  $k$ , which has its maximum in  $k = \text{deg}_G(v_i) - \frac{3}{2}$ , giving:

$$\text{deg}_G^*(v_i, L) \leq \frac{1}{2} \text{deg}_G(v_i)(\text{deg}_G(v_i) - 1) + \frac{9}{8}.$$

Since  $\text{deg}_G^*(v_i, L)$  and  $\text{deg}_G(v_i)$  are integers, we may strengthen this bound to  $\text{deg}_G^*(v_i, L) \leq \frac{1}{2} \text{deg}_G(v_i)(\text{deg}_G(v_i) - 1) + 1$ . Finally, we obtain:

$$\text{col}_2(G) \leq 1 + \max_{v \in V(G)} \text{deg}_G^*(v, L) \leq 2 + \frac{1}{2} \Delta(G)(\Delta(G) - 1). \quad \square$$

Let us recall that if  $\mathcal{H}$  is a family of graphs ( $H$  is a graph) then graph  $G$  is  $\mathcal{H}$ -free ( $H$ -free) if and only if none of its induced subgraphs is isomorphic to any element of  $\mathcal{H}$  (to  $H$ ).

**Theorem 2.3.** *For every  $C_3$ -free graph  $G$  in which every vertex belongs to at most  $t$  cycles of length 4 it holds that*

$$\text{col}_2(G) \geq \frac{1}{1 + \sqrt{8t + 1}} \frac{m^2(G)}{n^2(G)} + \left( 1 - \frac{1}{1 + \sqrt{8t + 1}} \right) \frac{m(G)}{n(G)} + 1.$$

**Proof.** Let  $L = (v_i)_{i=1,2,\dots,n(G)}$  be any ordering of vertices of  $G$ . Since  $G$  contains no  $C_3$  we have  $N_G^-(v_i, 1, L) \cap N_G^-(v_i, 2, L) = \emptyset$  and  $\text{deg}_G^*(v_i, L) = |N_G^-(v_i, 1, L)| + |N_G^-(v_i, 2, L)|$ . Moreover, we know that  $\sum_{i=1}^{n(G)} |N_G^-(v_i, 1, L)| = m(G)$  and  $|N_G^-(v_i, 1, L)| \leq \text{deg}_G(v_i)$ .

Now we find  $T$ , the number of triples  $(j, i, k)$  such that  $j < i < k$  and  $v_i v_k, v_j v_k \in E(G)$ . First, summing for all fixed  $k$  we have:

$$\begin{aligned}
T &= \sum_{k=1}^{n(G)} \binom{|N_G^-(v_k, 1, L)|}{2} \\
&= \frac{1}{2} \sum_{k=1}^{n(G)} |N_G^-(v_k, 1, L)|^2 - \frac{1}{2} \sum_{k=1}^{n(G)} |N_G^-(v_k, 1, L)| \\
&= \frac{1}{2} \sum_{k=1}^{n(G)} |N_G^-(v_k, 1, L)|^2 - \frac{1}{2} m(G).
\end{aligned}$$

Alternatively, for any fixed vertex  $v_i$  and any fixed  $v_j \in N_G^-(v_i, 2, L)$  we have at most  $t$  cycles  $C_4$  to which they both belong. Therefore, we have at most  $l$  paths of length 2 between them, where  $\binom{l}{2} \leq t$  – or, equivalently,  $2l \leq 1 + \sqrt{8t + 1}$ . Therefore any such  $i$  and  $j$  may correspond only to at most  $\frac{1}{2}(1 + \sqrt{8t + 1})$  triples  $(j, i, k)$ :

