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Kamil Żyliński, Aleksandra Korzec, Karol Winkelmann, and Jarosław Górski

Random Field Model of Foundations at the Example of Continuous Footing

Kamil Żyliński[1,2 a)], Aleksandra Korzec[3, b)], Karol Winkelmann[1, c)] and Jarosław Górski[1, d)]

[1]*Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland*
[2]*Industria Engineering, Azymutalna 9, 80-298 Gdańsk, Poland*
[3]*Institute of Hydro-Engineering, Polish Academy of Sciences, Kościerska 7, 80-328 Gdańsk, Poland*

[a)]Corresponding author: kamil.zylinski@pg.edu.pl, k.zylinski@ibg.gda.pl
[b)]a.korzec@ibwpan.gda.pl
[c)]karolwin@pg.edu.pl
[d)]jgorski@pg.edu.pl

Abstract. The purpose of the paper is to indicate an efficient method of analysis foundation settlement taking into account the variability of soil properties. The impact of the random variable distribution (Gauss or Lognormal) describing soil stiffness on foundation deposits was assessed. The Monte Carlo simulation method was applied in the computations. The settlements of the strip foundation with the subsoil described by a single random variable and a random field were compared.

INTRODUCTION

While the literature regarding foundation settlements is vast, the investigations are still active on optimal analytical methods to refer precisely to the real foundation conditions. It is made possible by means of software employing advanced material models [1-5]. Probabilistic methods are successively gaining more and more importance [6-18]. They are distinguished in the soil cases showcasing high parameter variability, compared to metals, concrete, even composites. Various probabilistic analytical methods are in current use, the Monte Carlo simulation (MC) is the most featured due to its easy implementation, result reliability and a possible engineer-oriented result interpretation [19]. The MC-based repetition of computations is available due to the increasing power of computational tools. The dedicated software makes it possible to implement algorithms governing series of simulations. Nowadays it seems a promising direction in soil mechanics.

The advanced algorithms are created in order to accelerate the crude MC routines, e.g. variance-reduction techniques (stratified and Latin hypercube sampling), the response surface method (RSM), targeted random sampling (TRS) and more. Their results are always uncertain, especially in the cases of a number of random variables due to the soil, multi-strata cases and complex boundary conditions.

The most important issue in the probabilistic analysis is a relevant random description of soil parameters. Single random variables are most frequently applied, but this approach is too simplistic and unrealistic. A much more advanced approach incorporates random fields. This version requires a precise definition of variables, their mean values, standard deviations, moreover, their correlation function. Such an approach have to be preceded by extensive soil investigations to properly assess single parameters and correlation length, necessary to define a random field with properly formed correlation function on the basis of a limited data set.

The Eurocodes require to employ probabilistic methods in structural reliability assessment and safety issues. These computations involve advanced, dedicated software operated by users familiar with probabilistic methodology, yet their direct introduction to standard engineering computations becomes a necessity [20, 21].

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The work attempts an insight on random soil parameters modelling, involving either a random variable or a random field. The work is aimed at pointing out optimal computational algorithms for engineering procedures.

CASE STUDY

An example of continuous foot in plane strain conditions is being investigated (Fig. 1). The model horizontal length equals 18 m, its vertical height 6 m. The thickness of the foundation equals 0.5 m.

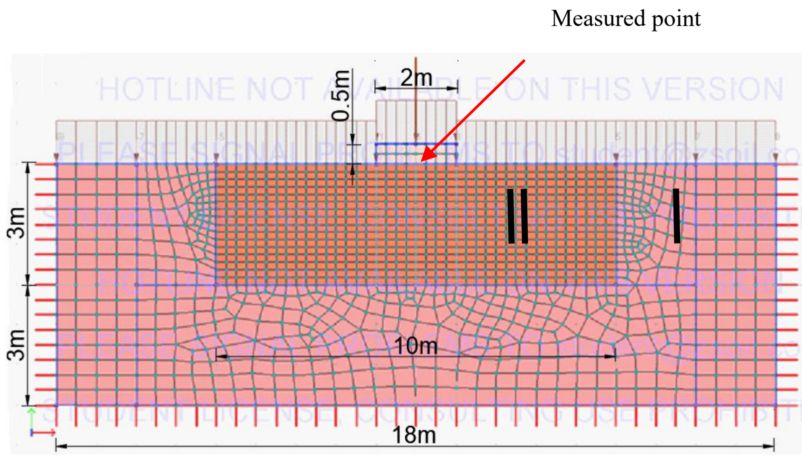

Figure 1. 2D model of plane strain case modelled in ZSoil software [22]

While the analysis character is dominantly probabilistic and the Monte Carlo simulation method is a basic computational tool, these random parameters partially impose certain model limitations. Thus the under-foundation domain was split into two sub-regions, due to analytical parameters applied: the first (no. 1), deterministic and the second (no. 2), random. The middle random region (II) of horizontal and vertical finite mesh 40×16 elements corresponds to finite element dimensions 0.25×0.1875 m. The remaining domain (no. 1) is defined by the unstructured mesh and deterministic parameters: E = 30 MPa, cohesion c = 10 kPa, and internal friction angle φ = 30°. Concrete is considered elastic, its parameters correspond to C30/37 concrete (with Young's modulus E = 33 GPa).

