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Transverse surface waves on a cylindrical surface with coating

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ABSTRACT

We discuss the propagation of transverse surface waves that are so-called whispering-gallery waves along a surface of an elastic cylinder with coating. The coating is modelled in the framework of linearized Gurtin–Murdoch surface elasticity. Other interpretations of the surface shear modulus are given and relations to so-called stiff interface and stiff skin model are discussed. The dispersion relations are obtained and analyzed.

Introduction

The thin rod-like elements are widely used in NEMS and MEMS, see, *e.g.*, Bhushan (2017). Nowadays, it is well-known that due to high surface-to-volume ratio in comparison with a size of considered nanosized rod-like elements, one needs in a certain enhancement of the classic elasticity. To this end, we mention the surface elasticity applications, see, *e.g.*, Duan, Wang, and Karihaloo (2008), Wang et al. (2011), Javili, McBride, and Steinmann (2013) and Eremeyev (2016), where for example, the so-called size effect was described. The mostly used model of the surface elasticity proposed by Gurtin and Murdoch (1975, 1978) has origin in earlier landscape works by Laplace (1805, 1806) and Young (1805) for fluids and Gibbs for solids (Longley & Van Name, 1928), see also review by Orowan (1970). From the mathematical point of view, the presence of surface stresses change solutions of the corresponding boundary-value problems, see, *e.g.*, the analysis of stress singularity near crack tips by Kim, Ru, and Schiavone (2013) and Gorbushin, Eremeyev, and Mishuris (2020), or even bring new phenomena as anti-plane waves (Eremeyev, Rosi, & Naili, 2016).

Here, we discuss the so-called transverse surface waves propagating along the boundary of a cylindrical solid body whose the cross-section is circular. This type of surface waves is an example of the whispering-gallery waves known from works by Lord Rayleigh, see Strutt (1945) and Rayleigh (1910). In the framework of the linear elasticity, such waves for an elastic circular cross-section cylinder were discussed by Victorov (1974) and later were confirmed experimentally on cylindrical specimens made of CdS (Cadmium sulfide) by Vas'kova et al. (1974). In order to capture the material behaviour at the nanoscale, we use the linearized Gurtin–Murdoch surface elasticity. So, the present paper focuses on the analysis of these

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surface waves taking into account surface stresses. As for cylindrical nanorods the axisymmetric and torsional waves were analyzed in some works, see, *e.g.*, Chen, Wu, Zhang, and Zhang (2014), Xu and Fan (2016) and Huang (2018), the presented results add new view to the picture of surface waves in elastic circular cross-section cylinders with surface stresses.

The paper is organized as follows. In Section 1, we recall in brief the basics of the linearized Gurtin–Gurtin model. In Section 2, we consider antiplane deformations and reduce the problem under consideration to the wave equation with nonclassic boundary conditions. We also discuss in Section 3 the similarities of these conditions with ones for which we have in the case of so-called rigidly stiff interface discussed by Mishuris, Movchan, and Movchan (2006b, 2010), and some other approaches to coating modelling. The dispersion relations are analyzed in Section 4.

1. The linearized Gurtin–Murdoch model

Let us consider an isotropic elastic solid body which occupies a volume V with smooth enough boundary $S=\partial V$. In the framework of the Gurtin–Murdoch model, we introduce a surface strain energy and surface stress tensor (Gurtin & Murdoch, 1975). The latter is a generalization of a scalar surface tension in fluids. Considering small deformations, we get the surface stress tensor $\boldsymbol{\tau}$ by the formulae

$$\boldsymbol{\tau}=\gamma\mathbf{P}+2(\mu_s-\gamma)\boldsymbol{\varepsilon}+(\lambda_s+\gamma)\mathbf{P}(\operatorname{tr}\boldsymbol{\varepsilon})+\gamma\nabla_s\mathbf{u},\quad \boldsymbol{\varepsilon}=\frac{1}{2}\left(\mathbf{P}\cdot(\nabla_s\mathbf{u})+(\nabla_s\mathbf{u})^T\cdot\mathbf{P}\right), \tag{1}$$

where $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is a displacement vector, $\mathbf{x}$ is the position vector, t is time, $\boldsymbol{\varepsilon}$ is the surface strain tensor, γ is a scalar coefficient interpreted as a residual surface tension, λ_s and μ_s are the surface Lamé *moduli*, ∇_s is the surface nabla operator related to the three-dimensional one through the formula $\nabla_s=\mathbf{P}\cdot\nabla$, $\mathbf{P}\equiv\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$, $\mathbf{n}$ is the unit vector of outer normal to S and $\mathbf{I}$ is the 3D unit tensor. Here T and tr denotes the transpose and trace of a second-order tensor, $\otimes$ stands for the dyadic product. Hereinafter, we use the direct tensor calculus as described by Simmonds (1994) and Eremeyev, Cloud, and Lebedev (2018).

