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Identification of Shear Modulus Parameters of Half-space Inhomogeneous by Depth

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Abstract. The paper propose a method for determining of the parameters of the exponential shear modulus of a functionally graded half-space based on the solution of the problem of a pure shear of an elastic functionally graded half-space by a strip punch. The solution of the integral equation of the contact problem is constructed by asymptotic methods with respect to the dimensionless parameter. The dependence of contact stresses on the parameters of the shear modulus is analyzed. The determination of the parameters of the shear modulus of a functionally graded half-space is based on the values of shear stresses at the contact. By choosing proper shear modulus parameters of the functionally graded half-space, "approximately homogeneous" area inside of the functionally graded half-space are developed.

INTRODUCTION

Nowadays, functionally graded materials are widely used in various industries [1-3]. Recently, much attention has been paid to the creation of calculation schemes of the indentation for determination the stress-strain state of functionally graded materials witch elastic moduli vary with the depth of the product material either according to exponential or power law [4-7]. These schemes are based on the solving contact problems of the theory of elasticity. On the other hand, methods for determining the parameters of elastic modules, the magnitude of which varies with depth, are also based on the solution of contact problems on the indentation of a material sample [8-11].

The paper proposes a theoretical-experimental technique for determining the shear modulus parameters of a functionally graded half-space, which shear modulus varies exponentially with depth. A secondary problem on the antiplane shear of the surface of a functionally graded half-space with a strip punch is formulated. The solution of the secondary contact problem is reduced to an integral equation of the first kind with a difference kernel. Approximate analytical solutions of the integral equation are constructed by asymptotic methods upon the dimensionless parameter of the problem. The dependence of contact stresses on the shear modulus parameters of the half-space is analyzed. Using the analytical solution of the contact problem, a method for determining the shear modulus parameters of the half-space from the contact shear stresses under the punch and from the static conditions is proposed.

PROBLEM STATEMENT

Let it be known, that the shear modulus of the elastic functionally graded material of the half-space changes with the depth according to the exponential law

$$\mu(y) = \mu_0 e^{2dy} \quad (0 \le y < \infty) \tag{1}$$

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where the parameter $\mu_0 = \mu(0)$ is the value of the shear modulus on the surface of the half-space and the parameter d determines the intensity of the shear modulus change with depth.

The task is to determine the shear modulus parameters μ_0 and d. To solve it a secondary contact problem is formulated. Let us consider the shear of the surface of the elastic half-space by the strip punch of $2a$ width ($|x| \le a$, $y = 0$) by the surface displacement ε ($w(x,0) = \varepsilon$, $|x| \le a$, where $w(x, y)$ is the displacement in the half-space $0 \le y < \infty$ along axis z). The half-space is made of a functionally graded material. Its shear modulus depends on the depth coordinate y according to (1) [12]. Contact stresses $\varphi(x) = -\sigma_{yz}(x,0)$ arising under the punch are to be determined. It is assumed, that the displacements and stresses in the half-space vanish at infinity.

The equations of the theory of elasticity in stresses [13] in the case of antiplane deformation have the form

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \tag{2}$$

Stresses in the case of a functional heterogeneity of the elastic medium are expressed in terms of strains in the form

$$\sigma_{zx} = \mu(y)\frac{\partial w}{\partial x}, \quad \sigma_{zy} = \mu(y)\frac{\partial w}{\partial y} \tag{3}$$

Passing from the equation of the theory of elasticity in stresses (2) to the equation of the theory of elasticity in displacements, taking into account representation (1), we obtain a differential equation with constant coefficients with respect to the displacement function

$$\frac{\partial^2 w}{\partial y^2} + 2d\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x^2} = 0 \tag{4}$$

The surface mixed boundary conditions of the formulated contact problem about the pure shear of the surface of the functionally graded half-space by the strip punch have the form

$$w(x,0) = \varepsilon \quad |x| \le a, \tag{5}$$

$$\sigma_{yz}(x,0) = \begin{cases} -\tilde{\varphi}(x) & |x| \le a, \\ 0 & a < |x| \le \infty. \end{cases} \tag{6}$$

Here $\tilde{\varphi}(x)$ is the contact stresses under the punch which has to be determined.