Aforementioned random layer no. 2 detects the following mechanical parameters: E = 50 MPa, c = 10 kPa, φ = 30°. The stiffness of random domain was assumed higher than in the surrounding one. The selection of elements and zones was intended to optimize the computational time of a single case, making it possible to implement the Monte Carlo simulation method. The Mohr-Coulomb shear criterion was chosen to assess the settlements of the footing. The uniform load intensity equals 18 kPa, the vertical loading is 1.75 MN (Fig. 1). The load was collected from the internal wall subjected to vertical loading only.

In the following case, deterministic footing settlement (Fig. 2) computed in ZSoil [22] assuming characteristic load values and mechanical parameters of layers equals u_{det}=0.1038 m. This deterministic, accessible procedure does not account for the scatter of soil material parameters and the field tests uncertainty. It is substantial to conduct a probabilistic computational routine here. They complement the mean, deterministic settlement values by their standard deviations, providing a broader engineering outlook on the results scatter, and finally, leading to reliability assessment. Material parameters are routinely modelled by single random variables.

SINGLE RANDOM VARIABLE DESCRIPTION

In the case when sufficient databases related to real foundation-oriented soil parameter distribution are not accessible, they are assumed a priori [23]. In the domain of single random variables, the most widespread are Normal (N) or Lognormal (LN) patterns. The single random variates generators are available in simple computational and spreadsheet software, further enabling random approach in standard design. Although they produce pseudorandom numbers, they meet the engineering computational aims well.

Young's modulus E has been assumed a sole random parameter of the problem. Its mean value is based on deterministic inquiries $\overline{E}$=50 MPa. The coefficient of variation (ν_E) of $\overline{E}$ (layer no. 2) is assumed fixed at the $\nu_E = 0.3$ level, this variability extent stems from the data in [24]. A maximum standard deviation taken for the analysis is intended to produce maximum foundation settlement scatter, i.e. values of sufficient safety margin due to engineering design.

In this approach, Normal and Lognormal random variables were chosen to model Young’s modulus of layer no. 2. At first Normal (Gaussian) case was conducted due to its easy implementation in closed formulae and relatively proper representation of most material properties. However the Gaussian probability density function (PDF) shows disadvantages, such as negative values appearance should the lower boundary for this case exceed (3-5)σ_E, where σ_E is the assumed standard deviation. In the considered example, no negative generated samples were detected, however, should more samples be generated, there is no certainty of the occurrence of positive-only samples. A definite solution to the problem is the selection of truncated Gaussian random variable.

Next, lognormal random variable was also incorporated to exclude negative values of Young’s modulus and to generate values of higher density within the mean value range.

The work is intended to directly compare the Normal and Lognormal variants, while the generated variates may exceed allowable material parameters. This analysis has been undertaken due to the high accessibility of Gaussian variables in engineering computations. These results are legible, intuitive, and have a key meaning in engineering. Any Gaussian variable is symmetrically distributed. Moreover, in the case where no laboratory-based database is available, the application of non-symmetrically distributed variables, e.g. Lognormal, is not advised. The latter type is popular in the problems of lower bound criteria. In these cases, the variable selection is not supported by field investigations.

Layer II (Fig. 1) which was considered random was described using two random variables. First of it, Gaussian one was used, with statistical parameters equal to the one denoted in the Table 1.

Table 1. Statistical moments of Normal and Lognormal random variables

Statistical moments	**Normal**	**Lognormal**
Mean value $\overline{E}$ [kPa]	50000	10.7767
Standard deviation σ_E [kPa]	15000	0.2936
Coefficient of variation ν_E [-]	0.3	0.3

In order to apply the same mean and variability coefficient in lognormal PDF simple transformations based on the following formulae are performed:

$$E_s(x_i)=\exp\left(\mu_{\ln(E_s)}+\sigma_{\ln(E_s)}G(x_i)\right) \tag{1}$$

$$\sigma^2_{\ln(E_s)}=\ln\left(1+\frac{\sigma^2_{E_s}}{\mu^2_{E_s}}\right)=\ln\left(1+COV^2_{E_s}\right) \tag{2}$$

$$\mu_{\ln(E_s)}=\ln\left(\mu_{\ln(E_s)}\right)-0.5\sigma^2_{\ln(E_s)} \tag{3}$$

where E_s denotes desired, Lognormal value transformed from normal distribution.