$$T \leq \frac{1}{2}(1 + \sqrt{8t + 1}) \sum_{i=1}^{n(G)} |N_G^-(v_i, 2, L)|.$$

Combining the above observations we receive

$$\begin{aligned}
\sum_{i=1}^{n(G)} \deg_G^*(v_i, L) &= \sum_{i=1}^{n(G)} |N_G^-(v_i, 2, L)| + \sum_{i=1}^{n(G)} |N_G^-(v_i, 1, L)| \\
&\geq \frac{2T}{1 + \sqrt{8t + 1}} + m(G) \\
&= \sum_{k=1}^{n(G)} \frac{|N_G^-(v_k, 1, L)|^2}{1 + \sqrt{8t + 1}} - \frac{m(G)}{1 + \sqrt{8t + 1}} + m(G) \\
&= \sum_{k=1}^{n(G)} \frac{|N_G^-(v_k, 1, L)|^2}{1 + \sqrt{8t + 1}} + \left(1 - \frac{1}{1 + \sqrt{8t + 1}}\right) m(G) \\
&\geq \frac{\left(\sum_{k=1}^{n(G)} |N_G^-(v_k, 1, L)|\right)^2}{n(G)(1 + \sqrt{8t + 1})} + \left(1 - \frac{1}{1 + \sqrt{8t + 1}}\right) m(G) \\
&= \frac{m^2(G)}{n(G)(1 + \sqrt{8t + 1})} + \left(1 - \frac{1}{1 + \sqrt{8t + 1}}\right) m(G)
\end{aligned}$$

and finally

$$\begin{aligned}
\max_{1 \leq i \leq n(G)} \deg_G^*(v_i, L) &\geq \frac{1}{n(G)} \sum_{i=1}^{n(G)} \deg_G^*(v_i, L) \\
&\geq \frac{1}{1 + \sqrt{8t + 1}} \frac{m^2(G)}{n^2(G)} + \left(1 - \frac{1}{1 + \sqrt{8t + 1}}\right) \frac{m(G)}{n(G)},
\end{aligned}$$

which completes the proof.  $\square$

**Corollary 2.4.** For every regular  $C_3$ -free graph  $G$  in which every vertex belongs to at most  $t$  cycles of length 4 it holds that

$$\text{col}_2(G) \geq \frac{\Delta(G)^2}{4 + 4\sqrt{8t + 1}} + \frac{1}{2} \left(1 - \frac{1}{1 + \sqrt{8t + 1}}\right) \Delta(G) + 1. \quad \square$$

**Corollary 2.5.** For every regular  $\{C_3, C_4\}$ -free graph  $G$  it holds that

$$\text{col}_2(G) \geq \frac{\Delta(G)^2}{8} + \frac{\Delta(G)}{4} + 1. \quad \square$$

Theorem 2.2 and Corollary 2.5 in conjunction give us the result that  $\text{col}_2(G) = \Theta(\Delta(G)^2)$  for  $G$  regular and  $\{C_3, C_4\}$ -free. This means that there exists a gap between  $\text{col}_2(G)$  and  $\chi_g(G)$  for such graphs, as it is obvious that  $\chi_g(G) \leq \Delta(G) + 1$ .

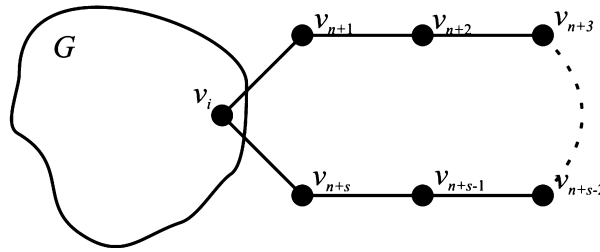


Fig. 1. Graph with a cycle  $C_{s+1}$  attached to the vertex  $v_i$ .

### 3. Subcubic graphs

First, let us state simple facts about  $\text{col}_2(G)$ , which will be used throughout the analysis below.

**Proposition 3.1.** *Let  $G$  be a connected graph. If  $\Delta(G) \leq 2$ , then exactly one of the following holds:*

- $G$  is empty and  $\text{col}_2(G) = 1$ ,
- $G$  is a path of length at least 2 and  $\text{col}_2(G) = 2$ ,
- $G$  is a cycle and  $\text{col}_2(G) = 3$ .  $\square$

**Proposition 3.2.** *If  $H$  is equal to  $G$  with a leaf attached to one of its vertices, then  $\text{col}_2(H) = \max\{\text{col}_2(G), 2\}$ .*

**Proof.** Let  $L$  be an optimal ordering for  $G$  and  $L'$  be equivalent to  $L$  with  $w$  appended at the end, where  $w$  is the leaf attached to  $G$  to obtain  $H$ . It is easy to see that  $\text{deg}_H^*(v, L') = \text{deg}_G^*(v, L')$  for any  $v \in V(G)$ . Moreover, we know that  $\text{deg}^*(w, L') = 1$ , hence  $\text{col}_2(H) \leq \max\{\text{col}_2(G), 2\}$ . But on the other hand, it is obvious that if  $G$  and  $P_2$  are subgraphs of  $H$ , then  $\text{col}_2(H) \geq \max\{\text{col}_2(G), 2\}$ , which completes the proof.  $\square$

