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ABSTRACT

We derive two equivalent analytical expressions for an lth partial-wave scattering length a_l for central potentials with long-range tails of the form $V(r) = -\frac{\hbar^2}{2m}\frac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} - \frac{\hbar^2}{2m}\frac{C}{r^2(r^{n-2}+R^{n-2})}$, $(r \geqslant r_s, R > 0)$. For $C = 0$, this family of potentials reduces to the Lenz potentials discussed in a similar context in our earlier works [R. Szmytkowski, Acta Phys. Pol. A **79**, 613 (1991); J. Phys. A: Math. Gen. **28**, 7333 (1995)]. The formulas for a_l that we provide in this paper depend on the parameters B, C, and R characterizing the tail of the potential, on the core radius r_s, as well as on the short-range scattering length a_{ls}, the latter being due to the core part of the potential. The procedure, which may be viewed as an analytical extrapolation from a_{ls} to a_l, is relied on the fact that the general solution to the zero-energy radial Schrödinger equation with the potential given above may be expressed analytically in terms of the *generalized* associated Legendre functions.

I. INTRODUCTION

Scattering lengths are among the most important parameters characterizing atomic collisional processes at ultralow energies.[1–4] Therefore, there is a need for developing reliable and effective methods for the calculation of these quantities and a variety of such procedures—analytical, numerical, or of a mixed character—have been proposed.[5–45]

Some time ago, in Ref. 19, we presented analytical formulas for partial-wave scattering lengths a_l for central potentials with the following three types of long-range tails: (i) the inverse power tail

$$V(r) = -\frac{\hbar^2}{2m}\frac{A_n}{r^n} \qquad (r \geqslant r_s), \tag{1.1}$$

(ii) the so-called Lennard-Jones $(n, 2n-2)$ tail

$$V(r) = -\frac{\hbar^2}{2m}\frac{A_n}{r^n} - \frac{\hbar^2}{2m}\frac{A_{2n-2}}{r^{2n-2}} \qquad (r \geqslant r_s), \tag{1.2}$$

and (iii) the so-called Lenz tail

$$V(r) = -\frac{\hbar^2}{2m}\frac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} \qquad (r \geqslant r_s, R > 0). \tag{1.3}$$

The expressions for a_l provided in Ref. 19 involve parameters characterizing tails of particular potentials, the core radius r_s, the short-range scattering lengths a_{ls} that are due to the core part of the potential and that usually have to be determined numerically, and also some of the well-known special functions of mathematical physics: the Bessel functions for the tail (1.1), the Whittaker functions for the tail (1.2), and the associated Legendre functions for the tail (1.3). In brief, the procedure may be viewed as an analytical extrapolation from a_{ls} to a_l, with the use of the fact that in the region $r \geqslant r_s$ the general solution to the zero-energy radial Schrödinger equation with the potentials given above are expressible in terms of the afore-mentioned special functions.

In the present paper, we consider a class of central potentials with still another functional form of the long-range tail, which is

$$V(r) = -\frac{\hbar^2}{2m}\frac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} - \frac{\hbar^2}{2m}\frac{C}{r^2(r^{n-2}+R^{n-2})} \qquad (r \geqslant r_s,\ R > 0). \tag{1.4}$$

This tail is seen to generalize the Lenz tail (1.3); moreover, asymptotically, it imitates the Lennard-Jones tail (1.2) since it falls off as

$$V(r) \overset{r\to\infty}{\longrightarrow} -\frac{\hbar^2}{2m}\frac{B+C}{r^n} - \frac{\hbar^2}{2m}\frac{(-2B-C)R^{n-2}}{r^{2n-2}} + O\big(r^{-3n+4}\big). \tag{1.5}$$

In the following, we shall prove that also for potentials with the tail (1.4), it is possible to extrapolate analytically from a_{ls} to a_l, but this time with the use of *generalized* associated Legendre functions.[46]

The paper is structured as follows: In Sec. II, a definition and some basic facts about partial-wave scattering lengths are reminded, and then, a particular method enabling one to calculate these quantities is sketched. This method is then used in Sec. III to derive two equivalent analytical expressions, displayed in Eqs. (3.16a) and (3.16b), for scattering lengths for potentials with the tail given in Eq. (1.4). Special cases when these two formulas simplify are discussed in Sec. IV. Finally, concluding remarks form Sec. V.