The parameters are presented in Table 1. The Python supplied generator [25] was applied to create the corresponding input data sets. Histograms of the generated sets of 10000 samples are presented in Figure 2. The statistical parameters of Normal and Lognormal variables are compared in Table 2.

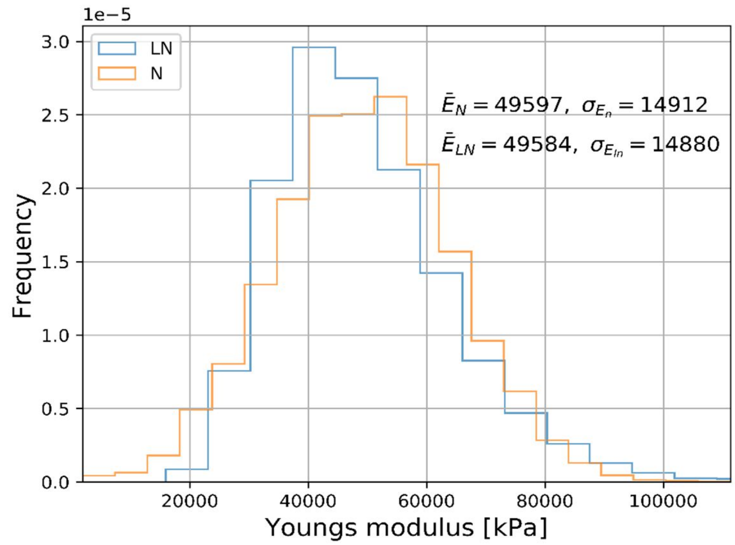

Figure 2. Histogram of Normal and Lognormal input data set (Python [25])

Table 2. Statistical parameters of Normal and Lognormal random variables for 10000 set size

Statistical parameters	Normal	Lognormal
E_{min} [kPa]	1890	15928
E_{max} [kPa]	111370	159175
$\bar{E}$ [kPa]	49597	49584
σ_E [kPa]	14912	14880

The work incorporates the crude Monte Carlo simulation variant. The prescribed series computation procedures were performed by means of Python and ZSoil packages. Here the computational time of a single realization is relatively short, allowing a mulitude of runs to be conducted, thus making the results reliable. A variety of alternative methods, e.g. variance-reduction techniques, response surface, are doubtful due to their result reliability.

The generated Young's modulus variates made it possible to conduct two computational series, each one including 10000 runs, resulting in two soil settlement response variables, corresponding to both Normal (Fig. 3) and Lognormal cases. Parameters of these output variables are collected in Table 3.

Significantly lower minimum settlement values are generated by the Normal pattern. On the other hand, larger maxima are produced by Lognormal generation variant. While the Lognormal variable is naturally low-bound, no such limitations appear due to extremes.

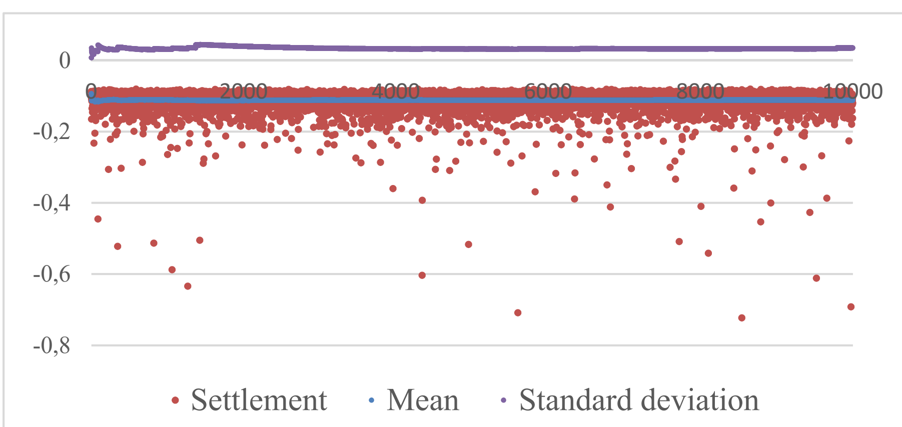

Figure 3. Divergence of settlements observed for the Normal input data of standard MC method (10000 samples)

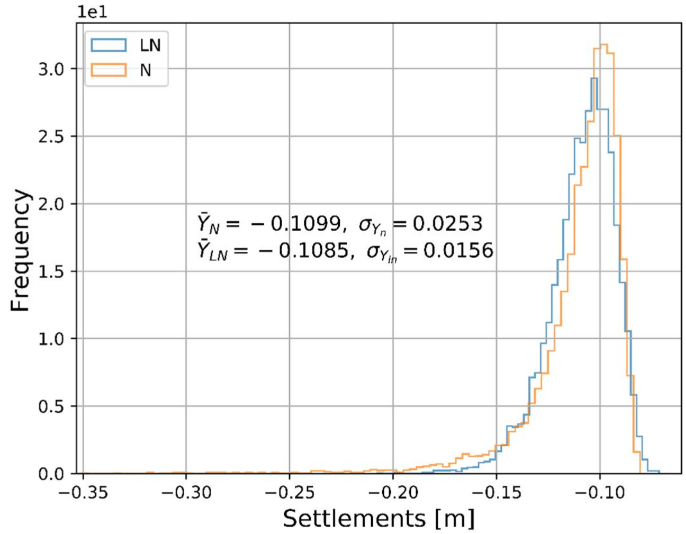

Figure 4. Comparison of model response with regard to Normal and Lognormal input data sets

