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SUBHARMONIC SOLUTIONS FOR A CLASS OF LAGRANGIAN SYSTEMS

Anouar Bahrouni

Department of Mathematics, Faculty of Sciences
University of Monastir
5019 Monastir, Tunisia

Marek Izydorek* and Joanna Janczewska

Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
Narutowicza 11/12, 80-233 Gdańsk, Poland

Abstract. We prove that second order Hamiltonian systems $-\ddot{u} = V_u(t, u)$ with a potential $V: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ of class C^1, periodic in time and superquadratic at infinity with respect to the space variable have subharmonic solutions. Our intention is to generalise a result on subharmonics for Hamiltonian systems with a potential satisfying the global Ambrosetti-Rabinowitz condition from [14]. Indeed, we weaken the latter condition in a neighbourhood of $0 \in \mathbb{R}^N$. We will also discuss when subharmonics pass to a nontrivial homoclinic orbit.

1. **Introduction.** A variational approach to the study of periodic solutions of Hamiltonian systems was initiated by Poincaré at the end of the XIX century. In the first half of the XX century, Morse and Lusternik-Shnirelman theories significantly contributed to the development of research in this direction. In the second half of the XX century, the mountain pass theorem, Ekeland's principle, linking theorems and Conley theory played an important role in the study of periodic orbits. In the last three decades, variational methods have been intensively developed and applied in the theory of ordinary and partial differential equations. Let us quote here only selected books: [1, 2, 4, 13, 15]. These methods rely on many variational principles in Hamiltonian dynamics, the two most important of which are Lagrangian and Hamiltonian action functionals.

The present work can be summarised by the following two aims:

- Prove the existence of subharmonics to a class of Hamiltonian systems by applying a classical approach based on the mountain pass theorem [3].
- Get a nontrivial homoclinic orbit for a slightly smaller class of Hamiltonian systems by a complementary approach based on the approximative method [11].

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* Corresponding author: Marek Izydorek.

Namely, in this paper we shall study the existence of subharmonic solutions for Lagrangian systems of the type

$$-\ddot{u}(t) = V_u(t, u(t)) \tag{1}$$

with a C^1-smooth potential $V\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ of the form

$$V(t,u) = -K(t,u) + F(t,u), \tag{2}$$

where $K, F\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are C^1-smooth mappings which are T-periodic in t (for some $T > 0$) and satisfy the following conditions:

$(V1)$ there are constants $b_1, b_2 > 0$ such that for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$,

$$b_1|u|^2 \le K(t,u) \le b_2|u|^2,$$

$(V2)$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$,

$$K(t,u) \le (K_u(t,u), u) \le 2K(t,u),$$

$(V3)$ there exist $r > 0$, $\mu > 2$ and $0 < \nu < b_1$ such that for all $t \in \mathbb{R}$,
  (i) $0 < \mu F(t,u) \le (F_u(t,u), u)$ if $|u| \ge r$,
  (ii) $2F(t,u) \le (F_u(t,u), u)$ and $|F(t,u)| \le \nu |u|^2$ if $|u| < r$.

Here and subsequently, $(\cdot,\cdot)\colon \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^N$ and $|\cdot|\colon \mathbb{R}^N \to [0,\infty)$ is the induced norm.

Clearly a solution u of (1) over $[-T, T]$ verifying

$$u(-T) - u(T) = \dot{u}(-T) - \dot{u}(T) = 0 \tag{3}$$

can be extended by $2T$-periodicity over $\mathbb{R}$ to give a $2T$-periodic solutions of (1). We shall show that (1) possesses solutions u such that

$$u(-kT) - u(kT) = \dot{u}(-kT) - \dot{u}(kT) = 0 \tag{4}$$

for some $k \ge 2$, where the minimal period is greater than $2T$. We will call such solutions *subharmonics*.

Let us briefly discuss our assumptions. Condition $(V1)$ is the pinching condition due to M. Izydorek and J. Janczewska [9]. The model potential satisfying $(V1)$ and $(V2)$ takes the form

$$K(t,u) = \frac{1}{2}(L(t)u, u),$$

where $L\colon \mathbb{R} \to \mathbb{R}^{N^2}$ is a continuous T-periodic matrix valued function such that $L(t)$ is positive definite and symmetric for every $t \in \mathbb{R}$. Condition $(V3)(i)$ is the superquadratic growth condition due to A. Ambrosetti and P.H. Rabinowitz [14]. This condition implies that F and V grow faster than $|u|^2$ as $|u| \to \infty$ (compare (5)). Condition $(V3)(ii)$ determines the behaviour of F at the neighbourhood of $0 \in \mathbb{R}^N$. It follows that $F_u(t,0) = 0$ for each $t \in \mathbb{R}$.

