

MAREK BEŚKA and MATEUSZ GAŁKA (Gdańsk)

## MOMENTS OF HERMITE–GAUSSIAN FUNCTIONALS

*Abstract.* Moments of finite products of Hermite–Gaussian functionals are expressed by covariances of a Gaussian sequence.

**Introduction.** Mixed moments of Hermite–Gaussian functionals play an important role in stochastic analysis of Wiener chaos (for extensive treatment of the ideas corresponding to Wiener chaos, also those regarding moments, see [J], [PT]). In this paper, we present a new method of computing such moments. It allows us to formulate a necessary and sufficient condition (see Proposition 2.1 below) for vanishing of a moment of even order in the case of non-negative correlations of Gaussian random variables from Wiener chaos.

**1. Hermite polynomials.** Let  $\mathbb{R}^d$  denote the  $d$ -dimensional Euclidean space, equipped with the standard inner product  $(\cdot, \cdot)_d$  and the Euclidean norm  $\|\cdot\|_d$ . Let  $(\Omega, \mathcal{F}, P)$  be a fixed probability space. The Hermite polynomial  $H_n$  of degree  $n \geq 1$  on  $\mathbb{R}$  is defined by

$$H_n(x) = (-1)^n \exp(x^2/2) \frac{d^n}{dx^n} (\exp(-x^2/2)), \quad x \in \mathbb{R}, n \geq 1.$$

Additionally, we assume that  $H_0 \equiv 1$ . The first Hermite polynomials are  $H_1(x) = x$ ,  $H_2(x) = x^2 - 1$ . The polynomials  $H_n$  divided by  $n!$  are the coefficients of the expansion in powers of  $t$  of the generating function  $w(t, x) = \exp(tx - t^2/2)$ ,  $x, t \in \mathbb{R}$ . In fact, we have

$$(1.1) \quad w(t, x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \quad x, t \in \mathbb{R}.$$

---

2020 *Mathematics Subject Classification:* Primary 60E05; Secondary 60G15.

*Key words and phrases:* Gaussian random vector, Hermite polynomials.

Received 25 October 2019; revised 16 March 2020.

Published online 10 June 2020.

Note that for a standard Gaussian variable  $\eta$  we have

$$w(t, x) = \overline{\exp(tx)} E \exp(it\eta) = E \exp(tx + it\eta) = E \sum_{n=0}^{\infty} \frac{t^n}{n!} (x + i\eta)^n.$$

Now, using the Lebesgue dominated convergence theorem and comparing the above expansion with (1.1), we get

$$(1.2) \quad H_n(x) = E(x + i\eta)^n, \quad x \in \mathbb{R}, n \geq 0.$$

Hence,

$$(1.3) \quad \left| \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \right| \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} E[(|x| + |\eta|)^n] \leq E \exp[|t|(|x| + |\eta|)] < \infty.$$

Therefore, the sum in (1.1) converges absolutely for all  $t, x \in \mathbb{R}$ .

Another well known relationship between Hermite polynomials and Gaussian random variables is the result below (see [N]).

LEMMA 1.1. *Let  $(X, Y)$  be a two-dimensional Gaussian vector such that  $E(X) = E(Y) = 0$ ,  $E(X^2) = E(Y^2) = 1$ ,  $E(XY) = \rho$ , where  $\rho$  is the correlation coefficient of  $X$  and  $Y$ . Then, for all  $n, m \geq 0$ ,*

$$E[H_n(X)H_m(Y)] = \begin{cases} n!\rho^n & \text{if } n = m, \\ 0 & \text{if } n \neq m. \blacksquare \end{cases}$$

Now, let  $X = (X_1, \dots, X_d)$  be a Gaussian random vector such that  $E(X_i) = 0$  and  $E(X_i^2) = 1$  for  $i = 1, \dots, d$ . The aim of this note is to compute the expectation

$$E[H_{n_1}(X_1)H_{n_2}(X_2) \cdots H_{n_d}(X_d)].$$

To formulate our result, we need some notations and definitions. For  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  and  $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d = (\mathbb{N} \cup \{0\})^d$ , we write

$$|x| = \sum_{i=1}^d x_i, \quad x^k = \prod_{i=1}^d x_i^{k_i}, \quad |k| = \sum_{i=1}^d k_i, \quad k! = \prod_{i=1}^d k_i!.$$