Table 3. Statistical parameters of response variable generated applying of 10000 samples taken from Normal and Lognormal distributions

Statistical parameters		Normal	Lognormal	Difference [%]
Min value Y_{min}	[m]	-0.2887	-0.207658	28.07%
Max value Y_{max}	[m]	-0.0843	-0.071268	15.46%
Mean $\bar{Y}$	[m]	-0.1099	-0.108564	1.22%
Standard deviation σ_Y	[m]	0.025325	0.0156266	38.30%
Skewness W_s	[-]	-3.96086	-0.913517	76.94%

The settlements obtained by means of Gaussian-distributed variables meet the expectations, i.e. the solution includes a number of values significantly lower than Young's modulus in comparison to the uncertainty covered by Lognormal random model. However, the mean values in both cases coincide. The standard deviations are higher in the Normal solution variant. It is not apparent that the input Gaussian variables produce unsymmetrically distributed settlement variables, while the outcome of Lognormal input is almost symmetric. Note that both Normal and Lognormal computational variants bring about approximate settlement distributions. Thus they may be considered equivalent, assuming a dedicated Normal variable generation (bounding negative values).

As mentioned before, material parameters of a vast soil volume under a foundation by means of a single random variable only is not enough to relevantly model the real footing conditions. It is suggested to apply a random field of material parameters to properly capture the dispersion of parameters governing the subsoil.

RANDOM FIELD APPLICATIONS

As far as random field generation is concerned it is a considerable task to apply an available, standard FEA software. It is not possible to appropriately match the random field correlation function, its parameters as well as to properly interpret the outcomes without a deep background on random field theory. This is the possible reason why different studies in the present time refer either to novel generation concepts or direct problem-oriented solutions.

Method of generations

The paper proposes a random field generation code too, addressed in detail in [26, 27]. The paper outlines the conditional generation method only. The software makes it possible to simulate discrete Gaussian random fields of an arbitrary correlation function. The entirety of random field points is split into two sets of known, prior generated values and unknown values, investigated in given steps, the dimensions of these subsets are equal p and n, respectively:

$$\mathbf{X}=\begin{Bmatrix}\mathbf{X}_u\\ \mathbf{X}_k\end{Bmatrix}\begin{matrix}n\\ p\end{matrix} \tag{4}$$

These variables are assumed their mean values

$$\bar{\mathbf{X}}=\begin{Bmatrix}\bar{\mathbf{X}}_u\\ \bar{\mathbf{X}}_k\end{Bmatrix}\begin{matrix}n\\ p\end{matrix} \tag{5}$$

The covariance matrix defined by an arbitrary correlation function is split according to the division of vector X:

$$\mathbf{K}=\begin{array}{c}\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\begin{matrix}n\\ p\end{matrix}\\ \begin{matrix}n & \quad p\end{matrix}\end{array} \tag{6}$$

Standard transformations lead to a function decisive in the generation of unknown values $\mathbf{X}_u$:

$$f_t(\mathbf{X}_u/\mathbf{X}_k)=(1-t)^{-\frac{m}{2}}\left(\det\mathbf{K}_c\right)^{-\frac{1}{2}}(2\pi)^{-\frac{m}{2}}\exp\left(-\frac{1}{2(1-t)}\left(\mathbf{X}_u-\bar{\mathbf{X}}_c\right)^{\mathrm{T}}\mathbf{K}_c^{-1}\left(\mathbf{X}_u-\bar{\mathbf{X}}_c\right)\right) \tag{7}$$

here $\mathbf{K}_c=\mathbf{K}_{11}-\mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}$ is a conditional correlation matrix, $\bar{\mathbf{X}}_c=\bar{\mathbf{X}}_u+\mathbf{K}_{12}\mathbf{K}_{22}^{-1}\left(\mathbf{X}_k-\bar{\mathbf{X}}_k\right)$ is a conditional vector of mean values, t is a truncation parameter.

The random fields were generated according to various correlation functions, e.g. Wiener, Brown and others. The software has been thoroughly checked and verified [26].