II. THE METHOD

The lth partial-wave scattering length a_l is defined through the limit relation[47]

$$a_l = -(2l-1)!!(2l+1)!!\lim_{k\to 0}\frac{\tan\delta_l(k)}{k^{2l+1}}, \tag{2.1}$$

[by definition, $(-1)!! = 1$], where $\delta_l(k)$ is the lth partial-wave scattering phase shift at the particle wave number k (notice that some authors prefer a definition of a_l with the double factorials omitted). It can be shown (Ref. 48, Sec. 12) that for potentials that asymptotically fall off as

$$V(r) \overset{r\to\infty}{\longrightarrow} \text{const} \times r^{-n} + O\big(r^{-n-\varepsilon}\big) \qquad (n > 3,\ \varepsilon > 0) \tag{2.2}$$

the limit in Eq. (2.1) is finite, and thus, a_l does exist, for partial waves with the angular momentum quantum number l constrained by the inequality

$$2l < n - 3. \tag{2.3}$$

The method of evaluation of a_l based on the direct use of the definition (2.1) is impractical, as it requires prior knowledge of the functional form of $\delta_l(k)$ in the neighborhood of the threshold point $k = 0$. The more convenient approach is the following one (cf. Ref. 19). Let $F_l(r)$ be a solution to the zero-energy radial Schrödinger equation in the outer domain,

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right]F_l(r) = 0 \qquad (r \geqslant r_s), \tag{2.4}$$

which at $r = r_s$ matches smoothly onto an inner-domain solution that is regular at $r = 0$. The asymptotic form of $F_l(r)$ is

$$F_l(r) \overset{r\to\infty}{\longrightarrow} \mathcal{A}_l\Big[r^{l+1} - a_l r^{-l}\Big], \tag{2.5}$$

where a_l is the scattering length and $\mathcal{A}_l$ is a multiplicative factor. Guided by the form of the right-hand side of Eq. (2.5), we introduce two auxiliary functions $\mathcal{A}_l(r)$ and $a_l(r)$ such that

$$F_l(r) = \mathcal{A}_l(r)\Big[r^{l+1} - a_l(r) r^{-l}\Big] \tag{2.6a}$$

and

$$\frac{\mathrm{d}F_l(r)}{\mathrm{d}r} = \mathcal{A}_l(r)\Big[(l+1)r^l + la_l(r)r^{-l-1}\Big]. \tag{2.6b}$$

It is evident that asymptotically, the function $a_l(r)$ tends to a_l,

$$a_l = \lim_{r\to\infty} a_l(r) \tag{2.7}$$

and that $a_l(r)$ may be expressed as

$$a_l(r) = r^{2l+1}\frac{rL_l(r)-(l+1)}{rL_l(r)+l}, \tag{2.8}$$

where

$$L_l(r) = \frac{1}{F_l(r)}\frac{\mathrm{d}F_l(r)}{\mathrm{d}r} \tag{2.9}$$

is the logarithmic derivative of $F_l(r)$. If $f_l(r)$ and $g_l(r)$ are any two linearly independent solutions to Eq. (2.4), the physical solution $F_l(r)$ is a linear combination of the two,

$$F_l(r) = \alpha_l f_l(r) + \beta_l g_l(r). \tag{2.10}$$

Hence, the logarithmic derivative $L_l(r)$ is

$$L_l(r) = \frac{f_l'(r)+\gamma_l g_l'(r)}{f_l(r)+\gamma_l g_l(r)}, \tag{2.11}$$

where the prime means differentiation with respect to r, while γ_l is the ratio of the coefficients appearing in Eq. (2.10),

$$\gamma_l = \frac{\beta_l}{\alpha_l}. \tag{2.12}$$

If in Eqs. (2.8) and (2.11) we set $r = r_s$ and solve the resulting system for γ_l, this gives

$$\gamma_l = -\frac{\big(r_s^{2l+1}-a_{ls}\big)r_s f_l'(r_s) - \big[(l+1)r_s^{2l+1}+la_{ls}\big]f_l(r_s)}{\big(r_s^{2l+1}-a_{ls}\big)r_s g_l'(r_s) - \big[(l+1)r_s^{2l+1}+la_{ls}\big]g_l(r_s)}, \tag{2.13}$$

where

$$a_{ls} = a_l(r_s) \tag{2.14}$$

is a scattering length due to the core part of the potential. Thus, we see that the scattering length a_l may be found from Eqs. (2.7) and (2.8) augmented with Eqs. (2.11) and (2.13). This method is adopted in the present work.