Conditions at infinity are

$$\lim_{\sqrt{x^2+y^2}\to\infty} \left\{ w, \frac{\partial w}{\partial x} \right\} = 0 \tag{7}$$

INTEGRAL EQUATION

Based on the integral Fourier transformation, taking into account the condition at infinity (7), we obtain the integral representation of the displacements

$$w(x, y) = -(2\pi\mu_0)^{-1} \int_{-\infty}^{\infty} \tilde{\varphi}^F(\alpha) e^{-\lambda_- y} \lambda_-^{-1} e^{-i\alpha x} d\alpha \tag{8}$$

where $\tilde{\varphi}^F(\alpha) = \int_{-\infty}^{\infty} \tilde{\varphi}(x) e^{i\alpha x} dx$ is the Fourier transform of $\tilde{\varphi}(x)$, and $\lambda_- = -d - \sqrt{d^2 + \alpha^2}$, $d > 0$.

Satisfying the mixed boundary conditions (5), (6) and using (8), the integral equation is obtained

$$\int_{-a}^{a}\widetilde{\varphi}(\xi)k(\xi-x)\mathrm{d}\xi = 2\pi\mu_0\varepsilon\,, \quad |x|\le a\,, \tag{9}$$

$$k(t)=\int_{-\infty}^{\infty}K(\alpha)e^{i\alpha t}\mathrm{d}\alpha\,, \quad K(\alpha)=-\lambda_{-}^{-1}\,. \tag{10}$$

It should be noted, that a classical integral equation for a homogeneous half-space is obtained in the case $d=0$, when $\mu(y)=const$.

The integral equation (9), (10) is reduced to the integral equation in dimensionless form

$$\int_{-1}^{1}\varphi(\xi)k\left(\frac{\xi-x}{\lambda}\right)\mathrm{d}\xi = 2\pi\,, \qquad |x|\le 1\,, \qquad \lambda=\frac{1}{da}\,,. \tag{11}$$

$$k(t)=\int_{-\infty}^{\infty}K(\alpha)e^{i\alpha t}\mathrm{d}\alpha\,, \quad K(\alpha)=\left(1+\sqrt{1+\alpha^2}\right)^{-1} \tag{12}$$

The integral equation (11) is an integral equation of the Fourier convolution type of the first kind with a difference kernel with respect to the unknown dimensionless contact stresses $\varphi(x)=a\varepsilon^{-1}\mu_0^{-1}\widetilde{\varphi}(x)$. The function $K(\alpha)$ is an even function, multivalued on the complex plane with branch points of algebraic type $\alpha=\pm i$, and has the following asymptotic properties

$$K(\alpha)=|\alpha|^{-1}+\mathrm{O}(\alpha^{-2}) \ \text{ at } \ |\alpha|\to\infty\,, \tag{13}$$

$$K(\alpha)=K(0)+\mathrm{O}(\alpha^{2}) \ \text{ at } \ |\alpha|\to 0\,, \tag{14}$$

Where $K(0)=0{,}5$. The kernel $k(t)$ after calculating the quadrature in (12) takes the form

$$k(t)=2e^{t}\left(K_0(t)-\int_{t}^{\infty}\mathrm{d}\tau\int_{\tau}^{\infty}K_0(\xi)\mathrm{d}\xi\right) \tag{15}$$

where $K_0(t)$ is the Macdonald function [7]. For small t the estimation of $k(t)$ can be:

$$\frac{1}{2}k(t)=-2\ln|t|+\mathrm{O}(1) \quad \text{at} \quad t\to 0\,. \tag{16}$$

The listed properties of the kernel $k(t)$ allow us to conclude [14], that the solution of the integral equation (11) exists only in the class of function $\varphi(x)=\omega(x)\left(1-x^2\right)^{-1/2}$, where $\omega(x)\in C^1[-1{,}1]$.

To determine an effective analytical solutions of the integral equation (11) we use an asymptotic methods upon the dimensionless parameter $\lambda=(da)^{-1}$ [17].