The method outlined above is marked by two distinct features. The first points out conditional algorithm to sequentially generate an unknown random field. Thus it is possible to capture the fields of arbitrarily large dimensions. The second feature is a field envelope making it possible to modify restraining conditions in order to adjust the created field to a given example. Hence restraints of a theoretical correlation function may be superimposed by additional envelopes regarding engineering boundary conditions

Random field calculations

Random fields are continuously gaining their application to analyse material soil parameters. The selection of correlation function and its parameters, including material data should be based on in situ investigations in the foundation site. These tests are expensive, thus prior parameter assumption is a routine.

The random field is given a standard, widely encountered correlation function:

$$\rho(x,y)=\mathrm{e}^{-d_1|x|-d_2|y|} \tag{8}$$

This form is reasonable while an under-foundation volume is filled by a single soil type. The key parameters to be estimated here are correlation lengths $d1$ and $d2$. According to their values, the internal random field correlation is exclusively affected by the distance between the points (homogeneous field). The impact of d parameter in the uni-dimensional case is shown in Fig. 5. The field correlation on both vertical and horizontal directions is usually taken separately.

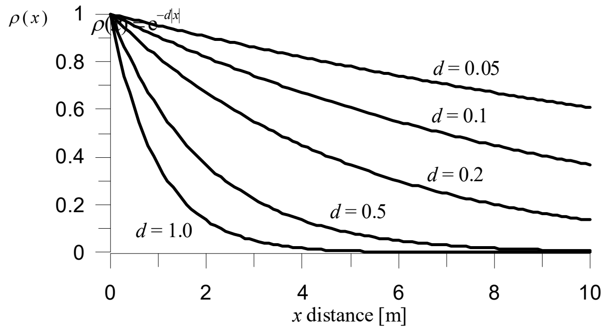

Figure 5. Correlation length with regard to different correlation parameters

In order to comprehensively outlook the correlation length impact on field performance, the strong and weak correlation cases may be compared. It was conducted applying a finite element 16×40 field discretization, assuming $d1 = d2$ (Eq. 8). In the case of high correlation parameter $d = 0.05$ in surrounding points, such information means that the closely located values are convergent, as presented in Figure 6. In case of different correlation length assumed in vertical and horizontal direction the shape may slightly differ, however the idea still remains the same.

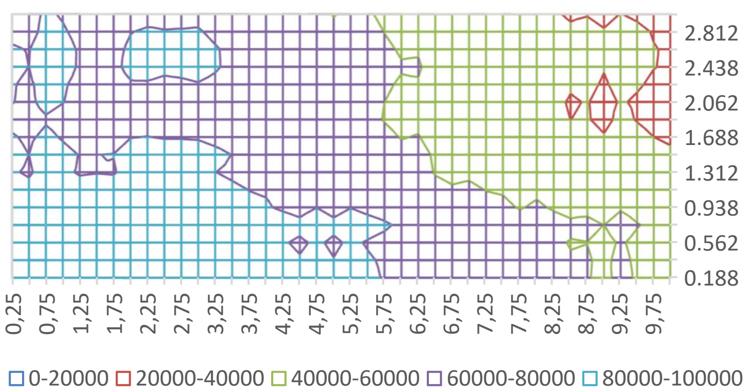

Figure 6. Example of highly correlated field ($d = 0.05$)

The less correlated field is presented in Fig. 7. As the visual representation of middle point of corresponding elements stiffness values differs a lot, smooth connection between these values is preserved. However comparing extremes, boundary regions differences of up to 2σ are allowed.

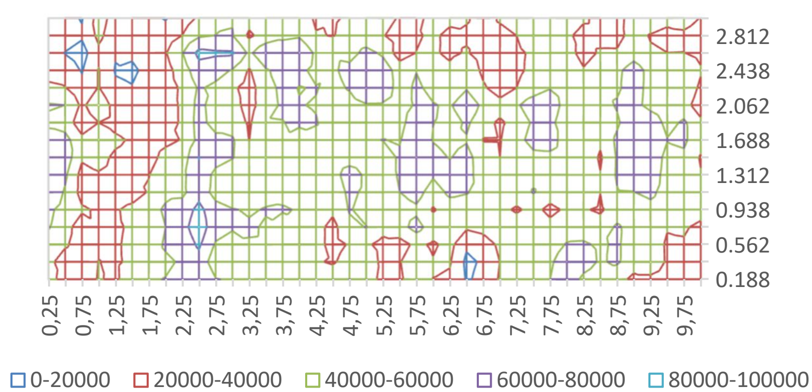

Figure 7. Example of low correlated field ($d = 1.0$)

The examples of a highly correlated random field in Fig. 6 and a slightly correlated case in Fig. 7 are compared with the totally random case, generated by white noise field algorithm presented in Fig. 8. Most differences are observable due to values at neighbouring points, totally irrational in the white noise generation type. On the other hand, only highly correlated fields find their real-case scenario fundamentals due to a soil natural homogeneity within regions, not only referring to local values. It seems correct, especially in the stiffness analysis.