Our intention is to generalise the Rabinowitz result on subharmonic solutions [14], where the author assumes that F satisfies the superquadratic growth condition $(V3)(i)$ also for $|u| \le r$. Let us remark that $(V3)(ii)$ instead of $(V3)(i)$ for $|u| < r$ allows F to be negative in the neighbourhood of $0 \in \mathbb{R}^N$ (compare Example 2).

Our result is as follows.

Theorem 1.1. *We assume that V satisfies the conditions $(V1) - (V3)$. Then the Hamiltonian system (1) possesses a sequence of subharmonic solutions, i.e. for each $k \in \mathbb{N}\setminus\{0\}$ there is a $2kT$-periodic solution u_k of (1) such that along a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ the minimal period of u_k tends to $+\infty$ when $k \to \infty$.*

We prove Theorem 1.1 in the next two sections. In Section 3 we show the existence of a nontrivial homoclinic orbit of (1) by some stronger assumptions on F. Finally, we discuss some examples.

2. **Preliminaries.**

Let us start with some preliminary facts, notions and notation.

Lemma 2.1. *Under the condition $(V3)(i)$, the following inequality holds:*

$$r^\mu F(t,u) \geq |u|^\mu \inf_{t\in\mathbb{R},|x|=r} F(t,x) \quad \text{if} \quad |u| \geq r. \tag{5}$$

Proof. It is readily seen by $(V3)(i)$ that for all $t \in \mathbb{R}$ and $|u| \geq r$ the map

$$[1,\infty) \ni \xi \longrightarrow \xi^{-\mu} F(t,\xi u)$$

is non-decreasing. Set $v = r\frac{u}{|u|}$. For all $t \in \mathbb{R}$ we obtain

$$\left(\frac{|u|}{r}\right)^{-\mu} F(t,u) \geq F(t,v),$$

which yields (5). □

Clearly, (5) implies that

$$\frac{F(t,u)}{|u|^2} \to +\infty$$

as $|u| \to \infty$ uniformly in $t \in \mathbb{R}$.

For each $k \in \mathbb{N}$, we let E_k be the Sobolev space $W^{1,2}_{2kT}(\mathbb{R},\mathbb{R}^N)$ of $2kT$-periodic $W^{1,2}$-functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ equipped with the norm

$$\|u\|_{E_k} = \left(\int_{-kT}^{kT} \left(|u(t)|^2 + |\dot{u}(t)|^2\right) dt\right)^{\frac{1}{2}}.$$

For $1 \leq q < +\infty$, let $L^q_{2kT}(\mathbb{R},\mathbb{R}^N)$ be the space of $2kT$-periodic L^q-functions with the norm

$$\|u\|_{L^q_{2kT}} = \left(\int_{-kT}^{kT} |u(t)|^q dt\right)^{\frac{1}{q}}.$$

Let $L^\infty_{2kT}(\mathbb{R},\mathbb{R}^N)$ be the space $2kT$-periodic, essentially bounded and measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with the norm

$$\|u\|_{L^\infty_{2kT}} = \operatorname{ess\,sup}\{|u(t)| \colon t \in [-kT,kT]\}.$$

We note for later reference that there is a positive constant $C > 0$ such that for each $k \in \mathbb{N}$,

$$\|u\|_{L^\infty_{2kT}} \leq C\|u\|_{E_k}. \tag{6}$$

Furthermore, if $u\colon \mathbb{R} \to \mathbb{R}^N$ is a continuous function and $\dot{u}$ is locally square integrable, then for every $t \in \mathbb{R}$,

$$|u(t)| \leq \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(|u(s)|^2 + |\dot{u}(s)|^2\right) ds\right)^{\frac{1}{2}}. \tag{7}$$

Both (6) and (7) are proved in [14, 9].