**Corollary 3.3.**  $\text{col}_2(G) = 2$  if and only if  $G$  is a nonempty forest.  $\square$

**Lemma 3.4.** *If  $H$  is equal to:*

1. graph  $G$  with an attached cycle to one of its vertices, then

$$\text{col}_2(H) = \max\{\text{col}_2(G), 3\},$$

2. graph  $G$  with an attached cycle to one of its edges, then

$$\text{col}_2(H) = \max\{\text{col}_2(G), 3\},$$

3. graph  $G$  with an attached path connecting its two non-adjacent vertices, then

$$\text{col}_2(H) = \max\{\text{col}_2(G), 2\}.$$

**Proof.** Let  $L = (v_i)_{i=1,2,\dots,n(G)}$  be an optimal ordering for  $G$  and  $s$  be the length of the attached cycle/path.

- (1) To get  $L'$ , we append new vertices at the end of  $L$  in the order corresponding to the path, as shown in Fig. 1. Clearly, we have  $\text{deg}_H^*(v_j, L') \leq 2$  for  $n + 1 \leq j \leq n + s$  and it is obvious that  $\text{deg}_H^*(v_j, L') = \text{deg}_G^*(v_j, L)$  for  $1 \leq j \leq n$ . Therefore  $\text{deg}_H^*(v, L') \leq \max\{\text{col}_2(G) - 1, 2\}$  for all  $v \in V(H)$ .
- (2) If  $s \geq 4$ , then we proceed exactly as above. If we attach  $C_3$  to an edge  $v_i v_j$  ( $i < j$ ), note that it is sufficient to use the ordering  $L'$ , which is constructed from  $L$  by appending a new vertex at the end. It is easy to see that  $\text{deg}_H^*(v_k, L') = \text{deg}_G^*(v_k, L)$  for all  $1 \leq k \leq n$ ,  $k \neq j$ . However,  $\text{deg}_H^*(v_j, L') = \text{deg}_G^*(v_j, L)$  too, since  $v_i$  was already counted in the back-neighborhood of  $v_j$  according to  $L$ . Therefore  $\text{deg}_H^*(v, L') \leq \max\{\text{col}_2(G) - 1, 2\}$  for all  $v \in V(H)$ .
- (3) We proceed exactly as in (1), the only difference is that we receive an ordering  $L'$  satisfying  $\text{deg}_H^*(v, L') \leq \max\{\text{col}_2(G) - 1, 1\}$  for all  $v \in V(H)$ .

To complete the proof it suffices to use the definition of the 2-coloring number and use the monotonicity property: this is, if  $G$  is a subgraph of  $H$ , then  $\text{col}_2(G) \leq \text{col}_2(H)$ .  $\square$

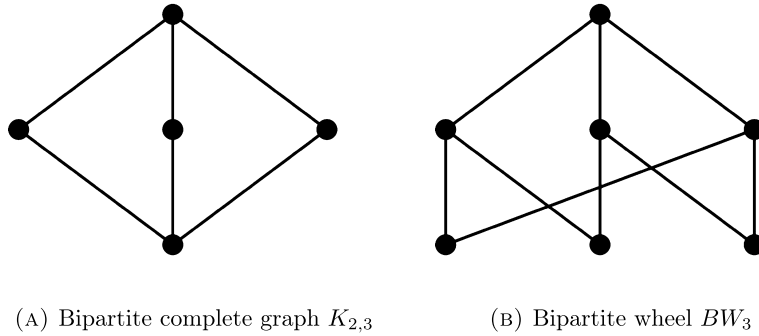


Fig. 2. Gadgets for Proposition 3.7.

Note that Proposition 3.1 solves the problem of determining  $\text{col}_2(G)$  for graphs with  $\Delta(G) \leq 2$ , therefore we need only to consider the case  $\Delta(G) = 3$ . Moreover, we note that from Theorem 2.2 we have the bound  $\text{col}_2(G) \leq \Delta(G) + 2$  for all subcubic graphs  $G$ .