III. SCATTERING LENGTHS FOR POTENTIALS WITH THE TAIL (1.4)

The zero-energy radial Schrödinger equation with the tail potential (1.4) may be written in the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} + \frac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} + \frac{C}{r^2(r^{n-2}+R^{n-2})}\right]F_l(r) = 0 \qquad (r \geqslant r_s) \tag{3.1}$$

(the constraint $R > 0$ is assumed to hold throughout the rest of the paper). Below, we shall show that this equation may be solved analytically in terms of known special functions. To this end, we switch from the independent variable r to the new one

$$\rho = \frac{r^{n-2}-R^{n-2}}{r^{n-2}+R^{n-2}} \qquad (\rho_s \leqslant \rho \leqslant 1), \tag{3.2}$$

with

$$\rho_s = \frac{r_s^{n-2} - R^{n-2}}{r_s^{n-2} + R^{n-2}}, \tag{3.3}$$

and from the function $F_l(r)$ to the function

$$\mathcal{F}_l(\rho) = r^{-1/2} F_l(r). \tag{3.4}$$

The new function $\mathcal{F}_l(\rho)$ is found to be a solution to the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}\rho}\left(1-\rho^2\right)\frac{\mathrm{d}}{\mathrm{d}\rho} + \lambda(\lambda+1) - \frac{\mu^2}{2(1-\rho)} - \frac{\nu^2}{2(1+\rho)}\right]\mathcal{F}_l(\rho) = 0 \qquad (\rho_s \leqslant \rho \leqslant 1), \tag{3.5}$$

with

$$\lambda = \frac{1}{2}\sqrt{1 + \frac{4B}{(n-2)^2 R^{n-2}}} - \frac{1}{2}, \tag{3.6a}$$

$$\mu = \frac{2l+1}{n-2} \tag{3.6b}$$

and

$$\nu = \sqrt{\left(\frac{2l+1}{n-2}\right)^2 - \frac{4C}{(n-2)^2 R^{n-2}}}. \tag{3.6c}$$

It should be observed that, in virtue of the inequality (2.3), the parameter μ defined above is constrained to obey

$$0 < \mu < 1. \tag{3.7}$$

Equation (3.5) is the *generalized* associated Legendre equation. Some investigations concerning its solutions had been carried out by Bateman[49] in the early 1900's, but systematic studies on the subject began only half a century later with the works of Kuipers and Meulenbeld;[50,51] a summary of relevant results obtained by various researchers up to the year 2000 may be found in the monograph.[46] The solution to Eq. (3.5) is

$$\mathcal{F}_l(\rho) = \alpha_l P_\lambda^{\mu,\nu}(\rho) + \beta_l P_\lambda^{-\mu,-\nu}(\rho), \tag{3.8}$$

where

$$P_\lambda^{\mu,\nu}(\rho) = \frac{1}{\Gamma(1-\mu)}\frac{(1+\rho)^{\nu/2}}{(1-\rho)^{\mu/2}}\,{}_2F_1\left(-\lambda - \frac{\mu-\nu}{2}, \lambda + 1 - \frac{\mu-\nu}{2}; 1-\mu; \frac{1-\rho}{2}\right) \tag{3.9}$$

is the *generalized* associated Legendre function of the first kind *on the cross-cut* $-1 \leqslant \rho \leqslant 1$; here, ${}_2F_1(\cdots)$ denotes the hypergeometric function. The functions $P_\lambda^{\mu,\nu}(\rho)$ and $P_\lambda^{-\mu,-\nu}(\rho)$ appearing in Eq. (3.8) are linearly independent since their Wronskian

$$W\left[P_\lambda^{\mu,\nu}(\rho), P_\lambda^{-\mu,-\nu}(\rho)\right] = -\frac{2\sin(\pi\mu)}{\pi(1-\rho^2)} \tag{3.10}$$

does not vanish by virtue of the constraint (3.7) obeyed by μ. Now, as $\rho \to 1-0$ (which corresponds to $r \to \infty$), the functions $P_\lambda^{\mu,\nu}(\rho)$ and $P_\lambda^{-\mu,-\nu}(\rho)$ behave as