For  $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$  the integer  $|k|$  will be called the *length* of the vector  $k$ . The set of all square matrices of dimension  $d$  with elements from  $\mathbb{R}$  (resp.  $\mathbb{N}_0$ ) is denoted by  $\mathcal{M}_d(\mathbb{R})$  (resp.  $\mathcal{M}_d(\mathbb{N}_0)$ ). If  $R \in \mathcal{M}_d(\mathbb{R})$ , the  $j$ th column and  $i$ th row are denoted by  $R_j$  and  $R^i$  respectively. From time to time, we shall use the shorthand notation  $R = [R_j^i]$ . As usual, we identify rows and columns of  $R$  with vectors from  $\mathbb{R}^d$ . If  $R \in \mathcal{M}_d(\mathbb{R})$  and  $K \in \mathcal{M}_d(\mathbb{N}_0)$ , we denote

$$|K| = (|K^1|, \dots, |K^d|), \quad |R| = (|R^1|, \dots, |R^d|),$$



$$K! = K^1! \dots K^d! = \prod_{i,j=1}^d K_j^{i!}, \quad R^K = R^{1K^1} \dots R^{dK^d} = \prod_{i,j=1}^d R_j^{iK_j^i}$$

with the convention  $0^0 = 1$ . For  $K = [K_j^i] \in \mathcal{M}_d(\mathbb{N}_0)$ , let  $u(K)$  denote the upper diagonal matrix of  $K$ , i.e.

$$u(K) := [U_j^i], \quad \text{where} \quad U_j^i := \begin{cases} K_j^i & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$$

For  $n \in \mathbb{N}_0^d$  let us introduce the following families of matrices:

$$\mathcal{M}_d^0(\mathbb{N}_0) = \{K \in \mathcal{M}_d(\mathbb{N}_0) : \text{diag}(K) = 0, K \text{ is symmetric}\},$$

$$\mathcal{M}_{d,n}^0(\mathbb{N}_0) = \{K \in \mathcal{M}_d^0(\mathbb{N}_0) : |K| = n\},$$

where  $\text{diag}(K)$  denotes the main diagonal of the matrix  $K$ .

The Hermite polynomials on  $\mathbb{R}^d$  are defined as tensor products of the Hermite polynomials on  $\mathbb{R}$ : for  $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$  and  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  we put

$$H_n(x) = \prod_{i=1}^d H_{n_i}(x_i).$$

Similarly to the one-dimensional case, the polynomials  $H_n$  divided by  $n!$  are the coefficients of expansion in powers of  $t = (t_1, \dots, t_d) \in \mathbb{R}^d$  of the generating function

$$w(t, x) = \exp(-\|t\|_d^2/2 + (t, x)_d), \quad t, x \in \mathbb{R}^d.$$

That is,

$$w(t, x) = \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} H_n(x), \quad t, x \in \mathbb{R}^d.$$

**2. Main result.** We can now formulate the main result of this note.

**THEOREM 2.1.** *Let  $X = (X_1, \dots, X_d)$ ,  $d \geq 2$ , be a Gaussian random vector such that  $E(X_i) = 0$  and  $E(X_i^2) = 1$  for  $i = 1, \dots, d$ . Then, for the Hermite polynomial  $H_n$  on  $\mathbb{R}^d$  of degree  $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$  we have*

$$(2.4) \quad \begin{aligned} EH_n(X) &= E[H_{n_1}(X_1)H_{n_2}(X_2) \cdots H_{n_d}(X_d)] \\ &= \begin{cases} \sum_{K \in \mathcal{M}_{d,n}^0} \frac{n!}{\sqrt{K!}} Q^{u(K)} & \text{if } \mathcal{M}_{d,n}^0 \neq \emptyset, \\ 0 & \text{if } \mathcal{M}_{d,n}^0 = \emptyset, \end{cases} \end{aligned}$$

where  $Q$  denotes the covariance matrix of  $X$ .