EFFECTIVE ANALYTICAL SOLUTIONS FOR SMALL λ

We use the Wiener-Hopf method to get the effective solution for the small characteristic parameter λ.The solution of the integral equation (11) is constructed as a Neumann series, the zeroth-order term of which is given by the relation [17]

$$\varphi(x) = \varphi_+\left((1+x)\lambda^{-1}\right) + \varphi_-\left((1-x)\lambda^{-1}\right) - \varphi_\infty(x\lambda^{-1}) \tag{17}$$

The functions $\varphi_\pm(x)$ are determined from the corresponding (11) integral equation on the half-axis [14, 16]

$$\int_0^\infty \varphi_\pm(\xi)k(\xi - x)\mathrm{d}\xi = \frac{2\pi}{\lambda}, \quad 0 < x < \infty . \tag{18}$$

The function $\varphi_\infty(x)$ is determined from the corresponding (11) integral equation of the convolution type on an infinite interval [14, 16]

$$\int_{-\infty}^\infty \varphi_\infty(\xi)k(\xi - x)\mathrm{d}\xi = \frac{2\pi}{\lambda}, \quad -\infty < x < \infty \tag{19}$$

The solution of the integral equation (19) is constructed using the generalized Fourier integral transformation [15] and has the form

$$\varphi_\infty(x) = \frac{1}{\lambda K(0)}, \quad -\infty < x < \infty \tag{20}$$

The solution of the integral equation (18) is constructed by Wiener-Hopf method [16]:

$$\varphi_\pm(x) = -\frac{1}{\lambda K_-(0)}\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} \frac{e^{-i\alpha x}}{i\alpha K_+(\alpha)}\mathrm{d}\alpha, \quad x > 0 \tag{21}$$

As well as the dimensionless displacements of the free surface of the half-space outside the contact area $w_\mp(x)$ is defined:

$$w_\mp(x) = \frac{1}{2\pi}\int_{-\infty+id}^{\infty+id} \left(\frac{1}{i\alpha} - \frac{1}{K_-(0)}\frac{K_-(\alpha)}{i\alpha}\right)e^{-i\alpha x}\mathrm{d}\alpha, \quad x < 0 \tag{22}$$

Here, functions $K_\pm(\alpha)$ are functions that make up the factorization of a function $K(\alpha)$:

$$K(\alpha) = K_+(\alpha)K_-(\alpha) \tag{23}$$

where function $K_+(\alpha)$ is regular in the upper half-plane ($\mathrm{Im}(\alpha) > \eta_-$), function $K_-(\alpha)$ is regular in the lower half-plane ($\mathrm{Im}(\alpha) < \eta_+$). At that $\eta_- < c < d < \eta_+$.

Note, that $w_\mp(x) = \varepsilon^{-1}v_\mp(x)$, where $v_\mp(x)$ is the dimensional horizontal displacements of the free surface of the half-space outside the contact area.

The integrands in (21), (22) depend on $K_+(\alpha)$ and $K_-(\alpha)$. To calculate these quadratures one has to factorize the transform of the kernel $K(\alpha)$ from (23). To obtain an effective solution an approximation of $K(\alpha)$ is used, the factorization of which is carried out by elementary means.

The simplest approximations of $K(\alpha)$, which both satisfy properties (13), (14) and do not contain zeros and poles within the band $\eta_- < \mathrm{Im}(\alpha) < \eta_+$, appear to be the functions

$$\text{a)}\ K(\alpha) = \frac{1}{\sqrt{A^2 + \alpha^2}}, \ A = \frac{1}{K(0)}, \tag{24}$$

$$\text{b)}\ K(\alpha) = \frac{1}{\sqrt{A^2 + \alpha^2}}\frac{B^2 + \alpha^2}{C^2 + \alpha^2}, \ \frac{1}{A}\frac{B^2}{C^2} = K(0). \tag{25}$$

For the approximation (24) $A = 2$. For the approximation (25) $A = 1{,}5344$, $B = 6{,}0332$, $C = 6{,}8881$. The errors of such approximations of $K(\alpha)$ within the band $|\mathrm{Im}(\alpha)| < \inf(\eta_-, \eta_+)$ along the real axis ($\mathrm{Im}(\alpha) = 0$) do not exceed 12% and 4% for (24) and (25) respectively. Factorization (23) of the given approximations (24), (25) is achieved by elementary means

$$K_\pm(\alpha) = \frac{1}{\sqrt{A \mp i\alpha}}, \tag{26}$$

$$K_\pm(\alpha) = \frac{1}{\sqrt{A \mp i\alpha}} \frac{B \mp i\alpha}{C \mp i\alpha} \tag{27}$$