$$P_\lambda^{\mu,\nu}(\rho) \xrightarrow{\rho\to 1-0} \frac{2^{\nu/2}}{\Gamma(1-\mu)}(1-\rho)^{-\mu/2} + O\left((1-\rho)^{-\mu/2+1}\right) \tag{3.11a}$$

and

$$P_\lambda^{-\mu,-\nu}(\rho) \xrightarrow{\rho\to 1-0} \frac{2^{-\nu/2}}{\Gamma(1+\mu)}(1-\rho)^{\mu/2} + O\left((1-\rho)^{\mu/2+1}\right), \tag{3.11b}$$

respectively. On combining Eqs. (3.4), (3.8) and (3.11), we see that the asymptotic behavior of the radial wavefunction $F_l(r)$ is

$$F_l(r) \xrightarrow{r\to\infty} \alpha_l \frac{2^{(\nu-\mu)/2}}{\Gamma(1-\mu)} \frac{r^{l+1}}{R^{l+1/2}} + \beta_l \frac{2^{(\mu-\nu)/2}}{\Gamma(1+\mu)} \frac{R^{l+1/2}}{r^l} + O\Big((r/R)^{l-n+3}\Big). \tag{3.12}$$

Hence, with the use of the method presented in Sec. II, it is found that the scattering length a_l is

$$a_l = R^{2l+1} 2^{\mu-\nu} \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\left(r_s^{2l+1} - a_{ls}\right)\left(1-\rho_s^2\right)\left[\mathrm{d}P_\lambda^{\mu,\nu}(\rho)/\mathrm{d}\rho\right]_{\rho=\rho_s} - \mu\left(r_s^{2l+1} + a_{ls}\right)P_\lambda^{\mu,\nu}(\rho_s)}{\left(r_s^{2l+1} - a_{ls}\right)\left(1-\rho_s^2\right)\left[\mathrm{d}P_\lambda^{-\mu,-\nu}(\rho)/\mathrm{d}\rho\right]_{\rho=\rho_s} - \mu\left(r_s^{2l+1} + a_{ls}\right)P_\lambda^{-\mu,-\nu}(\rho_s)}, \tag{3.13}$$

where a_{ls} is the short-range scattering length.

The presence of derivatives of the generalized Legendre functions makes the formula displayed in Eq. (3.13) impractical for use in actual applications. However, at this moment, we may exploit either the relation [Ref. 52, Eq. (25)]

$$(\lambda+1)\left(1-\rho^2\right)\frac{\mathrm{d}P_\lambda^{\mu,\nu}(\rho)}{\mathrm{d}\rho} = \left[(\lambda+1)^2\rho + \frac{\mu^2-\nu^2}{4}\right]P_\lambda^{\mu,\nu}(\rho) - \left(\lambda+1-\frac{\mu-\nu}{2}\right)\left(\lambda+1-\frac{\mu+\nu}{2}\right)P_{\lambda+1}^{\mu,\nu}(\rho) \tag{3.14a}$$

or the relation

$$\lambda\left(1-\rho^2\right)\frac{\mathrm{d}P_\lambda^{\mu,\nu}(\rho)}{\mathrm{d}\rho} = -\left(\lambda^2\rho + \frac{\mu^2-\nu^2}{4}\right)P_\lambda^{\mu,\nu}(\rho) + \left(\lambda+\frac{\mu-\nu}{2}\right)\left(\lambda+\frac{\mu+\nu}{2}\right)P_{\lambda-1}^{\mu,\nu}(\rho), \tag{3.14b}$$

where the latter emerges when the expression in Eq. (3.14a) is combined with the identity [Ref. 52, Eq. (7)]

$$(2\lambda+1)\left[\lambda(\lambda+1)\rho + \frac{\mu^2-\nu^2}{4}\right]P_\lambda^{\mu,\nu}(\rho) = \lambda\left(\lambda+1-\frac{\mu-\nu}{2}\right)\left(\lambda+1-\frac{\mu+\nu}{2}\right)P_{\lambda+1}^{\mu,\nu}(\rho) + (\lambda+1)\left(\lambda+\frac{\mu-\nu}{2}\right)\left(\lambda+\frac{\mu+\nu}{2}\right)P_{\lambda-1}^{\mu,\nu}(\rho). \tag{3.15}$$