*Proof.* From the definition of  $H_n$  and from (1.3), we conclude that

$$\begin{aligned} H_n(x) &= H_{n_1}(x_1)H_{n_2}(x_2) \cdots H_{n_d}(x_d) \\ &= E[(x_1 + i\eta_1)^{n_1}(x_2 + i\eta_2)^{n_2} \cdots (x_d + i\eta_d)^{n_d}], \end{aligned}$$



where  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ ,  $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$  and  $\eta_1, \dots, \eta_d$  is a sequence of independent standard Gaussian variables, independent of  $X$ . From the above and from (1.2) we deduce that (with  $E_{\eta_i}$  denoting the expectation with respect to  $\eta_i$ ,  $i = 1, \dots, d$ )

$$\begin{aligned} E \sum_{n \in \mathbb{N}_0^d} \left| \frac{t^n}{n!} H_n(X) \right| &= E \prod_{i=1}^d \sum_{n_i=0}^{\infty} \left| \frac{t_i^{n_i}}{n_i!} H_{n_i}(X_i) \right| \\ &\leq E[E_{\eta_1} e^{|t_1|(|X_1|+|\eta_1|)} E_{\eta_2} e^{|t_2|(|X_2|+|\eta_2|)} \dots E_{\eta_d} e^{|t_d|(|X_d|+|\eta_d|)}] \\ &\leq E_{\eta_1} e^{|t_1 \eta_1|} E_{\eta_2} e^{|t_2 \eta_2|} \dots E_{\eta_d} e^{|t_d \eta_d|} E e^{|t_1 X_1|+|t_2 X_2|+\dots+|t_d X_d|} < \infty. \end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem, we have

$$(2.5) \quad Ew(t, X) = E \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} H_n(X) = \sum_{n \in \mathbb{N}_0^d} \frac{t^n}{n!} E[H_n(X)].$$

On the other hand,

$$\begin{aligned} Ew(t, X) &= E \exp((t, X)_d - \|t\|_d^2/2) = \exp((Qt, t)_d/2 - \|t\|_d^2/2) \\ &= \exp\left(\frac{1}{2}((Q - I)t, t)_d\right) = \exp\left(\sum_{1 \leq i < j \leq d} \rho_{ij} t_i t_j\right), \end{aligned}$$

where  $I$  is the identity operator on  $\mathbb{R}^d$ . Consequently,

$$(2.6) \quad Ew(t, X) = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{1 \leq i < j \leq d} \rho_{ij} t_i t_j \right)^m.$$

Let us compute the components of the above sum. For simplicity, denote by  $S_m$  the set of all vectors

$$k = (k_{12}, \dots, k_{1d}, k_{23}, \dots, k_{2d}, \dots, k_{d-1,d}) \in \mathbb{N}_0^{d(d-1)/2}$$

such that  $|k| = m$ . It follows that

$$\begin{aligned} \left( \sum_{1 \leq i < j \leq d} \rho_{ij} t_i t_j \right)^m &= \sum_{k \in S_m} \frac{m!}{k!} \rho_{12}^{k_{12}} \dots \rho_{1d}^{k_{1d}} \rho_{23}^{k_{23}} \dots \rho_{2d}^{k_{2d}} \dots \rho_{d-1,d}^{k_{d-1,d}} \\ &\quad \times (t_1 t_2)^{k_{12}} \dots (t_1 t_d)^{k_{1d}} (t_2 t_3)^{k_{23}} \dots (t_2 t_d)^{k_{2d}} \dots (t_{d-1} t_d)^{k_{d-1,d}} \\ &= \sum_{\substack{K \in \mathcal{M}_d^0 \\ \|K\|=m}} \frac{m!}{\sqrt{K!}} Q^{u(K)} t^{|K|}, \end{aligned}$$

where  $\|K\| = |K^1| + \dots + |K^d|$ . From the above and from (2.6) we have

$$(2.7) \quad Ew(t, X) = \sum_{m=0}^{\infty} \sum_{\substack{K \in \mathcal{M}_d^0 \\ \|K\|=m}} \frac{1}{\sqrt{K!}} Q^{u(K)} t^{|K|} = \sum_{n \in \mathbb{N}_0^d} \sum_{K \in \mathcal{M}_{d,n}^0} \frac{1}{\sqrt{K!}} Q^{u(K)} t^n.$$

Now, comparing (2.5) and (2.7) we get (2.4), and the theorem follows. ■



We see at once that  $\mathcal{M}_{d,n}^0 = \emptyset$  if  $|n|$  is an odd integer. When  $|n|$  is even, we have the result below.