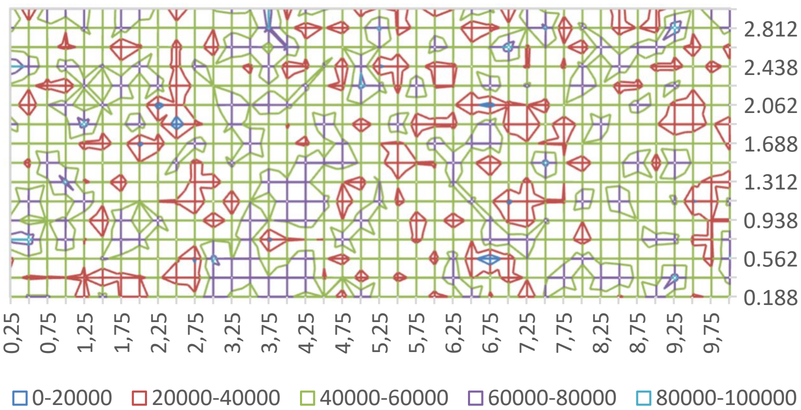

Figure 8. Non-correlated, white noise generated field

Apparently, both random field and single random variable analytical variants of soil parameter variation may be compared assuming a strong correlation case. The result convergence is expected here. A population of 5000 fields was generated, assuming $d1 = d2 = 0.05$ (Eq. 8). Properties of the conducted generation were additionally verified in the first place, starting with the check of all the achieved distribution at an arbitrarily chosen single point in the vicinity of the field center. Making use of an entire generated population of 5000 fields, the achieved distribution is shown in Fig. 9.

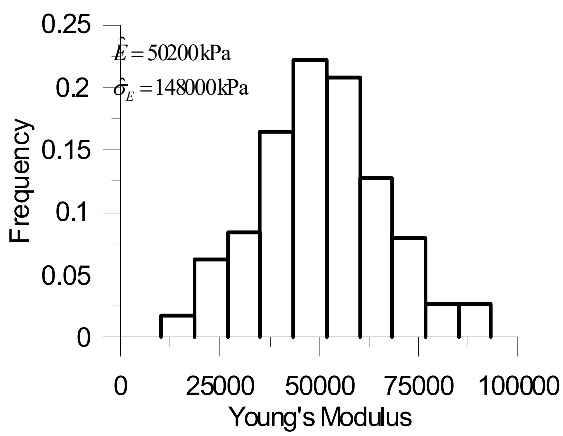

Figure 9. Frequency distributions from 5000 generated field (middle point of each field).

Next, mean values are estimated of the generated fields $\hat{\mathbf{E}}$, subsequently comes the covariance matrix $\hat{\mathbf{K}}_E$:

$$\hat{\mathbf{E}} = \frac{1}{500}\sum_{i=1}^{NR}\mathbf{E}_i \tag{9}$$

$$\hat{\mathbf{K}}_E = \frac{1}{500-1}\sum_{i=1}^{500}\left(\mathbf{E}_i - \hat{\mathbf{E}}\right)\left(\mathbf{E}_i - \hat{\mathbf{E}}\right)^T \tag{10}$$

The norm of the covariance matrix $\left\|\hat{\mathbf{K}}\right\|$ and the variance norm of this matrix are obtained as well $\left\|\hat{\mathbf{K}}_W\right\|$:

$$\left\|\hat{\mathbf{K}}\right\| = \sqrt{\left(\hat{\mathbf{K}}_E\right)^2} \tag{11}$$

$$\left\|\hat{\mathbf{K}}_W\right\| = \sqrt{tr\left(\hat{\mathbf{K}}_E\right)^2} \tag{12}$$

These norms are computed due to a theoretical model, so generation errors are available as follows: maximum error of a single element 8.44%, normal errors globally 3.87%, normal errors regarding variance 2.65%.

The analysis proves proper random field generation, thus the realizations may be a starting point of a subsequent MC simulation. A population of 200 samples was taken here. The convergence course is shown in Fig. 10 and Table 4. An apparent feature is observable in the result convergence, mean value and standard deviation