We now define for $k \in \mathbb{N}$ a functional $I_k\colon E_k \to \mathbb{R}$ by

$$I_k(u) = \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{u}(t)|^2 + K(t,u(t)) - F(t,u(t))\right) dt. \tag{8}$$

Then $I_k \in C^1(E_k, \mathbb{R})$ and, moreover,

$$I_k'(u)v = \int_{-kT}^{kT} ((\dot{u}(t), \dot{v}(t)) + (K_u(t, u(t)), v(t)) - (F_u(t, u(t)), v(t)))\, dt. \tag{9}$$

Clearly, critical points of I_k are classical $2kT$-periodic solutions of (1).

In the next section we will show the existence of a critical point of I_k by using the mountain pass theorem (see [1, 3, 13]). This theorem provides the minimax characterisation for a critical value which is important for our argument. Let us recall its statement for the convenience of the reader.

Theorem 2.2. *Let E be a real Banach space and $I\colon E \to \mathbb{R}$ a C^1-smooth functional. If I satisfies the following conditions:*

(i) *$I(0) = 0$,*

(ii) *every sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and $I'(u_n) \to 0$ in E^* as $n \to \infty$ contains a convergent subsequence (Palais-Smale condition),*

(iii) *there exist constants $\varrho, \alpha > 0$ such that $I|_{\partial B_\varrho(0)} \geq \alpha$,*

(iv) *there is some $e \in E \setminus \overline{B}_\varrho(0)$ such that $I(e) < 0$,*

where $B_\varrho(0)$ denotes the open ball in E of radius ϱ about 0, then I has a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E)\colon g(0) = 0,\ g(1) = e\}.$$

3. **Periodic solutions.** Now we will prove that (1) possesses subharmonic solutions.

Lemma 3.1. *For each $k \in \mathbb{N}$, the functional I_k given by (8) has the mountain pass geometry, i.e. it satisfies the conditions $(i) - (iv)$ in Theorem 2.2.*

Proof. Fix $k \in \mathbb{N}$. Clearly, $I_k(0) = 0$, which is (i). To prove the Palais-Smale condition (ii), we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E_k$ such that $\{I_k(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, and $I_k'(u_n) \to 0$ in E_k^* as $n \to \infty$. Thus there is a constant $d_k > 0$ such that for each $n \in \mathbb{N}$,

$$|I_k(u_n) - \frac{1}{\mu} I_k'(u_n)u_n| \leq d_k(1 + \|u_n\|_{E_k}).$$

Applying (8), (9), $(V1) - (V3)$ we get

$$I_k(u_n) - \frac{1}{\mu} I_k'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \min\{1, 2b_1\}\|u_n\|_{E_k}^2 + \frac{1}{\mu}\int_{-kT}^{kT} ((F_u(t, u_n(t)), u_n(t)) - \mu F(t, u_n(t)))\, dt,$$

$$I_k(u_n) - \frac{1}{\mu} I_k'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \min\{1, 2b_1\}\|u_n\|_{E_k}^2 + \frac{1}{\mu}\int_{\{t\in[-kT,kT]\colon |u_n(t)|<r\}} ((F_u(t, u_n(t)), u_n(t)) - \mu F(t, u_n(t)))\, dt,$$

$$I_k(u_n) - \frac{1}{\mu} I_k'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \min\{1, 2b_1\}\|u_n\|_{E_k}^2 - m_k.$$

Consequently,

$$\left(\frac{1}{2}-\frac{1}{\mu}\right)\min\{1,2b_1\}\|u_n\|^2_{E_k}\le d_k(1+\|u_n\|_{E_k})+m_k,$$

which yields that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E_k. Going to a subsequence if necessary, we can assume that there is $u\in E_k$ such that $u_n\rightharpoonup u$ in E_k as $n\to\infty$, and so $u_n\to u$ uniformly on $[-kT,kT]$, which implies in particular that $\|u_n-u\|_{L^2_{2kT}}\to 0$ as $n\to\infty$.

Using (9) we have

$$\begin{aligned}\|\dot{u}_n-\dot{u}\|^2_{L^2_{2kT}}&=(I_k'(u_n)-I_k'(u),u_n-u)\\&\quad+\int_{-kT}^{kT}(V_u(t,u_n(t))-V_u(t,u(t)),u_n(t)-u(t))\,dt\end{aligned}$$

for each $n\in\mathbb{N}$. As $I_k'(u_n)\to 0$ in E_k^*, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E_k and $u_n\to u$ uniformly on $[-kT,kT]$, we conclude that $\|\dot{u}_n-\dot{u}\|_{L^2_{2kT}}\to 0$ as $n\to\infty$. In consequence, $\|u_n-u\|_{E_k}\to 0$ as $n\to\infty$ and the Palais-Smale condition is shown.