PROPOSITION 2.1. *Let  $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$  be such that  $|n|$  is an even integer and let  $n_{i_0} = \max_{1 \leq i \leq d} n_i$ . Then*

$$\mathcal{M}_{d,n}^0 \neq \emptyset \iff n_{i_0} \leq \sum_{\substack{i=1 \\ i \neq i_0}}^d n_i.$$

*Proof.* Without loss of generality, we may assume that

$$n_1 \geq \dots \geq n_d.$$

( $\Rightarrow$ ) Assume that  $n_1 > n_2 + \dots + n_d$  and let  $K \in \mathcal{M}_{d,n}^0$ . Then the first row of  $K$  is

$$K^1 = (0, k_{12}, k_{13}, \dots, k_{1d}) \quad \text{and} \quad n_1 = |K^1| = k_{12} + k_{13} + \dots + k_{1d}.$$

Hence, there exists  $2 \leq i \leq d$  such that  $k_{1i} > n_i$ . Therefore,  $k_{i1} = k_{1i} > n_i$  and  $|K^i| > n_i$ . Consequently,  $|K| \neq n$  and this contradicts the assumption that  $K \in \mathcal{M}_{d,n}^0$ .

( $\Leftarrow$ ) Notice first that if  $n_1 = n_2 + \dots + n_d$  then the matrix

$$K = \begin{bmatrix} 0 & n_2 & n_3 & \dots & n_d \\ n_2 & 0 & 0 & \dots & 0 \\ n_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_d & 0 & 0 & \dots & 0 \end{bmatrix}_{d \times d}$$

belongs to  $\mathcal{M}_{d,n}^0$ , so  $\mathcal{M}_{d,n}^0 \neq \emptyset$ . Now, let  $n_1 < n_2 + \dots + n_d$ . Then  $p := n_2 + \dots + n_d - n_1 > 0$  and we see at once that  $p$  is even. For our further considerations, the following lemma will be necessary.

LEMMA 2.1. *Let  $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$  be a non-increasing sequence such that  $|n|$  is even and  $|n| > 2n_1$ . Then there exists a sequence  $m = (m_1, \dots, m_k) \in \mathbb{N}_0^k$  with  $2k + 1 \leq d$  such that*

$$s = (0, n_2 - m_1, n_3 - m_1, n_4 - m_2, n_5 - m_2, \dots, n_{2k} - m_k, n_{2k+1} - m_k, n_{2k+2}, \dots, n_d) \in \mathbb{N}_0^d$$

and  $|s| = n_2 + \dots + n_d - 2|m| = n_1$ .

*Proof of Lemma 2.1.* Let  $\{e_i\}_{i=1}^d$  be the standard basis in  $\mathbb{R}^d$  and

$$r := \begin{cases} (d+1)/2 & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \text{ is even.} \end{cases}$$

Moreover, define a sequence  $\{p_j\}_{j=1}^r$  by

$$p_1 = 0, \quad p_j = \sum_{i=2}^j n_{2i-1}, \quad j = 2, \dots, r,$$

and a sequence  $\{S^{(i)}\}_{i=0}^{p_r}$  of vectors in  $\mathbb{N}_0^d$  as follows:

$$S^{(0)} = (0, n_2, n_3, \dots, n_d), \quad S^{(p_j+l)} = S^{(p_j+l-1)} - e_{2j} - e_{2j+1},$$

where  $1 \leq l \leq n_{2j+1}$ ,  $j = 1, \dots, r-1$ . It can be seen that for  $j > 1$ ,

$$S^{(p_j+l)} = S^{(0)} - n_3 e_2 - n_3 e_3 - \dots - n_{2j-1} e_{2j-2} - n_{2j-1} e_{2j-1} - l e_{2j} - l e_{2j+1}.$$

From the definition of  $\{S^{(i)}\}_{i=0}^{p_r}$  we have

$$|S^{(p_j+l)}| = |S^{(p_j+l-1)}| - 2, \quad 1 \leq l \leq n_{2j+1}, \quad j = 1, \dots, r-1,$$

i.e. the lengths  $|S^{(p_j+l)}|$  decrease in arithmetic progression with common difference 2. By assumption,

$$|S^{(0)}| = n_2 + n_3 + \dots + n_d > n_1.$$

On the other hand, for  $r = (d+1)/2$ ,

$$\begin{aligned} |S^{(p_r)}| &= |S^{(0)}| - 2(n_3 + n_5 + \dots + n_d) \\ &= n_2 + n_3 + \dots + n_d - 2(n_3 + n_5 + \dots + n_d) \\ &= n_2 + (n_3 + n_4) + (n_5 + n_6) + \dots + (n_{d-2} + n_{d-1}) \\ &\quad + n_d - 2(n_3 + n_5 + \dots + n_d) \\ &\leq n_1 + 2n_3 + 2n_5 + \dots + 2n_{d-2} + 2n_d - 2(n_3 + n_5 + \dots + n_d) = n_1. \end{aligned}$$