It is necessary to stress that $K_+(\alpha)$ is regular in the upper half-plane ($\mathrm{Im}(\alpha) > \eta_-$), and $K_-(\alpha)$ is regular in the lower half-plane ($\mathrm{Im}(\alpha) < \eta_+$), and $\eta_\pm = \inf(A, B, C)$ in each case a) and b) respectively. Calculating the quadrature in (21), after the substitution of $K_+(\alpha)$ from (26), (27), the following solutions of the integral equation (18) are obtained

$$\text{a)}\ \ \varphi_\pm(x) = \frac{1}{\lambda K_-(0)} \Phi(x, A, 0), \quad x > 0, \tag{28}$$

$$\text{b)}\ \ \varphi_\pm(x) = \frac{1}{\lambda K_-(0)} \left(\frac{C}{B} \Phi(x, A, 0) + \frac{B - C}{B} \Phi(x, A, B) \right), \quad x > 0, \tag{29}$$

where $\Phi(x, A, B) = \frac{1}{\sqrt{\pi x}} e^{-Ax} + \sqrt{A - B}\, e^{-Bx} \operatorname{erf}\left(\sqrt{(A - B)x}\right)$ and function $\operatorname{erf}(x)$ is the error function [15].

Calculating the quadrature in (22), after substitution of $K_-(\alpha)$ from (26), (27), displacements outside the punch are obtained

$$\text{a) a)}\ \ w_\mp(x) = \operatorname{erfc}\left(\sqrt{A(-x)}\right), \quad x < 0,, \tag{30}$$

$$\text{b)}\ \ w_\mp(x) = 1 - \frac{1}{\sqrt{A}} \Phi(-x, A, 0) + \frac{\sqrt{A}C}{B(C - A)} \left(\frac{B - A}{A} \frac{1}{\sqrt{\pi(-x)}} e^{-A(-x)} - \frac{B - C}{C} \Phi(-x, A, C) \right),, \tag{31}$$

where $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ [15].

Substituting of $\varphi_\pm(x)$ from (24), (25) and $\varphi_\infty(x)$ from (20) into (17), the solution of the integral equation of the problem (11) is obtained. After returning to the old variable in formulas (30), (31), functions $w_\mp\left((1 \pm x)\lambda^{-1}\right)$ in (30), (31) provide horizontal displacements of the stress-free half-space surface outside the contact area at $1 \pm x < 0$.

Integral characteristic of the problem is the shear punch force $\widetilde{T}$, which is determined by the integration of the obtained solution

$$\widetilde{T} = \int_{-a}^{a} \widetilde{\varphi}(x) \mathrm{d}x = a \int_{-1}^{1} \widetilde{\varphi}(x) \mathrm{d}x = \varepsilon \mu_0 T, \tag{32}$$

Where

$$T = \int_{-1}^{1} \varphi(x) \mathrm{d}x, \tag{33}$$

is the dimensionless shear force. Taking $\varphi(x)$ in the form of superposition (17) and integrating it in the interval $[-1,1]$, T is obtained in the following form

$$\text{a)}\ \ T = T_+ + T_- - T_\infty,\ \ T_\pm = \int_{-1}^{1} \varphi_\pm\left(\frac{1 \pm x}{\lambda}\right) \mathrm{d}x,\ \ T_\infty = \int_{-1}^{1} \varphi_\infty\left(\frac{x}{\lambda}\right) \mathrm{d}x, \tag{34}$$

Using the solution of the integral equation (11) in the multiplicative form [14]

$$\varphi(x) = \varphi_+\left((1 + x)\lambda^{-1}\right) \varphi_-\left((1 - x)\lambda^{-1}\right) \varphi_\infty^{-1}(x\lambda^{-1}), \tag{35}$$

where $\varphi_{\pm}\left((1\pm x)\lambda^{-1}\right)$ is from (28), (29), and $\varphi_{\infty}(x\lambda^{-1})$ is from (20), allows us to obtain analytically more convenient formulas for T for all cases of approximation of $K(\alpha)$ (24), (25) of the same order of accuracy (32):

$$\text{a) } T = 1 + A\gamma, \quad \gamma = \frac{2}{\lambda} = 2ad, \tag{36}$$

$$\text{b) } T = \left(\frac{C}{B}\right)^2 \left(1 + 2\frac{A}{C}\frac{B-C}{B} + A\gamma\right) + \left(\frac{C-B}{B}\right)^2 \left(1 + 2\frac{C}{B}\frac{A-B}{C-B} + (A-B)\gamma\right) e^{-\gamma B}. \tag{37}$$

EFFECTIVE ANALYTICAL SOLUTIONS FOR LARGE λ

When constructing the solution of the integral equation of the problem (11) in the case of large values λ, it is necessary to take into account the fact that $K(\alpha)$ is represented by a series of

$$K(\alpha) = \sum_{k=0}^{4} c_k |\alpha|^{-k-1} + O\left(|\alpha|^{-6}\right) \quad \text{at} \quad |\alpha| \to \infty, \tag{38}$$

where $c_0 = 1$, $c_1 = -1$, $c_2 = \dfrac{1}{2}$, $c_3 = 0$, $c_4 = -\dfrac{1}{8}$.