This allows us to replace the formula in Eq. (3.13) with either of the following two:

$$a_l = R^{2l+1}2^{\mu-\nu}\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\left\{\begin{aligned} &\Big\{r_s^{2l+1}\left[(\lambda+1)^2\rho_s - \mu(\lambda+1) + \left(\mu^2-\nu^2\right)/4\right] \\ &\quad - a_{ls}\left[(\lambda+1)^2\rho_s + \mu(\lambda+1) + \left(\mu^2-\nu^2\right)/4\right]\Big\}P_\lambda^{\mu,\nu}(\rho_s) \\ &\quad - \left(r_s^{2l+1} - a_{ls}\right)[\lambda+1-(\mu-\nu)/2][\lambda+1-(\mu+\nu)/2]P_{\lambda+1}^{\mu,\nu}(\rho_s)\end{aligned}\right\}}{\left\{\begin{aligned} &\Big\{r_s^{2l+1}\left[(\lambda+1)^2\rho_s - \mu(\lambda+1) + \left(\mu^2-\nu^2\right)/4\right] \\ &\quad - a_{ls}\left[(\lambda+1)^2\rho_s + \mu(\lambda+1) + \left(\mu^2-\nu^2\right)/4\right]\Big\}P_\lambda^{-\mu,-\nu}(\rho_s) \\ &\quad - \left(r_s^{2l+1} - a_{ls}\right)[\lambda+1+(\mu-\nu)/2][\lambda+1+(\mu+\nu)/2]P_{\lambda+1}^{-\mu,-\nu}(\rho_s)\end{aligned}\right\}} \tag{3.16a}$$

or

$$a_l = R^{2l+1}2^{\mu-\nu}\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{\left\{\begin{aligned} &\Big\{r_s^{2l+1}\left[\lambda^2\rho_s + \mu\lambda + \left(\mu^2-\nu^2\right)/4\right] \\ &\quad - a_{ls}\left[\lambda^2\rho_s - \mu\lambda + \left(\mu^2-\nu^2\right)/4\right]\Big\}P_\lambda^{\mu,\nu}(\rho_s) \\ &\quad - \left(r_s^{2l+1} - a_{ls}\right)[\lambda+(\mu-\nu)/2][\lambda+(\mu+\nu)/2]P_{\lambda-1}^{\mu,\nu}(\rho_s)\end{aligned}\right\}}{\left\{\begin{aligned} &\Big\{r_s^{2l+1}\left[\lambda^2\rho_s + \mu\lambda + \left(\mu^2-\nu^2\right)/4\right] \\ &\quad - a_{ls}\left[\lambda^2\rho_s - \mu\lambda + \left(\mu^2-\nu^2\right)/4\right]\Big\}P_\lambda^{-\mu,-\nu}(\rho_s) \\ &\quad - \left(r_s^{2l+1} - a_{ls}\right)[\lambda-(\mu-\nu)/2][\lambda-(\mu+\nu)/2]P_{\lambda-1}^{-\mu,-\nu}(\rho_s)\end{aligned}\right\}}. \tag{3.16b}$$

Equations (3.16a) and (3.16b) constitute the main result of this paper. In Sec. IV, we shall investigate particular cases when these two expressions may be simplified.

IV. CASES WHEN EQS. (3.16a) AND (3.16b) SIMPLIFY

A. The case of $B = 0$

For $B = 0$, the tail potential (1.4) is

$$V(r) = -\frac{\hbar^2}{2m}\frac{C}{r^2(r^{n-2}+R^{n-2})} \qquad (r \geqslant r_s), \tag{4.1}$$

and it holds that

$$\lambda = 0 \tag{4.2}$$

[cf. Eq. (3.6a)]. As a consequence, Eq. (3.16a) becomes

$$a_l = R^{2l+1}2^{\mu-\nu}\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)}\frac{\left\{\begin{array}{l}\Big\{r_s^{2l+1}\big[\rho_s-\mu+\big(\mu^2-\nu^2\big)/4\big] \\ \quad -\ a_{ls}\big[\rho_s+\mu+\big(\mu^2-\nu^2\big)/4\big]\Big\}P_0^{\mu,\nu}(\rho_s) \\ \quad -\ \big(r_s^{2l+1}-a_{ls}\big)[1-(\mu-\nu)/2][1-(\mu+\nu)/2]P_1^{\mu,\nu}(\rho_s)\end{array}\right\}}{\left\{\begin{array}{l}\Big\{r_s^{2l+1}\big[\rho_s-\mu+\big(\mu^2-\nu^2\big)/4\big] \\ \quad -\ a_{ls}\big[\rho_s+\mu+\big(\mu^2-\nu^2\big)/4\big]\Big\}P_0^{-\mu,-\nu}(\rho_s) \\ \quad -\ \big(r_s^{2l+1}-a_{ls}\big)[1+(\mu-\nu)/2][1+(\mu+\nu)/2]P_1^{-\mu,-\nu}(\rho_s)\end{array}\right\}}, \tag{4.3}$$