Now we prove the crucial lemma for this section – improving by one the general bound if the coloring number of the graph is small:

**Lemma 3.5.** *If  $G$  is a subcubic graph with  $\text{col}(G) \leq 3$ , then  $\text{col}_2(G) \leq \Delta(G) + 1$ .*

**Proof.** Let  $L = (v_i)_{i=1,2,\dots,n(G)}$  be a Smallest Last ordering of the vertices of  $G$ . Let also  $G_i := G[\{v_1, v_2, \dots, v_i\}]$ .

From our assumption that  $\text{col}(G) \leq 3$  it follows that  $\text{deg}_{G_i}(v_i) \leq 2$ . Now, it is sufficient to prove that:

$$\forall 1 \leq i \leq n \forall j \leq i \text{ deg}_{G_i}^*(v_j, L) \leq \text{deg}_{G_i}(v_j).$$

This inequality is obvious for  $i = 1$ . Suppose that it holds for all  $i < k$  and let us consider a vertex  $v_k$ . There are 3 possible cases:

1.  $\text{deg}_{G_k}(v_k) = 0$  – then  $\text{deg}_{G_k}^*(v_k, L) = 0$  and for all  $1 \leq j < k$  it holds that  $\text{deg}_{G_k}^*(v_j, L) = \text{deg}_{G_{k-1}}^*(v_j, L) \leq \text{deg}_{G_{k-1}}(v_j) = \text{deg}_{G_k}(v_j)$ ,
2.  $\text{deg}_{G_k}(v_k) = 1$  – then  $\text{deg}_{G_k}^*(v_k, L) = 1$  and for all  $1 \leq j < k$  it holds that  $\text{deg}_{G_k}^*(v_j, L) = \text{deg}_{G_{k-1}}^*(v_j, L) \leq \text{deg}_{G_{k-1}}(v_j) \leq \text{deg}_{G_k}(v_j)$ ,
3.  $\text{deg}_{G_k}(v_k) = 2$  – then  $\text{deg}_{G_k}^*(v_k, L) = 2$ . Let  $v_a, v_b$  be the neighbors of  $v_k$  ( $a < b$ ). We have  $\text{deg}_{G_k}^*(v_a, L) = \text{deg}_{G_{k-1}}^*(v_a, L) \leq \text{deg}_{G_{k-1}}(v_a) < \text{deg}_{G_k}(v_a)$  and  $\text{deg}_{G_k}^*(v_b, L) \leq \text{deg}_{G_{k-1}}^*(v_b, L) + 1 \leq \text{deg}_{G_{k-1}}(v_b) + 1 = \text{deg}_{G_k}(v_b)$ . For all  $j \in \{1, 2, \dots, k-1\} \setminus \{a, b\}$  we have, as in previous case,  $\text{deg}_{G_k}^*(v_j, L) = \text{deg}_{G_{k-1}}^*(v_j, L) \leq \text{deg}_{G_{k-1}}(v_j) \leq \text{deg}_{G_k}(v_j)$ .

And since  $\text{deg}_G(v_j) \leq \Delta(G)$  for all  $j$ , we obtain directly that  $\text{col}_2(G) \leq \Delta(G) + 1$ .  $\square$

**Proposition 3.6.** *If  $G$  is a connected cubic graph containing  $K_3$ , then it holds that  $\text{col}_2(G) \leq 4$ .*

**Proof.** Let  $u, v, w$  induce  $K_3$  in  $G$ . Let also  $H = G \setminus \{u, v, w\}$ . Then every connected component  $H'$  of  $H$  is subcubic with  $\delta(H') = 2$ , so  $\text{col}(H) \leq 3$  and we may apply Lemma 3.5 to prove that  $\text{col}_2(H) \leq 4$  and, since the proof is constructive, to obtain an ordering  $L$  for  $H$  which realizes this bound.

Then, we just put  $u, v, w$  at the end of  $L$  to obtain  $L'$ . It is clear that  $\text{deg}_G^*(x, L') = \text{deg}_H^*(x, L)$  for any  $x \in V(H)$  (since each  $u, v, w$  has exactly one neighbor in  $H$ ) and  $\max\{\text{deg}_G^*(u, L'), \text{deg}_G^*(v, L'), \text{deg}_G^*(w, L')\} \leq 3$ .  $\square$

**Proposition 3.7.** *Let  $G$  be a connected  $K_3$ -free cubic graph. Then  $\text{col}_2(G) = 4$  if and only if  $G$  contains one of the subgraphs presented in Fig. 2.*

**Proof.** ( $\Rightarrow$ ) Let  $L = (v_i)_{i=1,2,\dots,n(G)}$  be an optimal ordering for  $G$ . Let us pick  $t$  such that  $G[\{v_{t+1}, v_{t+2}, \dots, v_n\}]$  is empty and  $G[\{v_t, v_{t+1}, \dots, v_n\}]$  is not empty. Then  $v_t$  has  $k \geq 1$  neighbors in  $\{v_{t+1}, v_{t+2}, \dots, v_n\}$ . Note that if  $k = 1$ , then due to the fact that  $G$  is  $K_3$ -free we would have  $\text{deg}^*(v_t, L) = 4$  – a contradiction.