With the decomposition (38) and the ratio (14), function $k(t)$ is represented by a series of [17]

$$\frac{1}{2}k(t) = -\ln|t| + a_{20}|t| + a_{11}t^2 \ln|t| + a_{21}|t|^3 + a_{31}t^2 + O\left(t^4 \ln|t|\right) \quad \text{at} \quad t \to 0, \tag{39}$$

Where $a_{20} = -\dfrac{\pi}{2}c_1$, $a_{11} = \dfrac{1}{2}c_2$, $a_{21} = \dfrac{1}{12}\pi c_3$, $a_{31} = -\dfrac{3}{4}c_2 + \dfrac{1}{2}\displaystyle\int_0^\infty \left[\alpha^2 - \alpha^3 K(\alpha) + c_1\alpha + c_2(1-e^{-\alpha})\right]\alpha^{-1} d\alpha$.

Substituting of (39) into (11) and differentiating it with respect to an external variable x, a singular integral equation with respect to $\varphi(x)$ is obtained. After inversion of the singular integral equation, an equivalent integral equation of the second kind is formed [17]. The solution of the latter is constructed by the method of successive approximations in the form of a double functional series

$$\varphi(x) = \frac{T}{a\sqrt{1-x^2}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \omega_{mn}(x)\lambda^{-m} \ln^n \lambda, \qquad n \le \left[\frac{m}{2}\right]., \tag{40}$$

Functions $\omega_{mn}(x)$ have the form

$$\begin{aligned}
&\omega_{00}(x) = \pi^{-1}, && \omega_{10}(x) = 4\pi^{-3}a_{20}S_1(x), \\
&\omega_{11}(x) = 0, && \omega_{01}(x) = 0, \\
&\omega_{20}(x) = \pi^{-1}\left(a_{131}\left(1-2x^2\right) + 32\pi^{-4}a_{20}^2\left(S_2(x) - 0{,}1508\right)\right), && \omega_{21}(x) = -\pi^{-1}a_{11}\left(1-2x^2\right), \\
&\omega_{31}(x) = -2\pi^{-3}a_{11}a_{20}S_4(x), && \omega_{30}(x) = \pi^{-3}\left(\frac{8}{3}a_{21} + \sum_{k=0}^{4} b_{1k}(x)S_{1+k}(x)\right),
\end{aligned} \tag{41}$$

where

$$b_{12} = \frac{8}{9}a_{11}a_{20}, \quad b_{10}(x) = 6a_{21}\left(1+2x^2\right) - 19{,}3024\pi^{-4}a_{20}^3, \quad b_{11} = 0,$$

$$b_{13} = 9a_{21} + 2a_{131}a_{20}, \quad a_{131} = a_{11}\left(\frac{3}{2} - \ln 2\right) + a_{31}, \quad b_{14}(x) = 64\pi^{-4}a_{20}^3,$$

$$S_1(x) = \left(1-2x^2\right) + 2\sqrt{1-x^2}\sum_{k=1}^{\infty} c_{2k+1}(x), \qquad c_n(x) = n^{-2}\sin(n \arccos x),$$

$$S_2(x) = \left(0{,}4356 + 0{,}1321x^2\right)\left(1-x^2\right) + 0{,}2494\, x\, l(x),$$

$$S_3(x) = -\left(1-2x^2\right) + 144\sqrt{1-x^2}\sum_{k=1}^{\infty}(2k-1)^{-2}(2k+3)^{-2}c_{2k+1}(x),$$

$$S_4(x) = \frac{1}{3} + \left(1-2x^2\right) + x\,l(x), \qquad l(x) = \left(1-x^2\right)\ln\frac{1-x}{1+x},$$

$$S_5(x) = r_1(x) + x\,l(x)\,r_2(x) - 0{,}04156\, l^2(x) + 0{,}3026\, S_1(x),$$

$$r_1(x) = 0{,}3547 - 0{,}8463x^2 + 0{,}3442x^4, \qquad r_2(x) = 0{,}1180 + 0{,}03305x^2.$$