while Eq. (3.16b) leads to an expression for a_l of the $0/0$ type since it holds that

$$P_{-1}^{\pm\mu,\pm\nu}(\rho_s) = P_0^{\pm\mu,\pm\nu}(\rho_s). \tag{4.4}$$

B. The case of $C = 0$

For $C = 0$, the tail potential (1.4) reduces to the Lenz one displayed in Eq. (1.3),

$$V(r) = -\frac{\hbar^2}{2m}\frac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} \qquad (r \geqslant r_s). \tag{4.5}$$

From Eqs. (3.6b) and (3.6c), one infers that now the parameters μ and ν are equal,

$$\nu = \mu. \tag{4.6}$$

Since it holds that

$$P_\lambda^{\mu,\mu}(\rho) = P_\lambda^{\mu}(\rho), \tag{4.7}$$

where $P_\lambda^{\mu}(\rho)$ is the well-known associated Legendre function of the first kind *on the cross-cut* $-1 \leqslant \rho \leqslant 1$ (Ref. 53, Sec. 4.3), in the case under study, Eq. (3.16a) and (3.16b) simplify and go over into

$$a_l = R^{2l+1}\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)}\frac{\big\{r_s^{2l+1}[(\lambda+1)\rho_s-\mu]-a_{ls}[(\lambda+1)\rho_s+\mu]\big\}P_\lambda^{\mu}(\rho_s)-\big(r_s^{2l+1}-a_{ls}\big)(\lambda+1-\mu)P_{\lambda+1}^{\mu}(\rho_s)}{\big\{r_s^{2l+1}[(\lambda+1)\rho_s-\mu]-a_{ls}[(\lambda+1)\rho_s+\mu]\big\}P_\lambda^{-\mu}(\rho_s)-\big(r_s^{2l+1}-a_{ls}\big)(\lambda+1+\mu)P_{\lambda+1}^{-\mu}(\rho_s)} \tag{4.8a}$$

and

$$a_l = R^{2l+1}\frac{\Gamma(1-\mu)}{\Gamma(1+\mu)}\frac{\big[r_s^{2l+1}(\lambda\rho_s+\mu)-a_{ls}(\lambda\rho_s-\mu)\big]P_\lambda^{\mu}(\rho_s)-\big(r_s^{2l+1}-a_{ls}\big)(\lambda+\mu)P_{\lambda-1}^{\mu}(\rho_s)}{\big[r_s^{2l+1}(\lambda\rho_s+\mu)-a_{ls}(\lambda\rho_s-\mu)\big]P_\lambda^{-\mu}(\rho_s)-\big(r_s^{2l+1}-a_{ls}\big)(\lambda-\mu)P_{\lambda-1}^{-\mu}(\rho_s)}, \tag{4.8b}$$

respectively. Up to notational differences, Eq. (4.8a) coincides with Eq. (52) in Ref. 19.

C. The hard-core potential

The next class of potentials we wish to consider are those with hard cores,

$$V(r) = \begin{cases} +\infty & \text{for } r < r_s, \\ -\dfrac{\hbar^2}{2m}\dfrac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} - \dfrac{\hbar^2}{2m}\dfrac{C}{r^2(r^{n-2}+R^{n-2})} & \text{for } r \geqslant r_s. \end{cases} \tag{4.9}$$

Then, the short-range scattering length is simply

$$a_{ls} = r_s^{2l+1} \tag{4.10}$$

so that either of Eq. (3.16a) or Eq. (3.16b) reduces to

$$a_l = R^{2l+1} 2^{\mu-\nu} \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{P_\lambda^{\mu,\nu}(\rho_s)}{P_\lambda^{-\mu,-\nu}(\rho_s)}. \tag{4.11}$$

The application of the identity [Ref. 46, Eq. (4.2)]

$$P_\lambda^{\mu,\nu}(\rho) = 2^\nu P_\lambda^{\mu,-\nu}(\rho) \tag{4.12}$$

casts Eq. (4.11) into

$$a_l = R^{2l+1} 2^{\mu} \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \frac{P_\lambda^{\mu,-\nu}(\rho_s)}{P_\lambda^{-\mu,-\nu}(\rho_s)}. \tag{4.13}$$

The latter formula will be used in Sec. IV D.