Let us note that $\boldsymbol{\tau}$ is the linearized first Piola–Kirchhoff surface stress tensor and the linearization was performed in the vicinity of an initial uniformly stressed state. Various formulations of the constitutive equation for $\boldsymbol{\tau}$ were discussed by Huang and Wang (2006), Duan et al. (2008) and Ru (2010). In particular, for solids, γ is usually smaller than λ_s and μ_s and can be neglected (Duan et al., 2008). In the literature, one can find simplified versions of (1) such as

$$\boldsymbol{\tau}=\gamma\mathbf{P}+2\mu_s\boldsymbol{\varepsilon}+\lambda_s\mathbf{P}(\operatorname{tr}\boldsymbol{\varepsilon}),\quad\text{or}\quad \boldsymbol{\tau}=2\mu_s\boldsymbol{\varepsilon}+\lambda_s\mathbf{P}(\operatorname{tr}\boldsymbol{\varepsilon}). \tag{2}$$

Eq. $(2)_2$ was derived by Gurtin and Murdoch (1975) as linearization in vicinity of natural reference placement.

In addition to the surface stresses, we introduce the surface kinetic energy density (Gurtin & Murdoch, 1978)

$$K=\frac{1}{2}m\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}, \tag{3}$$

where m is the surface mass density, the overdot stands for derivative with respect to t, whereas the centered dot denotes the scalar product.

In the bulk, we have Hooke's law for an istropic material

$$\boldsymbol{\sigma}=2\mu\mathbf{e}+\lambda\mathbf{I}\operatorname{tr}\mathbf{e},\quad \mathbf{e}=\frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right),$$

where $\boldsymbol{\sigma}$ and $\mathbf{e}$ are the stress and infinitesimal strain tensor, respectively, and λ and μ are the Lamé elastic moduli.

The corresponding boundary-value problem consists of the equation of motion

$$\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{f}=\rho\ddot{\mathbf{u}},\quad \forall\mathbf{x}\in V, \tag{4}$$

and the so-called generalized Laplace–Young equation (Duan et al., 2008), which plays a role of dynamic boundary condition

$$\mathbf{n}\cdot\boldsymbol{\sigma}=-\nabla_s\cdot\boldsymbol{\tau}-m\ddot{\mathbf{u}}+\mathbf{g},\quad \forall\mathbf{x}\in S, \tag{5}$$

where $\mathbf{f}$ and $\mathbf{g}$ are mass force and traction vectors. Obviously, the latter equation differs from the classic traction condition in linear elasticity. Eq. (5) contains second derivatives of $\mathbf{u}$ with respect to tangent spatial coordinates and the inertia term.

Let us note that the term $\gamma\mathbf{P}$ with constant γ results in normal pressure on S which can be treated as an external loading. Indeed, using the formula $\nabla_s\cdot\mathbf{P}=2H\mathbf{n}$, where $H=-1/2\operatorname{tr}\nabla_s\mathbf{n}$ is the mean curvature, this term transforms in (5) into $-2\gamma H\mathbf{n}$, that is to capillary pressure according to the classic the Young–Laplace equation (Adamson & Gast, 1997). So, one can simply add it to $\mathbf{g}$. As, in the following, we consider anti-plane deformations, for simplicity, we neglect it.

2. Anti-plane problem formulation

In order to describe transverse shear waves, let us consider an infinite elastic cylinder V of radius a with the boundary $S=\partial V$, see Fig. 1. Mass forces and surface traction are neglected, $\mathbf{f}=\mathbf{0}$ and $\mathbf{g}=\mathbf{0}$. The term $\gamma\mathbf{P}$ results in a constant hydrostatic pressure $-\gamma/a\mathbf{n}$ which corresponds to uniform constant displacement field. Omitting it, we restrict ourselves by displacements in the form