Let us introduce a shorthand  $A(v_i) := N_G(v_i) \setminus \{v_t\}$  for  $i > t$ . By the choice of  $v_t$  we know that for every  $v_i$  and every  $v_j \in A(v_i)$  it is true that  $j < t$ .

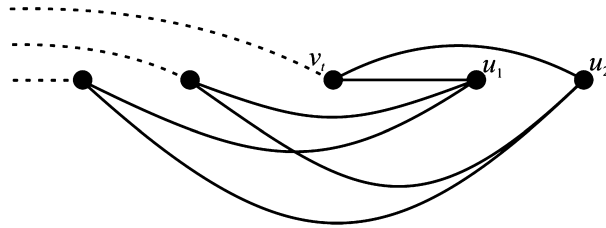
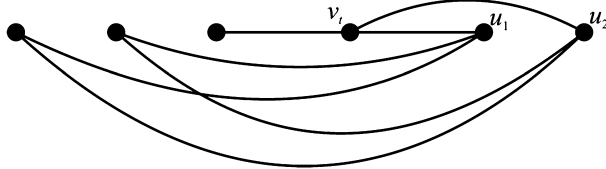


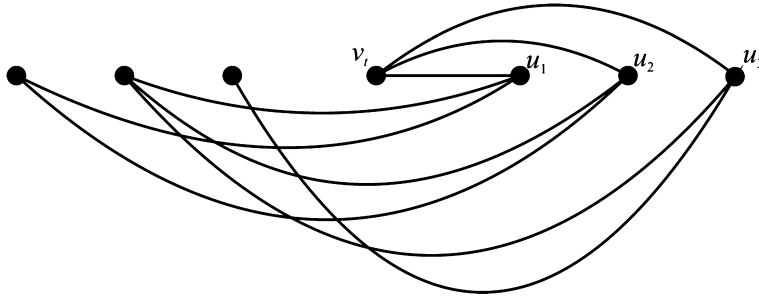
Fig. 3. Auxiliary ordering used when  $K_{2,3}$  is a subgraph of  $G$ .

If  $k = 2$ , then  $v_i$  has two neighbors,  $u_1$  and  $u_2$ , so we have a following situation:



Clearly if  $|A(u_1) \cup A(u_2)| > 2$ , then  $\text{deg}^*(v_i, L) > 3$ . However, if  $|A(u_1) \cup A(u_2)| = 2$ , then it has to be true that  $A(u_1) = A(u_2)$ , therefore  $G[A(u_1) \cup A(u_2) \cup \{v_i, u_1, u_2\}]$  is isomorphic to  $K_{2,3}$ .

Finally, if  $k = 3$ , then we have the following situation:



Graph is cubic, so  $|A(u_1)| = |A(u_2)| = |A(u_3)| = 2$ . Since  $\text{deg}^*(v_i, L) \leq 3$ , we know that  $|A(u_1) \cup A(u_2) \cup A(u_3)| \leq 3$ . This leaves us with only two possibilities:

- either it is true that  $A(u_1) = A(u_2)$  – then  $G$  contains  $K_{2,3}$  with partitions  $\{u_1, u_2\}$  and  $A(u_1) \cup \{v\}$ ,
- or  $|A(u_1) \cup A(u_2)| = 3$  – but then  $A(u_3) \subseteq A(u_1) \cup A(u_2)$  so either  $A(u_3) = A(u_i)$  for  $i \in \{1, 2\}$  and  $G$  has a subgraph  $K_{2,3}$ , or  $|A(u_3) \cap A(u_1)| = |A(u_3) \cap A(u_2)| = 1$  and  $G$  has a subgraph  $BW_3$ .

In either case we are done, since  $G$  contains  $K_{2,3}$  or  $BW_3$ .