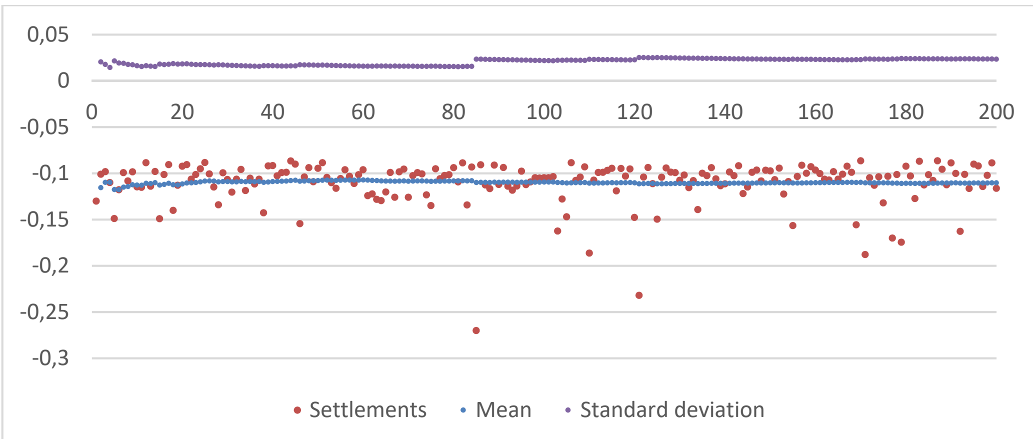

Figure 10. Divergence of response and its parameters in the case of correlated random fields (Monte Carlo, 200 samples)

In addition, computations were conducted for a population of 200 white noise runs (Fig. 11, Table 4). While a white noise use to capture soil features seems unrealistic, the results are intended to form an anticipated reference level for correlated random fields. The results are summarized in Table 4.

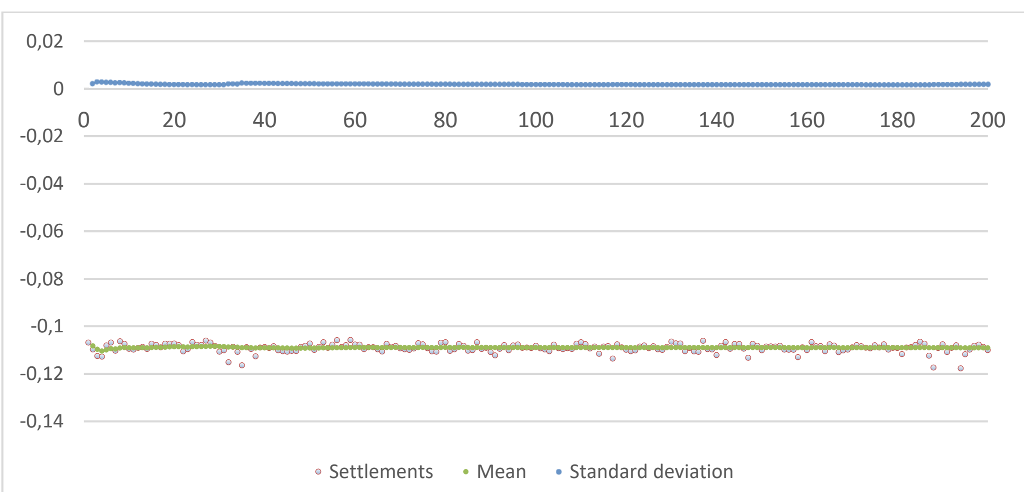

Figure 11. Divergence of response and its parameters in the case of uncorrelated (white noise) random fields

Table 4. Comparison between standard Monte Carlo sampling for Normal (N) and Lognormal (LN) histograms (10000 samples) and random field (200 samples) generated by highly correlated (RF) and White Noise (WN) functions.

Statistical parameters		10000 N	10000 LN	200 RF	200RF WN
Min value Y_{min}	[m]	-0.2887	-0.207658	-0.27013	-0.11769
Max value Y_{max}	[m]	-0.0843	-0.071268	-0.08649	-0.1057
Mean $\bar{Y}$	[m]	-0.1099	-0.108564	-0.11039	-0.1090646
Standard deviation σ_Y	[m]	0.025325	0.0156266	0.023384	0.0018344
Skewness W_s	[-]	-3.96086	-0.913517	-3.10202	-1.54488

All the random field application results are rapidly convergent, thus performing a great computational effort is not needed. However, the variance reduction methods may be incorporated as well, due to prior classification of generated fields. Such an approach substantially reduces the sample space required to get a reliable result.

CONCLUSIONS

The paper addresses the case of continuous footing situated on a random subsoil solved as a plane strain system. The general purpose of the paper was to implement an extended analysis of the random subsoil, using different probability density functions to describing the stiffness modulus of soil. Two independent paths were compared – the first, considering an entire domain beneath the foundation random, represented by a single random variable only, the second, applying a random field methodology to model spatial variability of soil parameters. The model was built in the FEM code ZSoil, cooperating with the external programming language Python.

The standard Monte Carlo method comparing both Normal and Lognormal variants proved the former one the easiest and relevant in such simple, shallow foundation case, with regard to serviceability limit state function. Different sampling domain sizes were regarded, the reference domain of 10000 samples was finally adopted. Considering both types of the input variables, even though the minimum and maximum values differ, the resultant histograms strongly resemble each other. In the cases of similar results of both variants, there is no need to use the Lognormal variables due to their complicated use in engineering application and the difficulties in interpretation.