We will prove now that there exist $\varrho>0$ and $\alpha>0$ independent of k such that $I_k|_{\partial B_\varrho(0)}>\alpha$, which is (iii).

Set

$$\varrho=\frac{r}{2C},$$

where $C>0$ and $r>0$ are defined by (6) and $(V3)$, respectively. We assume that $\|u\|_{E_k}\le\varrho$. From (6) it follows that $\|u\|_{L^\infty_{2kT}}<r$. Applying (8), $(V1)$ and $(V3)(ii)$ we get

$$\begin{aligned}I_k(u)&\ge\int_{-kT}^{kT}\left(\frac{1}{2}|\dot{u}(t)|^2+b_1|u(t)|^2-\nu|u(t)|^2\right)dt\\&\ge\min\left\{\frac{1}{2},b_1-\nu\right\}\|u\|^2_{E_k},\end{aligned}$$

and hence, if $\|u\|_{E_k}=\varrho$,

$$I_k(u)\ge\min\left\{\frac{1}{2},b_1-\nu\right\}\varrho^2\equiv\alpha.$$

It remains to show (iv). To this aim, we assume that $Q\in E_1$, $Q(\pm T)=0$ and $\min_{|t|\le T}|Q(t)|>r$. Combining (8), $(V1)$ and (5) we have for every $s>1$,

$$I_1(sQ)\le\frac{s^2}{2}\max\{1,2b_2\}\|Q\|^2_{E_1}-\frac{s^\mu}{r^\mu}\inf_{t\in\mathbb{R},|x|=r}F(t,x)\int_{-T}^{T}|Q(t)|^\mu dt. \tag{10}$$

As $\mu>2$ and $\inf_{t\in\mathbb{R},|x|=r}F(t,x)>0$, there is $s>1$ such that $\|sQ\|_{E_1}>\varrho$ and $I_1(sQ)<0$.

Set $e_1(t)=sQ(t)$. For each $k\in\mathbb{N}$, let $e_k(t)=e_1(t)$ if $|t|\le T$, and $e_k(t)=0$ if $T<|t|\le kT$. Then $e_k\in E_k$, $\|e_k\|_{E_k}=\|e_1\|_{E_1}>\varrho$ and $I_k(e_k)=I_1(e_1)<0$, which completes the proof. □

Consequently, by Theorem 2.2 and Lemma 3.1, for each $k\in\mathbb{N}$ the action functional I_k has a critical value $c_k\ge\alpha$ given by

$$c_k=\inf_{g\in\Gamma_k}\max_{s\in[0,1]}I_k(g(s)), \tag{11}$$

where

$$\Gamma_k=\{g\in C([0,1],E_k)\colon g(0)=0,\ g(1)=e_k\}.$$

Hence for each $k \in \mathbb{N}$ there is $u_k \in E_k$ such that

$$I_k(u_k) = c_k, \quad I_k'(u_k) = 0. \tag{12}$$

Proof of Theorem 1.1. In order to prove Theorem 1.1, it suffices to show that the sequence of norms $\{\|u_k\|_{E_k}\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Then there is a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ of subharmonic solutions of (1).

For this purpose, let us define

$$M = \max_{s\in[0,1]} I_1(se_1).$$

Since $I_k(se_k) = I_1(se_1)$ for all $k \in \mathbb{N}$ and $s \in [0,1]$, and $se_k \in \Gamma_k$, we have

$$c_k \le \max_{s\in[0,1]} I_k(se_k) = \max_{s\in[0,1]} I_1(se_1) = M. \tag{13}$$

Applying (8) and (9) we get

$$\begin{aligned} c_k &= I_k(u_k) - \frac{1}{2} I_k'(u_k)u_k \\ &\ge \int_{-kT}^{kT} \left(\frac{1}{2}(F_u(t,u_k(t)), u_k(t)) - F(t,u_k(t)) \right) dt \end{aligned}$$

for each $k \in \mathbb{N}$, and by $(V3)$,

$$c_k \ge \left(\frac{\mu}{2} - 1 \right) \int_{\{t\in[-kT,kT]\colon |u_k(t)|\ge r\}} F(t,u_k(t))dt.$$