Similarly for  $r = d/2$ ,

$$\begin{aligned} |S^{(p_r)}| &= |S^{(0)}| - 2(n_3 + n_5 + \dots + n_{d-1}) \\ &= n_2 + n_3 + \dots + n_d - 2(n_3 + n_5 + \dots + n_{d-1}) \\ &= n_2 + (n_3 + n_4) + (n_5 + n_6) + \dots + (n_{d-1} + n_d) - 2(n_3 + n_5 + \dots + n_{d-1}) \\ &\leq n_1 + 2n_3 + 2n_5 + \dots + 2n_{d-1} - 2(n_3 + n_5 + \dots + n_{d-1}) = n_1. \end{aligned}$$

We conclude that there exists  $1 \leq i_0 \leq p_r$  such that  $|S^{(i_0)}| = n_1$ . Put  $a = S^{(0)} - S^{(i_0)}$ . Then we can define

$$2k := \#\{a_i : a_i \neq 0\}$$

and

$$\begin{aligned} m &:= (m_1, \dots, m_k) := (n_2, n_4, \dots, n_{2k}) - (S_2^{(i_0)}, S_4^{(i_0)}, \dots, S_{2k}^{(i_0)}) \\ &= (n_3, n_5, \dots, n_{2k+1}) - (S_3^{(i_0)}, S_5^{(i_0)}, \dots, S_{2k+1}^{(i_0)}). \end{aligned}$$

From the construction of  $m$  and the definition of the vector  $s$ , we obtain

$$|s| = |S^{(0)}| - 2|m| = |S^{(0)}| - (|S^{(0)}| - |S^{(i_0)}|) = |S^{(i_0)}| = n_1. \quad \blacksquare$$

Using Lemma 2.1 we can construct a vector  $m = (m_1, \dots, m_k) \in \mathbb{N}_0^k$ , where  $2k + 1 \leq d$ , such that  $m_i \leq n_{2i+1}$  for  $i = 1, \dots, k$  and  $2|m| = p$ , i.e.  $n_1 = n_2 + \dots + n_d - 2|m|$ . Therefore, we can find a matrix  $K$  which belongs to  $\mathcal{M}_{d,n}^0$ . Namely, we set  $K = A - B + C$ , where

$$A = \begin{bmatrix} 0 & n_2 & n_3 & \dots & n_d \\ n_2 & 0 & 0 & \dots & 0 \\ n_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_d & 0 & 0 & \dots & 0 \end{bmatrix}_{d \times d}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}_{d \times d},$$

where  $B_{11} = [0]_{1 \times 1}$ ,  $B_{12} = [m_1 \ m_1 \ m_2 \ m_2 \ \dots \ m_k \ m_k \ 0 \ \dots \ 0]_{1 \times (d-1)}$  and  $B_{22}$  is a null  $(d - 1) \times (d - 1)$  matrix, and finally

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \dots & \dots & \dots & 0 & 0 & m_k & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & m_k & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{d \times d}$$

Therefore,  $\mathcal{M}_{d,n}^0 \neq \emptyset$  and the proof of Proposition 2.1 is complete. ■

### References

- [J] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Univ. Press, 1997.
- [N] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, 2006.
- [PT] G. Peccati and M. S. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams*, Springer, Milan, 2011.

Marek Beška  
 Faculty of Applied Mathematics and Physics  
 Gdańsk University of Technology  
 Narutowicza 11/12  
 80-233 Gdańsk, Poland  
 ORCID: 0000-0003-0088-9850  
 E-mail: marbeska@pg.edu.pl

Mateusz Gałka  
 Faculty of Mathematics,  
 Physics and Informatics  
 University of Gdańsk  
 Wita Stwosza 57  
 80-308 Gdańsk, Poland  
 ORCID: 0000-0003-3065-7807  
 E-mail: mgalka@mat.ug.edu.pl