( $\Leftarrow$ ) It is not hard to verify that for any cubic graph  $\text{col}_2(G) \geq 4$  – as any ordering  $L$  the last vertex  $v$  has  $\text{deg}_G^*(v, L) = 3$ . Since  $G$  is connected, if  $K_{2,3}$  is a subgraph of  $G$ , then  $K_{2,3}$  is an induced subgraph of  $G$  and  $G \neq K_{2,3}$ . Let  $H$  be the graph induced in  $G$  by vertices that do not belong to this  $K_{2,3}$  subgraph. Then  $\delta(H) \leq 2$  and  $\text{col}(H) \leq 3$  and it is sufficient to find an ordering  $L$  which proves that  $\text{col}_2(H) \leq 4$  (possible by Lemma 3.5) and put the vertices of  $K_{2,3}$  at the end of a sequence in the following way to obtain  $L'$ , as shown in Fig. 3.

We see that for all vertices  $u \in V(K_{2,3})$  we have  $\text{deg}_G^*(u, L') \leq 3$  and, moreover, for all vertices  $u \in V(H)$  we have  $\text{deg}_G^*(u, L') = \text{deg}_G^*(u, L) \leq \text{col}(H) - 1 = 3$ .

If  $BW_3$  is a subgraph of  $G$ , then we proceed with similar argument. The only difference is in the way we obtain  $L'$  – see Fig. 4. Therefore in both cases we can construct a required ordering which proves that  $\text{col}_2(G) \leq 4$ .  $\square$

**Proposition 3.8.** *If  $G$  is a subcubic graph such that every vertex with degree 2 is adjacent only to non-adjacent vertices of degree 3, then either it contains induced  $C_4$  such that  $\text{col}_2(G) \leq \max\{\text{col}_2(G \setminus V(C_4)), 3\}$  or  $\text{col}_2(G) \geq 4$ .*

**Proof.** Let  $L$  be an optimal ordering for  $\text{col}_2(G)$ . Let  $v$  be a last vertex with degree 3 in this ordering.

If  $|N_G^-(v, 1, L)| \geq 2$ , then  $\text{deg}_G^*(v, L) \geq 3$  and  $\text{col}_2(G) \geq 4$ , because either it has 3 neighbors in the back or it has 2 neighbors in the back, but the neighbor in the front has degree 2 (by definition of  $v$ ) and its other neighbor  $u$  is different from  $v$  or its neighbors (as  $v$  and  $u$  are not adjacent by assumption).

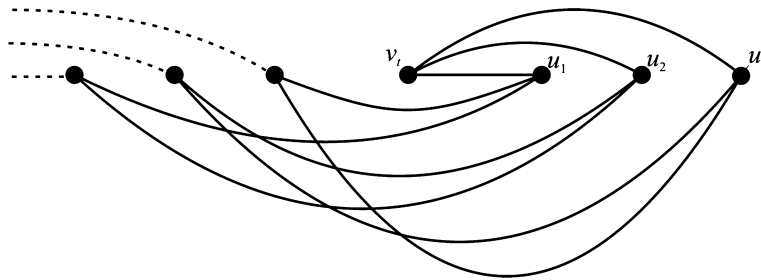


Fig. 4. Auxiliary ordering used when  $BW_3$  is a subgraph of  $G$ .