The shear dimensionless force from (33) takes the form

$$\begin{gathered} T = \pi\Big((\ln(2\lambda)(1 - a_{11}\lambda^{-2} + 0{,}1801a_{11}a_{20}\lambda^{-3}) + a_{31} + , \\ +0{,}8106a_{20}\lambda^{-1} + \left(a_{31} + a_{11} - 0{,}03287a_{20}^2\right)\lambda^{-2} + \\ +\left(1{,}442a_{21} - 0{,}2702a_{11}a_{20} - 0{,}1807a_{31}a_{20} - 0{,}02450a_{20}^3\right)\lambda^{-3} + O\left(\lambda^{-4}\right)\Big)^{-1} \end{gathered} \tag{42}$$

An analysis of numerical results shows, that the docking of asymptotic solutions, obtained by asymptotic methods for small and large values λ, occurs in the interval $\lambda \in (1,2)$.

IDENTIFICATION OF SHEAR MODULUS PARAMETERS

The obtained analytical asymptotic solutions allow us to analyze the relationship of the mechanical parameters of the problem. Table 1 presents the values of the reduced stresses $aT^{-1}\varphi(0)$ in the middle of the punch at $x=0$ for different values of the parameter d for $a=1$.

For small values of d, for example, at $d=0{,}001$ ($\lambda=1000$) the reduced stress value $aT^{-1}\varphi(0)$ practically coincides with the reduced stress value at $x=0$ in the classical solution for the homogeneous half-space $aT_0^{-1}\varphi_0(0) = \pi^{-1} = 0{,}3183$, while $aT_0^{-1}\varphi_0(x) = \left(\pi\sqrt{1-x^2}\right)^{-1}$ [17]. At $d=10$ ($\lambda=0{,}1$) the reduced stress has the value which is 1,53 higher than the value of reduced stress at $d=0{,}001$ ($\lambda=1000$), which is nearly identical to the classical one.

TABLE 1

λ	d (m^{-1})	$aT^{-1}\varphi(0)$	$\dfrac{T^{-1}\varphi(0)}{T_0^{-1}\varphi_0(0)}$	h (m)
0,1	10	0,4878	1,53	0,0005
0,2	5	0,4762	1,50	0,0010
1,0	1	0,4068	1,28	0,0053
5	0,2	0,3450	1,08	0,0263
10	0,1	0,3329	1,05	0,0527
10^3	10^{-3}	0,3185	1,00	5,27

Analysis of table 1 shows that the value of the parameter d of the shear modulus can be determined by the values of the reduced contact stresses $aT^{-1}\varphi(0)$ in the middle of the contact area. Thus, the parameter $0<d<\infty$ of the shear modulus of the functionally graded half-space is determined from the solution of the equation transcendental with respect to d (for $a=1$)

$$\varphi(0) = T\sum_{m=n}^{\infty}\sum_{n=0}^{\infty}\omega_{mn}(0)d^m(-1)^n\ln^n d, \qquad 0<d<1,, \tag{43}$$

$$\varphi(0) = T(\varphi_+(d) + \varphi_-(d) - \varphi_\infty(0)), \qquad d > 1, , \tag{44}$$

where $\omega_{mn}(x)$ are given in (41), $\varphi_\pm(x)$ are given in (28), (29), $\varphi_\infty(x)$ is given in (20). Value T for (43) is determined by formula (42), and value T for (44) is determined by formulas (36), (37).

To determine the parameter μ_0 of the shear modulus, which coincides with the magnitude of the shear modulus on the surface of the half-space $\mu(0)$, we use the static condition (32), from which

$$\mu_0 = \frac{\widetilde{T}}{\varepsilon T}, \tag{45}$$

where value $\widetilde{T}$ is the dimension shear force, which should be determined from the experiment, and T is the dimensionless shear force, which is determined by (43) and (44) for $0 < d < 1$ and $d > 1$ respectively.

HOMOGENEITY AS A LIMITING CASE OF FUNCTIONALLY GRADED INHOMOGENEITY

For different values of the parameter d the values of the parameter h are presented in Table 1. Parameter h is the depth of the functionally graded half-space such that in the interval $[0,h]$ the shear modulus $\mu(y)$ differs from the shear modulus μ_0 on the half-space surface by not more than 1%. In other words, in the surface layer with the depth h the shear modulus $\mu(y)$ of the elastic material is practically constant. This means that we can create a layer in the half-space which is in a nearly homogeneous state with the error of 1% (or any other predetermined error). In such situations we can say that we are dealing with an “approximately homogeneous” area inside the functionally graded half-space.