D. The pure potential with $C < 0$

Finally, we wish to consider a potential that is of the form (1.4) throughout the whole space $\mathbb{R}^3$, i.e., such that

$$V(r) = -\frac{\hbar^2}{2m}\frac{Br^{n-4}}{(r^{n-2}+R^{n-2})^2} - \frac{\hbar^2}{2m}\frac{C}{r^2(r^{n-2}+R^{n-2})} \qquad (r \geqslant 0), \tag{4.14}$$

under an additional constraint that it is repulsive near the origin,

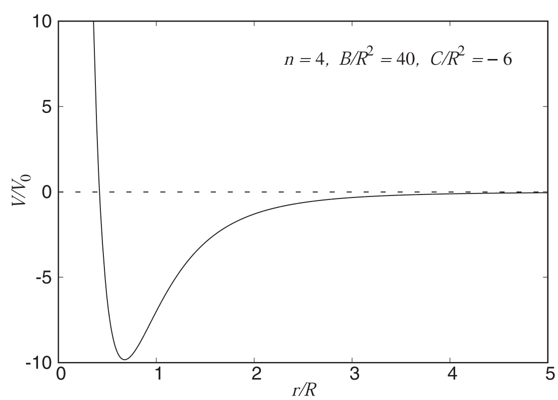

FIG. 1. A sample pure potential (4.14) with $n = 4$, $B/R^2 = 40$, and $C/R^2 = -6$. The potential normalization parameter V_0 equals $\hbar^2/2mR^2$.

$$C < 0 \tag{4.15}$$

(a sample potential function of that sort is depicted in Fig. 1). An expression for the scattering length for such a potential may be derived most conveniently from Eq. (4.13) by taking the limit $\rho_s \to -1 + 0$ (which corresponds to the limit $r_s \to 0$). On using Eq. (3.9) and the Gauss' identity (Ref. 53, p. 40)

$${}_2F_1(a_1, a_2; b; 1) = \frac{\Gamma(b)\Gamma(b - a_1 - a_2)}{\Gamma(b - a_1)\Gamma(b - a_2)} \qquad [\mathrm{Re}(b - a_1 - a_2) > 0], \tag{4.16}$$

we eventually find that the lth partial-wave scattering length for the pure potential (4.14) constrained by Eq. (4.15) is

$$a_l = R^{2l+1} \frac{\Gamma(1-\mu)\Gamma\left(\lambda + 1 + \frac{\mu+\nu}{2}\right)\Gamma\left(-\lambda + \frac{\mu+\nu}{2}\right)}{\Gamma(1+\mu)\Gamma\left(\lambda + 1 - \frac{\mu-\nu}{2}\right)\Gamma\left(-\lambda - \frac{\mu-\nu}{2}\right)}. \tag{4.17}$$

V. CONCLUDING REMARKS

The aim of this paper has been to show that there exists still another class of central potentials—those with the long-range tails (1.4) and the asymptotic representation (1.5)—for which partial-wave scattering lengths a_l may be obtained in analytical forms. Whilst expressions for a_l for potentials with the tails (1.1)–(1.3) considered earlier in Ref. 19 contain Bessel, Whittaker, and the associated Legendre functions, respectively, the present case involves lesser-known *generalized* associated Legendre functions.

In two particular cases, namely, for $n = 4$ and for $n = 6$, the potentials (1.4) may find applications in atomic physics. If $n = 4$, the resulting tail potential

$$V(r) = -\frac{\hbar^2}{2m}\frac{B}{(r^2 + R^2)^2} - \frac{\hbar^2}{2m}\frac{C}{r^2(r^2 + R^2)} \qquad (r \geqslant r_s) \tag{5.1}$$

may be used to model a long-range polarization interaction between a charged particle and an atom. On the other hand, with $n = 6$ one obtains the potential function

$$V(r) = -\frac{\hbar^2}{2m}\frac{Br^2}{(r^4 + R^4)^2} - \frac{\hbar^2}{2m}\frac{C}{r^2(r^4 + R^4)} \qquad (r \geqslant r_s), \tag{5.2}$$

which may imitate the van der Waals attraction between two atoms.

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