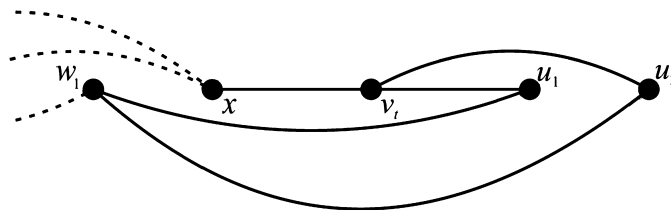
**Algorithm 1** Algorithm for  $\text{col}_2(G)$  for subcubic graphs.

```

1: function SCS(G)
2:   if G has no vertices then return 0
3:   if G has connected components  $G_1, G_2, \dots, G_k, k \geq 2$  then
4:     return  $\max\{\text{SCS}(G_1), \text{SCS}(G_2), \dots, \text{SCS}(G_k)\}$ 
5:   if G is empty then return 1
6:   if G is a forest then return 2
7:   if  $\delta(G) = 3$  then
8:     if G contains induced  $K_3, K_{2,3}$  or  $BW_3$  then return 4
9:     else return 5
10:  if  $\delta(G) = 2$  then
11:    if G contains  $C = uv_1 \dots v_k u, \text{deg}(u) = 3, \text{deg}(v_i) = 2$  then
12:      return  $\max\{\text{SCS}(G \setminus V(C)), 3\}$ 
13:    if G contains  $C = uv_1 \dots v_k w u, \text{deg}(u) = \text{deg}(w) = 3,$ 
14:       $\text{deg}(v_i) = 2$  then
15:        return  $\max\{\text{SCS}(G \setminus V(C)), 3\}$ 
16:    if G contains  $P = uv_1 \dots v_k w, \text{deg}(u) = \text{deg}(w) = 3,$ 
17:       $uw \notin E(G), \text{deg}(v_i) = 2, k \geq 2$  then
18:        return  $\max\{\text{SCS}(G \setminus V(P)), 3\}$ 
19:    if G contains  $C_4 = u_1 v_1 u_2 w_1$  as shown in Fig. 4 then
20:      return  $\max\{\text{SCS}(G \setminus V(C_4)), 3\}$ 
21:    else return 4
22:  Let v be any vertex with degree 1
23:  return  $\max\{\text{SCS}(G \setminus v), 2\}$ 

```

If  $|N_G^-(v, 1, L)| = 1$ , then  $v$  has two neighbors in the front,  $u_1$  and  $u_2$  – by definition of  $v$  they have both degree 2. If they have different other neighbors  $w_1$  and  $w_2$ , then  $\text{deg}_G^*(v) \geq 3$  – as it counts  $w_1, w_2$  and  $x$ , the only back-neighbor of  $v$ . Note that  $w_1, w_2$  are distinct from  $x$ , because neighbors of  $u_1$  and  $u_2$  have to be non-adjacent. However, if  $w_1 = w_2$ , then  $w_1, v, u_1$  and  $u_2$  form  $C_4$  such that given any optimal ordering of  $G \setminus V(C_4)$  we may show that  $\text{col}_2(G) \leq \max\{\text{col}_2(G \setminus V(C_4)), 3\}$ :



Similarly, if  $|N_G^-(v, 1, L)| = 0$ , then either all neighbors of  $v$  have distinct other neighbors (and then  $\text{deg}_G^*(v, L) \geq 3$ ), or at least one pair have a common one (and then we have  $C_4$  with two vertices with degree 2 in  $G$ ).  $\square$

Finally, we may construct an algorithm computing the exact value in all cases. Note that for simplicity of the algorithm we assumed that  $G$  can have zero vertices. It is not hard to modify this algorithm to exclude this case.

**Theorem 3.9.** Algorithm 1 returns  $\text{col}_2(G)$  for any subcubic graph  $G$ .

**Proof.** If  $\Delta(G) \leq 2$  or  $G$  is a forest, then the algorithm returns correct solution, due to Proposition 3.1 and Corollary 3.3, respectively.



If  $\Delta(G) = 3$  and  $G$  contains a cycle, then we have three cases:

- (1)  $\delta(G) = 3$  – then it is obvious that  $\text{col}_2(G) \geq 4$  as the last vertex in any ordering has  $\deg_G^*(v, L) = 3$ . Moreover, from the Theorem 2.2 we may directly obtain a general bound for subcubic graphs  $\text{col}_2(G) \leq 5$ .  
From Propositions 3.6 and 3.7 we know that  $\text{col}_2(G) = 4$  if and only if  $G$  contains  $K_3$  or either of gadgets  $K_{2,3}$ ,  $BW_3$ . Otherwise it must be the case that  $\text{col}_2(G) = 5$ .
- (2)  $\delta(G) = 2$  – then it is obvious that  $\text{col}_2(G) \geq 3$  as the last vertex in any ordering has  $\deg_G^*(v, L) \geq 2$ . Moreover, from Lemma 3.5 we know that  $\text{col}_2(G) \leq 4$ .  
Let  $G$  be the smallest counterexample to the correctness of this algorithm. Clearly, it does not contain a path induced by the vertices of degree 2, as it would violate minimality by Proposition 3.4. Therefore, it has to be a graph, in which every vertex of degree 2 is connected to two non-adjacent vertices of degree 3.  
But then, from Proposition 3.8 we know that either we can remove  $C_4$  and obtain smaller counterexample – or it has to be the case that  $\text{col}_2(G) = 4$  (and then we know the correct answer with certainty without falling into recursion). In both cases we get a contradiction, which proves this case.
- (3)  $\delta(G) = 1$  – then we may remove all leaves from the graph and, due to Proposition 3.2, we finally fall into one of the previous cases.