Combining this with (8), $(V1)$ and $(V3)$, for each $k \in \mathbb{N}$, we have

$$c_k \ge \int_{-kT}^{kT} \left(\frac{1}{2}|\dot u_k(t)|^2 + b_1|u_k(t)|^2 \right) dt - \frac{2c_k}{\mu - 2} - \int_{-kT}^{kT} \nu |u_k(t)|^2 dt,$$

and, finally by (13),

$$M + \frac{2M}{\mu-2} \ge \min\left\{ \frac{1}{2}, b_1 - \nu \right\} \|u_k\|^2_{E_k}.$$

□

4. **Homoclinic orbits.** We recall first that $u\colon \mathbb{R} \to \mathbb{R}^N$ is called a homoclinic (to 0) orbit of (1) if $u(t) \to 0, \dot u(t) \to 0$ as $t \to \pm\infty$.

Theorem 4.1. *We assume that $V\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is of the form (2), where K satisfies $(V1) - (V2)$, and F satisfies the following two conditions:*

$(V3')$ *there exist $r > 0$, $\mu > 2$ and $0 < \nu < b_1$ such that for all $t \in \mathbb{R}$,*
 (i) $0 < \mu F(t,u) \le (F_u(t,u),u)$ if $|u| \ge r$,
 (ii) $0 < 2F(t,u) \le (F_u(t,u),u)$ and $F(t,u) \le \nu|u|^2$ if $0 < |u| < r$.

$(V4)$ *$F_u(t,u) = o(|u|)$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}$.*

Then (1) possesses a nontrivial homoclinic orbit.

Of course, $(V3')$ implies $(V3)$. Furthermore, it is easily seen that combining $(V4)$ with $(V3')(ii)$ we get $F(t,u) = o(|u|^2)$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}$.

Since homoclinics are important objects in the understanding of the global behaviour of Hamiltonian systems, it is desirable to study their existence. See for example in [5, 12, 16, 17]. The technical difficulties encountered in looking for homoclinics go beyond those of the periodic setting in at least two ways:

- An action functional associated with a given problem may be infinite on the natural class of functions, and so one has to find a renormalized functional.

- There is a loss of compactness due to the fact that solutions are defined on $\mathbb{R}$, and this fact complicates the study of Palais-Smale sequences.

To overcome these difficulties, one can apply approximation methods, Lions' principle on concentration-compactness, and the LS-index introduced in [6] and developed in [7, 8]. Moreover, the shadowing chain lemma of [10] often allows to prove the existence of homoclinics for planar Lagrangian systems.

To prove Theorem 4.1 we use the approximative scheme by J. Janczewska [11].

Let E be the Sobolev space $W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ with the standard norm

$$\|u\|_E = \left(\int_{-\infty}^{\infty} \left(|u(t)|^2 + |\dot{u}(t)|^2 \right) dt \right)^{\frac{1}{2}}.$$

Let us denote by $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ the space of C^2-smooth functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the topology of almost uniformly convergence of functions and all derivatives up to the order 2.

Theorem 4.2. *Let $V \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}^N$ satisfy the following conditions:*

$(C1)$ *V is C^1-smooth with respect to all variables and T-periodic with respect to t, $T > 0$,*

(C_2) *f is bounded, continuous and square integrable.*

Assume also that for each $k \in \mathbb{N}$, the Newtonian system

$$\ddot{u}(t) + V_u(t, u(t)) = f_k(t) \tag{14}$$

has a $2kT$-periodic solution $u_k \in E_k$, where $f_k \colon \mathbb{R} \to \mathbb{R}^N$ is a $2kT$-periodic extension of f restricted to the interval $[-kT, kT)$ over $\mathbb{R}$.

Then, if the sequence of real numbers $\{\|u_k\|_{E_k}\}_{k\in\mathbb{N}}$ is bounded, there exist a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ and a function $u \in E$ such that

$$u_{k_j} \to u, \text{ as } j \to \infty,$$

in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ and u is an almost homoclinic solution of the Newtonian system

$$\ddot{u}(t) + V_u(t, u(t)) = f(t), \tag{15}$$

i.e. $u(t) \to 0$ as $t \to \pm\infty$.