$$\mathbf{u}=u(r,\varphi,t)\mathbf{e}_z. \tag{6}$$

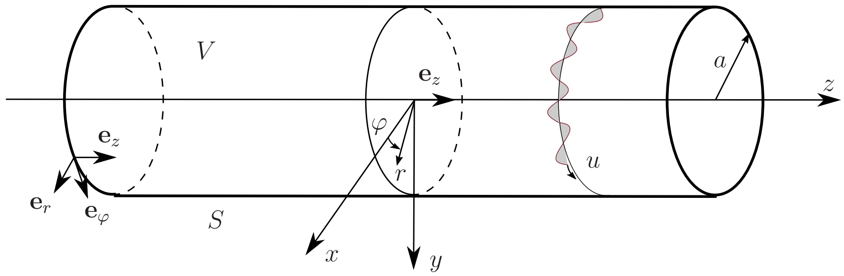

Fig. 1. An elastic cylinder with propagating transverse surface wave.

Note that this field differs from the torsional waves where the vector of displacements has the form

$$\mathbf{u}_t = u_\varphi(r, z, t)\mathbf{e}_\varphi, \tag{7}$$

and from axisymmetric waves with $\mathbf{u}$ in the form

$$\mathbf{u}_a = u_r(r, z, t)\mathbf{e}_r + u_z(r, z, t)\mathbf{e}_z. \tag{8}$$

Hereinafter, r, φ and z are the cylindrical coordinates with corresponding unit base vectors ($\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$), see, *e.g.*, Eremeyev et al. (2018). In this case, we have

$$\begin{aligned}
\mathbf{P} &= \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_z \otimes \mathbf{e}_z, \quad \mathbf{n} = \mathbf{e}_r,\\
\nabla &= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z},\\
\nabla_s &= \mathbf{e}_\varphi \frac{1}{a}\frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z},\\
\nabla \mathbf{u} &= (\nabla u) \otimes \mathbf{e}_z = \left(\frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi\right) \otimes \mathbf{e}_z,\\
\nabla_s \mathbf{u} &= (\nabla_s u) \otimes \mathbf{e}_z = \frac{1}{a}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi \otimes \mathbf{e}_z,\\
\boldsymbol{\sigma} &= \sigma_{rz}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \sigma_{\varphi z}(\mathbf{e}_\varphi \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\varphi)\\
&= \mu \frac{\partial u}{\partial r}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \mu \frac{1}{r}\frac{\partial u}{\partial \varphi}(\mathbf{e}_\varphi \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\varphi),\\
\boldsymbol{\tau} &= \tau_{\varphi z}\mathbf{e}_\varphi \otimes \mathbf{e}_z + \tau_{z\varphi}\mathbf{e}_z \otimes \mathbf{e}_\varphi\\
&= \mu_s \frac{1}{a}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi \otimes \mathbf{e}_z + (\mu_s - \gamma)\frac{1}{a}\frac{\partial u}{\partial \varphi}\mathbf{e}_z \otimes \mathbf{e}_\varphi.
\end{aligned}$$

Using these formulae and substituting (6) into (4), we get the wave equation in the polar coordinates

$$\mu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right] = \rho \ddot{u}. \tag{9}$$

The boundary condition (5) takes the form

$$\sigma_{rz} = \frac{1}{a}\frac{\partial \tau_{\varphi z}}{\partial \varphi} - m\ddot{u}, \quad r = a, \tag{10}$$

or in displacements,

$$\mu \frac{\partial u}{\partial r} = \mu_s \frac{1}{a^2}\frac{\partial^2 u}{\partial \varphi^2} - m\ddot{u}, \quad r = a. \tag{11}$$

Note that Eq. (11) has the same form as in the case of half-space and it is responsible for appearance of anti-plane surface waves (Eremeyev et al., 2016). Moreover, the boundary conditions at $r = a$ for shear stresses σ_{rz} have the same form also for the constitutive relations (2).

3. Remarks on the constitutive relations

In the framework of the Gurtin–Murdoch model, the surface stresses were introduced using the so-called direct approach. In fact, the membrane-type constitutive Eq. (1) was postulated. In literature, one can find extensive discussion on the nature