The same effect in creating areas of an “approximately homogeneous” state in a functionally graded half-space was achieved in [18] using the parameter h of the hyperbolic shear modulus $\mu(z) = \mu h(h-z)^{-1}$.

While solving the shear problem for a homogeneous half-space it is impossible to determine a unique correlation between the displacements $w(x,y)$ and the contact stresses $\varphi_0(x)$

$$w(x,y) = \frac{\varepsilon}{\pi}\int_{-1}^{1} \varphi_0(\xi) k\left(\frac{\xi - x}{\lambda}, y\right) \mathrm{d}\xi, \qquad \lambda = \frac{1}{ad}, \quad |x| < \infty, \quad y > 0, \tag{46}$$

$$k(t,y) = \int_0^\infty |\alpha|^{-1} e^{-y} \cos\alpha t \, \mathrm{d}\alpha, \tag{47}$$

because the integral in (47) diverges.

Formulas for the displacements $w(x,y)$ of a homogeneous half-space can be obtained using the formula (8) for $w(x,y)$ of the considered problem on shear of a functionally graded half-space at $d = 0$. The solution of the considered shear problem for a functionally graded half-space with the modulus kind of (1) leads to a unique and specific relationship between $w(x,y)$ and $\varphi(x)$ which is determined by the operator

$$w(x,y) = \frac{\varepsilon}{\pi}\int_{-1}^{1} \varphi(\xi) \theta\left(\frac{\xi - x}{\lambda}, yd\right) \mathrm{d}\xi, \qquad \lambda = \frac{1}{ad}, \quad |x| < \infty, \quad y > 0 ,, \tag{48}$$

$$\theta(u,v) = \int_0^\infty \zeta^{-1}(\alpha) \exp(-\zeta(\alpha) v) \cos u\alpha \, \mathrm{d}\alpha, \quad \zeta(\alpha) = 1 + \sqrt{1+\alpha^2} ,, \tag{49}$$

where the integral in (49) converges at $d > 0$.

The formula (48) can be used to approximately solve the problem for a homogeneous half-space, because it is possible to pick the value of the parameter d of the shear modulus $\mu(y)$ in such a way that the half-space will be "approximately homogeneous" up to some depth h with the predetermined error.

The stresses $\sigma_{yz}(x,y)$ in the half-space are uniquely defined by the contact stresses $\varphi(x)$ as follows

$$\sigma_{yz}(x,y) = -\frac{\varepsilon d\mu(y)}{\pi}\int_{-1}^{1}\varphi(\xi)\kappa\left(\frac{\xi - x}{\lambda}, yd\right)\mathrm{d}\xi\,, \quad |x| < \infty\,, \quad y > 0\,,, \tag{50}$$

$$\kappa(u,v) = \int_{0}^{\infty}\exp(-\zeta(\alpha)v)\cos u\alpha\,\mathrm{d}\alpha\,, \tag{51}$$

The obtained solution (48)-(51) allows to analyze the behavior of $w(x,y)$ and $\sigma_{yz}(x,y)$ inside the half-space and to specify their dependence on the shear modulus $\mu(y)$ and depth y. Representing $\zeta(\alpha)$ in (48)-(51) as $\zeta(\alpha) = 2 + \alpha^2\zeta^{-1}(\alpha)$ we obtain that the shear stress $\sigma_{yz}(x,y)$ and the displacement $w(x,y)$ in the functionally graded half-space decrease with the depth proportionally to $\frac{1}{\sqrt{y}}$ and $\frac{1}{\sqrt{y\mu(y)}}$ correspondingly.

CONCLUSIONS

The obtained asymptotic solutions of the problem on the shear of the elastic functionally graded half-space by the strip punch allows:

– to study the effect of parameters d and μ_0 of the shear modulus $\mu(y)$ on the contact stresses;

– to obtain the formulas which is needed to determine the shear modulus parameters of the functionally graded half-space by the contact stresses in the middle of a contact and the indentation force;

– to determine the area of an "approximately homogeneous" state of a functionally graded half-space. In an "approximately homogeneous" area of a functionally graded half-space a unique relationship between contact stresses and displacements exists. This is impossible when solving the problem of a shear for a homogeneous half-space.

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