The approximative method was formulated and proved by J. Janczewska [11] for inhomogenous second order Hamiltonian systems ($f \not\equiv 0$). The proof for $f \equiv 0$ is similar. However, in our case a homoclinic solution obtained by applying Theorem 4.2 may be trivial and we have to prove that it is not. To this aim, by the use of $(V3')(ii)$ we will introduce a certain auxiliary real function Y. A similar auxiliary function was applied by Rabinowitz [14] based on the global condition $(V3')(i)$.

Proof of Theorem 4.1. From Theorem 1.1 and its proof it follows that (1) possesses a sequence $\{u_k\}_{k\in\mathbb{N}} \subset E_k$ of subharmonic solutions defined by (12) and there is $\tilde{M} > 0$ such that for every $k \in \mathbb{N}$,

$$\|u_k\|_{E_k} \leq \tilde{M}.$$

By Theorem 4.2 we conclude that there is $u \in E$ such that going to a subsequence if necessary $u_k \to u$ as $k \to \infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ and u is an almost homoclinic solution of (1).

To finish the proof it remains to show that u is nontrivial and $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

Assume that $u \equiv 0$. Then $u_k \to 0$ uniformly on each compact subset of $\mathbb{R}$. As V is T-periodic in $t \in \mathbb{R}$, without loss of generality we can assume that for each $k \in \mathbb{N}$, u_k achieves its maximum at the interval $[-T, T]$. Hence

$$\|u_k\|_{L^\infty_{2kT}} = \max_{|t|\le kT} |u_k(t)| = \max_{|t|\le T} |u_k(t)| \to 0 \tag{16}$$

as $k \to \infty$. It follows that for $k \in \mathbb{N}$ sufficiently large, $\|u_k\|_{L^\infty_{2kT}} \le r$.

Let Y be a real function from $[0, r]$ into $\mathbb{R}$ given by $Y(0) = 0$, and

$$Y(s) = \max_{t\in[0,T], 0<|\xi|\le s} f(t,\xi) \quad \text{if } 0 < s \le r,$$

where

$$f(t,\xi) = \frac{(\xi, F_u(t,\xi))}{|\xi|^2}.$$

Let us remark that Y is non-negative. Indeed. By $(V3')(ii)$, for $0 < s \le r$, $t \in [0,T]$ and $0 < |\xi| \le s$, we have

$$Y(s) \ge \frac{(\xi, F_u(t,\xi))}{|\xi|^2} \ge \frac{2F(t,\xi)}{|\xi|^2} > 0.$$

Moreover, Y is non-decreasing. Fix $0 < s_1 < s_2 \le r$. As

$$\{f(t,\xi) \colon t \in [0,T],\ 0 < |\xi| \le s_1\} \subset \{f(t,\xi) \colon t \in [0,T],\ 0 < |\xi| \le s_2\},$$

we obtain $Y(s_1) \le Y(s_2)$.

Finally, Y is continuous. Fix $0 < s_0 \le r$ and $\varepsilon > 0$. There is $\delta > 0$ such that for all $t \in [0,T]$ and $0 < s \le r$, $|f(t,s) - f(t,s_0)| < \varepsilon$ if $|s - s_0| < \delta$.

If $s_0 < s < s_0 + \delta$, then $Y(s) \ge Y(s_0)$ and

$$\begin{aligned} Y(s) &= \max\left\{Y(s_0), \max_{t\in[0,T], s_0\le|\xi|\le s} f(t,\xi)\right\} \le \max\left\{Y(s_0), \max_{t\in[0,T]} f(t,s_0) + \varepsilon\right\} \\ &\le Y(s_0) + \varepsilon, \end{aligned}$$

and hence $Y(s) - Y(s_0) \le \varepsilon$.

If $s_0 - \delta < s < s_0$, then $Y(s) \le Y(s_0)$ and

$$\begin{aligned} Y(s_0) &= \max\left\{Y(s), \max_{t\in[0,T], s\le|\xi|\le s_0} f(t,\xi)\right\} \le \max\left\{Y(s), \max_{t\in[0,T]} f(t,s) + 2\varepsilon\right\} \\ &\le Y(s) + 2\varepsilon, \end{aligned}$$

and so $Y(s_0) - Y(s) \le 2\varepsilon$.