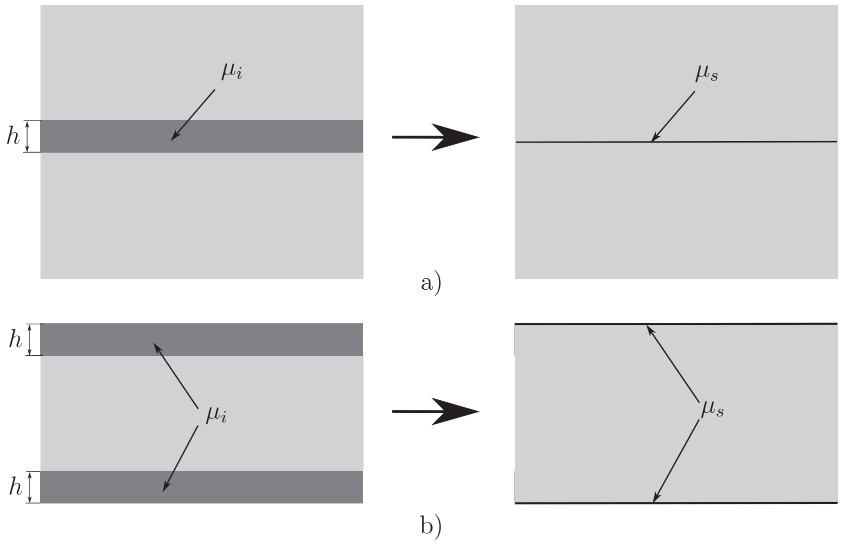

Fig. 2. Transition from a layer of finite thickness to a surface: a) stiff interface between two materials; b) near-surface layers of three-layered plate.

of surface stresses and surface energy, see, *e.g.* Adamson and Gast (1997), Rusanov (2005) and Murdoch (2005) and references therein. In particular, it is quite important to have properly determined material parameters of the models, that are ρ_s, γ, λ_s and μ_s. To this end, we mention atomistic calculations of the surface elasticity *moduli* and their straightforward experimental determination, see, *e.g.*, Miller and Shenoy (2000), Shenoy (2005) and Cuenot, Frétigny, Demoustier-Champagne, and Nysten (2004), Jing et al. (2006), Xu et al. (2017), respectively.

On the other side, let us also note that the Gurtin–Murdoch models describes the behaviour of a thin near-surface or near-interface layer. With thinness assumptions, it seems to be rather natural to consider also asymptotic techniques applied to such thin layers. We mention here *stiff interface* model discussed by Benveniste and Miloh (2001, 2007), see also Mishuris, Öchsner, and Kuhn (2006a) and Mishuris et al. (2006b, 2010), where the static and dynamic problems of crack growth along the stiff interface between two materials was analyzed. In particular, Eq. (11) coincides up to notations with the transmission condition on the stiff interface, see Mishuris et al. (2006b, Eq. (2.15)). This similarity gives us another possibility of interpretation of the surface Lamé *moduli*. For example, using Mishuris et al. (2006b, Eq. (2.15)) we can conclude that

$$\mu_s \approx \mu_i h, \tag{12}$$

where μ_i is the shear *modulus* of material of the interfacial layer and h is the thickness of the interface, see Fig. 2a). Results by Mishuris et al. (2006a) give also the approximate formula for λ_s. In other words, with this model, we can replace the problem with a thin layer of finite thickness by the problem with new non-classic boundary conditions.

Let us also note that the same relation for the surface shear *modulus* was obtained by Altenbach, Eremeev, and Morozov (2010) and Altenbach, Eremeyev, and Morozov (2012) considering bending and stretching of three-layered plates and shells. Here μ_s is also given by (12), where μ_i and h are now shear *modulus* and thickness of near-surface layers, see Fig. 2b). Using results on asymptotics for sandwich plates and shells, we can conclude that the presence of surface stresses corresponds to so-called *hard-skin* plates and shells (Berdichevsky, 2010a; 2010b).

Comparing anti-plane surface waves in an elastic half-space with surface stresses and in a square lattice with surface row of material particles different from ones in the bulk, Eremeyev and Sharma (2019) also proposed similar scaling law related lattice parameters with μ_s.

4. Dispersion relations

We are looking for a steady state solution of (9) in the form

$$u = U(r)\exp(ik\varphi - i\omega t), \tag{13}$$

where ω is a angular frequency, k is an positive number, and $i = \sqrt{-1}$. As a result, using (13) from (9), we get the Bessel equation (Abramowitz & Stegun, 1972)

$$U'' + \frac{1}{r}U' + \left(\frac{\rho}{\mu}\omega^2 - \frac{k^2}{r^2}\right)U = 0, \tag{14}$$

where the prime denotes the derivative with respect to r. In what follows, we consider the dimensionless form of (14) introducing the dimensionless variable $\bar{r} = r/a$, so $\bar{r} \in [0, 1]$. Keeping in mind that $\bar{r}$ is dimensionless, we omit the bar over r.