Since all the theorems above are constructive, the algorithm may be easily modified not only the value of  $\text{col}_2(G)$ , but also a respective ordering – therefore it surely returns a feasible solution, which completes the proof.  $\square$

**Theorem 3.10.** *The running time of Algorithm 1 for  $G$  on  $n$  vertices is  $O(n^2)$ .*

**Proof.** Evaluating each if clause takes  $O(n)$  time – since graph is cubic, for every vertex  $v$  one may check its  $k$ -neighborhood (and find required gadgets containing  $v$ ) for  $k = O(1)$  in constant time. If one has to find a cycle or a path, it is sufficient to find a vertex of degree 2 and follow from it in both directions until we find its ends (that is, vertices with degree 3).

Since we directly return an answer (lines 2, 5, 6, 8 – 9, and 19), we recurse on at least two smaller subproblems with total size not greater than the current number of vertices (line 4) or we're left with one smaller subproblem (lines 12, 14, 16, 18 and 21), it is straightforward that the total complexity of the algorithm is  $O(n^2)$ .  $\square$

#### 4. Conclusion

We provide several results for  $\text{col}_2(G)$  when  $\Delta(G) \leq 3$ . Moreover, we derive general upper and lower bound for values of  $\text{col}_2(G)$  in the worst case, proving that  $\text{col}_2(G) = \Theta(\Delta(G)^2)$  even for  $C_3$ -free graphs.

One may ask what is the minimum value of  $c$  such that  $\text{col}_2(G) \leq c\Delta(G)^2 + o(\Delta(G)^2)$ . We proved that  $c \in [\frac{1}{8}, \frac{1}{2}]$ . Moreover, there is a question whether computation of  $\text{col}_2(G)$  can be done in polynomial time for  $\Delta(G) > 3$  – or maybe at some point this problem becomes NP-hard.

#### Declaration of competing interest

The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript.

#### References

- [1] T. Bartnicki, J. Grytczuk, H.A. Kierstead, X. Zhu, The map-coloring game, *Am. Math. Mon.* 114 (2007) 793–803.
- [2] G. Chen, R.H. Schelp, W.E. Shreve, A new game chromatic number, *Eur. J. Comb.* 18 (1997) 1–9.
- [3] R. Diestel, *Graph Theory*, Springer, 2006.
- [4] Z. Dvořák, A. Kabela, T. Kaiser, Planar graphs have two-coloring number at most 8, *J. Comb. Theory, Ser. B* 130 (2018) 144–157.
- [5] M. Grohe, S. Kreutzer, R. Rabinovich, S. Siebertz, K. Stavropoulos, Colouring and covering nowhere dense graphs, in: *Proceedings of the 41st International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2015)*, in: *Lecture Notes in Computer Science*, vol. 9224, 2016, pp. 325–338.
- [6] J. van den Heuvel, P. Ossona de Mendez, D. Quiroz, R. Rabinovich, S. Siebertz, On the generalised colouring numbers of graphs that exclude a fixed minor, *Electron. Notes Discrete Math.* 49 (2015) 523–530.
- [7] H.A. Kierstead, A.V. Kostochka, Efficient graph packing via game coloring, *Comb. Probab. Comput.* 18 (2009) 765–774.
- [8] H.A. Kierstead, B. Mohar, S. Spacapan, D. Yang, X. Zhu, The two-coloring number and degenerate colorings of planar graphs, *SIAM J. Discrete Math.* 23 (2009) 1548–1560.
- [9] H.A. Kierstead, W.T. Trotter, Planar graph coloring with an uncooperative partner, *J. Graph Theory* 18 (1994) 569–584.
- [10] H.A. Kierstead, D. Yang, Orderings on graphs and game coloring numbers, *Order* 20 (2003) 255–264.
- [11] D. Matula, L. Beck, Smallest-last ordering and clustering and graph coloring algorithms, *J. ACM* 30 (1983) 417–427.
- [12] X. Zhu, Coloring graphs with bounded generalized coloring number, *Discrete Math.* 309 (2009) 5562–5568.