In order to assure the consistency with the fundamentals of soil mechanics, Gaussian random variables may be generated with a lower bound, not to allow the generation of negative values of Young's modulus. Moreover, if both upper and lower bounds are introduced, it is possible to provide material parameters generated according to in-situ tests or general engineering design remarks. Although the trimming operation causes the input variable set not to match the assumed Gaussian theoretical pattern, the results are still directly applicable in real-life design. Such a realistic, engineering-oriented input dataset described by means of a modified Normal distribution seems a straightforward route to implement a probabilistic approach in standard engineering routines.

For the random field cases, two variants were analysed. At first, completely uncorrelated white noise generation was conducted. While 200 runs were performed, the results convergence was perfect, thus the sequence was stopped. The second, reference calculations assumed high correlation between surrounding 2D elements. In this case 5000 different fields were created, and the results convergence was high, limiting the sequence to 200 runs only.

As a final result, both Monte Carlo and random field histograms were compared, a clear conclusion was driven proving that there is no necessity to incorporate developed statistical methods in the case of small footings. However, it can be significant, in the case of more complex structures.

In the extension of the work engineering, in-situ measurements may be applied to build the correlation matrix of the random field, to model the spatial variability of properties of the natural soils beneath the foundation.

REFERENCES

1. A. Truty, Studia Geotechnica et Mechanica, 40(2), 86-95 (2018)
2. S. Pietruszczak, M. Oulapour, Journal of Geotechnical and Geoenvironmental Engineering, 125, 7, 576-582 (1999)
3. P. Przecherski, S. Pietruszczak, Acta Geotechnica, 1-10 (2019)
4. B. Wrana, N. Pietrzak, Studia Geotechnica et Mechanica, 35, 1, 246-258 (2013)
5. P. Baranowski, K. Damaziak, Ł. Mazurkiewicz, P. Mertuszka, W. Pytel, J. Małachowski, B. Pałac-Walko, T. Jones, Shock and Vibration, 1-11, (2019)
6. W. Puła, M. Chwała, Computers and Geotechnics 68, 128–136 (2015)
7. J.M. Pieczynska-Kozłowska, W. Puła, D.V. Griffiths, G.A. Fenton, Computers and Geotechnics 67, 229–238 (2015)
8. R. Ghanem, W. Brzakala, Journal of Engineering Mechanics, 122, 4, 361-369 (1996).
9. G. A. Fenton, D. Griffiths, *Risk Assessment in Geotechnical Engineering* (John Wiley & Sons, 2008).
10. G. A. Fenton, D. V. Griffiths, Journal of Geotechnical and Geoenvironmental Engineering, Vol. 128, No. 5, May 1, 2002.
11. D. V. Griffiths, G.A. Fenton, H.R. Ziemann, Georisk 473 2, 113–121 (2008).
12. D. V. Griffiths, G. A. Fenton, Journal of Geotechnical and Geoenvironmental Engineering, 135, 11, 2009.
13. S. O. Akbas, F. H. Kulhawy, Journal of Geotechnical and Geoenvironmental Engineering, 135, 12, 1779-1788 (2009).
14. P. Allahverdizadeh, D. V. Griffiths, G. A. Fenton, GEORISK, 1-9, (2015)
15. C. L. Chan, B. K. Low, International Journal for Numerical and Analytical Methods in Geomechanics, 36, 1387–1409 (2012).
16. J. Ching, K. K. Phoon, Y.-K. Pan, Structural Safety **66**, 106–117 (2017).
17. S. Drakos, G.N. Pande, Studia Geotechnica et Mechanica, 38, 1, 33- 43 (2016).
18. Y. Liu, F.-H. Lee, S.-T. Quek, M. Beer, Probabilistic Engineering Mechanics. 38, 42-53 (2014).
19. B.K. Low, Structural Safety 49, 56–64 (2014).
20. B. K. Low, K. K. Phoon, Computers and Geotechnics 65, 30–44 (2015)
21. K.K. Phoon, J.V. Retief, J. Ching, M. Dithinde, T. Schweckendiek, Y. Wangf, L.M. Zhang, Structural Safety 62, 24–33 (2016).
22. Zimmermann Th., Truty A., Podleś K., Numerics in geotechnics and structures, Lausanne (2018).
23. M. Beer, Y. Zhang, S. T. Quek, K. K. Phoon, Structural Safety 41, 1–10 (2013).
24. J. Przewłócki, *Problems of stochastic soil mechanics – reliability analysis.* Wrocław: Lower Silesia Educational Publisher, in Polish, (2006).
25. Python Manual – 2.7.17.
26. E. Bielewicz, J. Górski: International Journal of Non-linear Mechanics 37, 4-5, 777-784 (2002).
27. J. Tejchman, J. Górski: International Journal of Numerical and Analytical Methods in Geomechanics. 35, 2, 226-243 (2011).