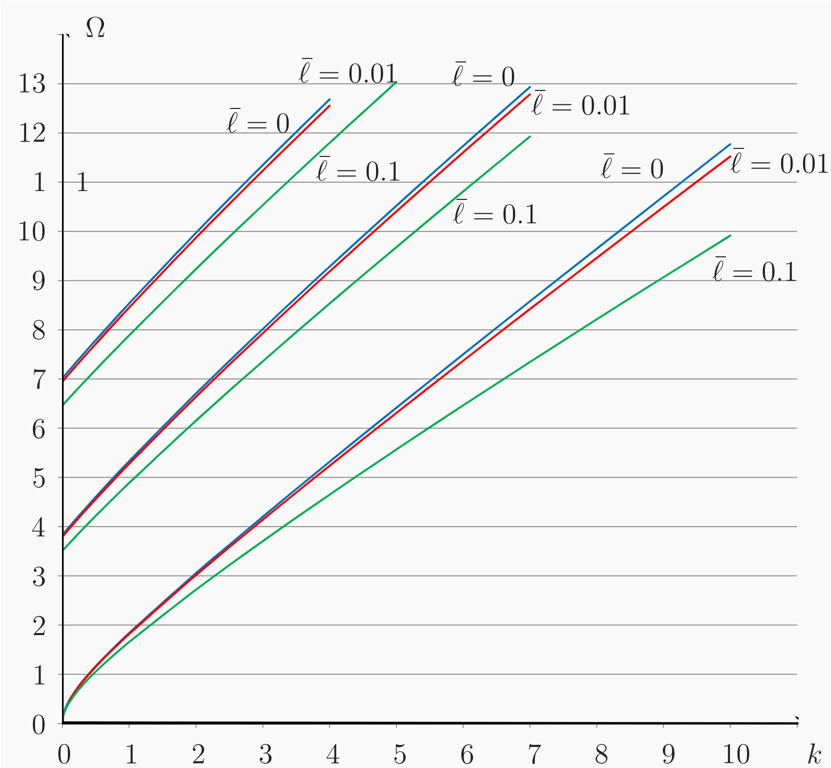

Fig. 3. Dispersion curves for different values of characteristic length $\bar{\ell}$. Here $\eta \equiv c_s/c_T = 0.75$.

The solution of (14) bounded at $r = 0$ has the form

$$U(r) = U_k J_k(\Omega r), \tag{15}$$

where U_k is a constant, J_k is the Bessel functions of the first kind, $\Omega = a\omega/c_T$, and $c_T = \sqrt{\mu/\rho}$ is the shear wave speed.

For (13), the boundary condition (11) transforms into the following equation

$$\frac{\mu}{a} U'(1) = \left[-k^2 \mu_s + m\omega^2\right] U(1). \tag{16}$$

Using (15) from (16), we get

$$\frac{\mu}{a} \Omega J_k'(\Omega) = \left[-k^2 \frac{\mu_s}{a^2} + m\omega^2\right] J_k(\Omega). \tag{17}$$

Finally, we have the dimensionless form of (17)

$$\Omega J_k'(\Omega) = \frac{\ell}{a}\left[\Omega^2 - \frac{c_s^2}{c_T^2} k^2\right] J_k(\Omega), \tag{18}$$

where $\ell = m/\rho$ and $c_s = \sqrt{\mu_s/m}$ are the dynamic characteristic length and the shear wave speed in the framework of the Gurtin–Murdoch model, respectively.