In order to prove the continuity of Y at 0, it is sufficient to show that $f(t,\xi) \to 0$ as $|\xi| \to 0$ uniformly in $t \in [0,T]$, which follows from $(V4)$ and the estimation below:

$$f(t,\xi) = \left(\frac{\xi}{|\xi|}, \frac{F_u(t,\xi)}{|\xi|}\right) \le \frac{|F_u(t,\xi)|}{|\xi|}.$$

By definition of Y, for $k \in \mathbb{N}$ sufficiently large,

$$Y(\|u_k\|_{L^\infty_{2kT}})|u_k(t)|^2 \ge (u_k(t), F_u(t, u_k(t))).$$

Integrating both sides we have

$$\left(\int_{-kT}^{kT} |u_k(t)|^2 dt\right) Y(\|u_k\|_{L^\infty_{2kT}}) \ge \int_{-kT}^{kT} (u_k(t), F_u(t, u_k(t)))dt.$$

From this and $(V1) - (V2)$ we get

$$Y(\|u_k\|_{L^\infty_{2kT}}) \cdot \|u_k\|^2_{E_k} \ge \min\{1, b_1\}\|u_k\|^2_{E_k},$$

and so $Y(\|u_k\|_{L^\infty_{2kT}}) \geq \min\{1, b_1\} > 0$, which contradicts (16).

Now, by (7), for each $t \in \mathbb{R}$,

$$|\dot{u}(t)| \leq \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(|\dot{u}(s)|^2 + |\ddot{u}(s)|^2\right) ds\right)^{\frac{1}{2}}. \tag{17}$$

Since $u(t) \to 0$ as $t \to \pm\infty$, combining (17) with (1) we conclude that $\dot{u}(t) \to 0$ as $t \to \pm\infty$, which completes the proof. □

5. **One-dimensional examples.**

Finally, we illustrate our results by the following one-dimensional examples.

Example 1. Let $K\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be given by $K(t,x) = (1+\cos^2 t)x^2$ and $F\colon \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$F(x) = \begin{cases} \frac{1}{2}e^{-2(x+1)} & \text{if } x \leq -1, \\ \frac{1}{2}x^2 & \text{if } |x| \leq 1, \\ \frac{1}{2}e^{2(x-1)} & \text{if } x \geq 1. \end{cases}$$

One can easily check that $(V1)-(V3)$ are satisfied with $b_1 = 1$, $b_2 = 2$, $r = \frac{3}{2}$, $\nu = \frac{1}{2}$ and $\mu = 3$.

Example 2. Take $K(t,x) = (2+\cos^2 t)x^2$, $x, t \in \mathbb{R}$, and set

$$F(x) = \begin{cases} \frac{1}{16}\left(x + \frac{\pi}{2} - \pi^{\frac{2}{3}}\right)^4 - \frac{1}{16}\pi^{\frac{8}{3}} & \text{if } x \leq -\frac{\pi}{2}, \\ -x^2\cos x & \text{if } |x| \leq \frac{\pi}{2}, \\ \frac{1}{16}\left(x - \frac{\pi}{2} + \pi^{\frac{2}{3}}\right)^4 - \frac{1}{16}\pi^{\frac{8}{3}} & \text{if } x \geq \frac{\pi}{2}. \end{cases}$$

A trivial verification shows that $(V1)-(V3)$ hold with $b_1 = 2$, $b_2 = 3$, $r = \frac{\pi}{2}$, $\nu = 1$ and $\mu = 2\pi^{\frac{1}{3}}$.

Example 3. Let $K(t,x) = (1+\sin^2 t)x^2$ for all $t, x \in \mathbb{R}$ and $F\colon \mathbb{R} \to \mathbb{R}$ be given by

$$F(x) = \begin{cases} \frac{1}{4}e^{-4(x+1)} & \text{if } x \leq -1, \\ \frac{1}{4}x^4 & \text{if } |x| \leq 1, \\ \frac{1}{4}e^{4(x-1)} & \text{if } x \geq 1. \end{cases}$$

One can immediately check that $(V1)-(V2)$, $(V3')$ and $(V4)$ are fulfilled with $b_1 = 1$, $b_2 = 2$, $r = 1$, $\nu = \frac{1}{4}$ and $\mu = 4$.

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E-mail address: bahrounianouar@yahoo.fr
E-mail address: marek.izydorek@pg.edu.pl
E-mail address: joanna.janczewska@pg.edu.pl