Eq. (18) is the dispersion relation that relates Ω and k. Without surface stress that is when $\ell = 0$, Eq. (18) takes the simple form $J_k'(\Omega) = 0$ for $\Omega \neq 0$ and was analyzed, *e.g.*, by Rayleigh (1910) and Victorov (1974). For each k, Eq. (18) has a series of roots that is $\Omega_1(k) = \Omega_1(k; \bar{\ell}, \eta)$, $\Omega_2(k) = \Omega_2(k; \bar{\ell}, \eta)$, $\Omega_3(k) = \Omega_3(k; \bar{\ell}, \eta)$, $\ldots$, where $\bar{\ell} = \ell/a$, $\eta = c_s/c_T$. In the framework of the Gurtin–Murdoch model, we have an additional length-scale parameter $\ell_s = \mu_s/\mu$, which is independent on ℓ. So, η relates to the ratio of ℓ and ℓ_s, $\eta^2 = \ell_s/\ell$. At $k = 0$ and $\ell = 0$, these roots coincide with roots of $J_1(z)$. Three families of dispersion curves are given in Fig. 3. Here, we assume that $\eta = 3/4$, whereas $\bar{\ell} = 0$; 0.01; 0.1. It is seen that the coating that is when $\ell \neq 0$ shifts the dispersion curves for any k. Unlike the problem for a half-plane (Eremeyev et al., 2016), for a cylinder, the waves exist also when $c_s > c_T$, that is when $\eta > 1$. In Fig. 4, we present the dispersion curves for $\bar{\ell} = 0$ (solid blue curves); for $\bar{\ell} = 0.1$ and $\eta = 0.25$ (dash-dot green curves); and for $\bar{\ell} = 0.1$ and $\eta = 2$ (dashed red curves). Obviously, the coating influences on Ω for any values of k except of Ω_1 where such influence is negligible when $k \ll 1$. The first dispersion curve given by $\Omega = \Omega_1(k; \bar{\ell}, \eta)$ begins at the point (0,0). Next curves $\Omega = \Omega_m(k; \bar{\ell}, \eta)$, $m = 2, 3, \ldots$, begin at $(0, \Omega_m(0; \bar{\ell}, \eta))$, where $\Omega_m(0; \bar{\ell}, \eta) < \Omega_m(0; 0, \eta)$ if $\bar{\ell} \neq 0$.

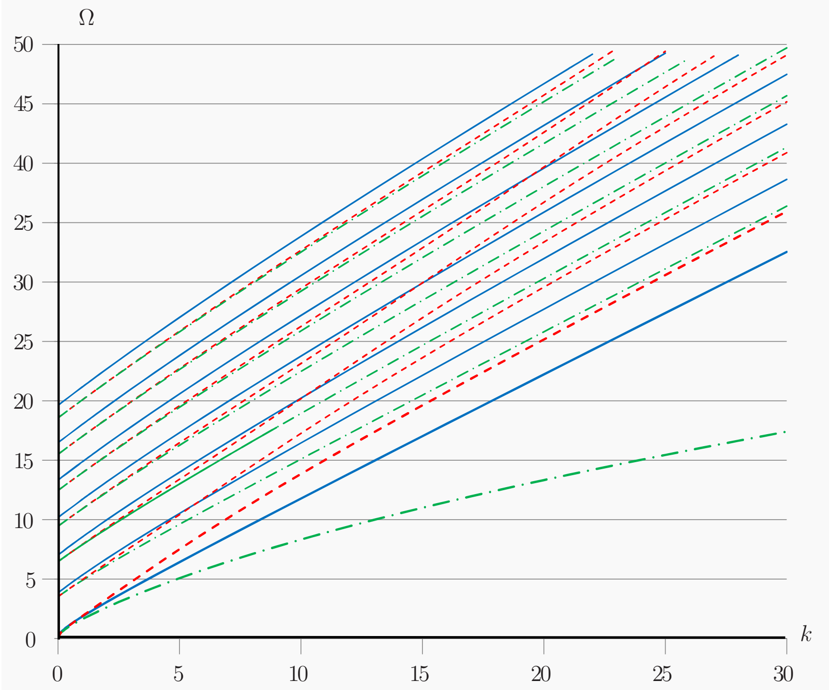

Fig. 4. Seven families of dispersion curves. Solid blue curves relate to $\ell = 0$, dashed red curves correspond to $\bar{\ell} = 0.1$ and $\eta = 2$, and dash-dot green curves stand for $\bar{\ell} = 0.1$ and $\eta = 0.25$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

5. Conclusions

In order to discuss the surface waves in nanostructured materials, we have discussed the propagation of a transverse surface shear wave along a cylindrical boundary considering an elastic surface coating. This type of waves known as so-called whispering-gallery waves guided by the effect of curvature. In order to capture the material behaviour at the nanoscale, we have used the linearized Gurtin–Murdoch surface elasticity model. The considered waves are similar to the anti-plane surface waves in a half-space with surface energy (Eremeyev et al., 2016). Unlike to the half-space problem where such waves exist when $c_s < c_T$, here they exist for wider range of parameters of the coating. We have derived the dispersion relation and have presented corresponding dispersion curves for a set of material parameters. As we have mentioned about some similarities between the generalized Young–Laplace equation and the transmission conditions through a stiff interface, the presented results can be also reformulated for surface shear waves propagating along such stiff interfaces.

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