




# On the well posedness of static boundary value problem within the linear dilatational strain gradient elasticity

Victor A. Eremeyev , Sergey A. Lurie, Yury O. Solyaev and Francesco dell'Isola

**Abstract.** In this paper, it is proven an existence and uniqueness theorem for weak solutions of the equilibrium problem for linear isotropic dilatational strain gradient elasticity. Considered elastic bodies have as deformation energy the classical one due to Lamé but augmented with an additive term that depends on the norm of the gradient of dilatation: only one extra second gradient elastic coefficient is introduced. The studied class of solids is therefore related to Korteweg or Cahn–Hilliard fluids. The postulated energy naturally induces the space in which the aforementioned well-posedness result can be formulated. In this energy space, the introduced norm does involve the linear combination of some specific higher-order derivatives only: it is, in fact, a particular example of anisotropic Sobolev space. It is also proven that aforementioned weak solutions belongs to the space  $H^1(\text{div}, V)$ , i.e. the space of  $H^1$  functions whose divergence belongs to  $H^1$ . The proposed mathematical frame is essential to conceptually base, on solid grounds, the numerical integration schemes required to investigate the properties of dilatational strain gradient elastic bodies. Their energy, as studied in the present paper, has manifold interests. Mathematically speaking, its singularity causes interesting mathematical difficulties whose overcoming leads to an increased understanding of the theory of second gradient continua. On the other hand, from the mechanical point of view, it gives an example of energy for a second gradient continuum which can sustain externally applied surface forces and double forces but cannot sustain externally applied surface couples. In this way, it is proven that couple stress continua, introduced by Toupin, represent only a particular case of the more general class of second gradient continua. Moreover, it is easily checked that for dilatational strain gradient continua, balance of force and balance of torques (or couples) are not enough to characterise equilibrium: to this aim, externally applied surface double forces must also be specified. As a consequence, the postulation scheme based on variational principles seems more suitable to study second gradient continua. It has to be remarked finally that dilatational strain gradient seems suitable to model the experimentally observed behaviour of some material used in 3D printing process.

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## 1. Introduction

Strain gradient elasticity deals with those models of continuum media where a strain energy density depends on the first and second gradients of placements. The mechanical motivation which requires the introduction of such an energy is related to the existence, at lower scales with respect to the scale where continuum behaviour is observed, of longer distance interactions among the basic constituents of the considered deformable body, see, e.g. [58, 59, 68].

Strain gradient elasticity has been originated not so recently. It is a particular case of the higher gradient continua already considered in the works by Gabrio Piola, who also recognised the mechanical mechanism, at lower scales, which may require the introduction of this class of generalised continua. Later, Le Roux and other scholars see, e.g. [21, 51, 52] and the reference therein, continued the line of thought started by Piola and, more recently, also by using the by using a more modern formalism and the powerful tools given by functional analysis, Paul Germain gave a further impulse to generalised continuum mechanics, see [28, 42, 43].

In the case of fluids, the strain gradient model, which was also at first formulated by Piola, was studied by Korteweg [49] and by Cahn and Hilliard [15, 16] see also [8, 29, 67]. It has to be remarked that Piola deduced the strong form of the evolution equations for continua (and in particular for fluids) using a variational principle, by calculating the corresponding Euler-Lagrange stationarity conditions in the referential (Lagrangian) description and finally transporting (using so-called Piola transport) these conditions in the current (Eulerian) description [8, 29].

Within rational mechanics approach, the corresponding and needed boundary value problems were fully formulated by Toupin [73, 74] and Mindlin [53, 54] by using some techniques from the differential geometry of surfaces, see also [25] for  $N$ th gradient media. More recently, the model was also applied for modelling materials at the micro- and nanometer scale [20, 40], or composites with high contrast (at a lower scale) in material properties [2, 22, 24, 26, 37, 61, 62, 69, 77].

In statics, the boundary value problems, which are deduced from the stationarity condition for a second gradient energy functional, constitute a system of fourth-order partial differential equations complemented by consistent boundary conditions. Consistent boundary conditions, as already recognised by Lagrange and Piola, can be easily deduced by formulating a variational principle (either a minimisation principle or a stationarity condition) and then integrating by parts to get the so-called strong formulation of equilibrium condition.

For infinitesimal deformations one assumes that a strain energy density  $W$  depends on the first and second gradients of displacements  $\mathbf{u}$ , so that we have  $W = W(\nabla\mathbf{u}, \nabla\nabla\mathbf{u})$ , where  $W$  is a quadratic form of its arguments. We call a model *gradient complete* if the following inequality is valid

$$W(\mathbf{u}) \geq C_1 \|\nabla\mathbf{u}\|^2 + C_2 \|\nabla\nabla\mathbf{u}\|^2, \quad \forall \mathbf{u},$$

where  $C_1$  and  $C_2$  are positive constants which are independent on  $\mathbf{u}$  and  $\|\cdot\|$  denotes the Euclidean norm. In this case, the analysis of the well posedness is based on the general theory of elliptic PDEs of higher order [4, 38, 48]. For the strain gradient elasticity such analysis was provided in [47, 50]. In this case, we have a unique weak solution which belongs to  $H^2(V)$ .

It is however remarkable that some interesting mechanical problems require the introduction of non-complete second gradient continuum models [9, 13, 24, 26, 32]. Therefore, we must conclude that in addition to gradient complete models such as those included in the so-called Toupin–Mindlin strain gradient elasticity, one must consider also models where a strain energy density depends on a subset of second derivatives of displacements or on particular combinations of them. We call such models *gradient incomplete*. It is worth to mention explicitly now the so-called couple stress theory [56, 78] with the strain energy in the form  $W = W(\nabla\mathbf{u}, \nabla \times \mathbf{u})$ , where “ $\times$ ” stands for the cross product. This special case of second gradient continuum models have a peculiarity which made them more acceptable by those who want to postulate continuum mechanics on the primitive concept of “balance laws” for forces and couples (or torques). In facts, [21] in the framework of couple-stress theories, the admissible boundary conditions (naturally deduced from the adapted variational principle) assume a particular for which double-forces are vanishing, see [28, 43] for a detailed discussion of the concept of double force and Alibert et al. [5] for its interpretation at micro-scales. Let us also note that the concept of double forces and other hypermoments are known in structural mechanics, see comments in [31, pp. 282–283]. As it is rather difficult to include the boundary conditions involving double forces in the postulation scheme based on balance laws, then its supporters have the tendency to deny the physical importance of said double forces. Instead, we believe in the importance of the role of double forces in continuum mechanics and, following the point of view expressed by Toupin [73, 74], we believe that the postulation scheme based on variational principles is more efficient in producing generalised continuum models. Moreover, if one considers the need of exploiting numerical integration schemes for getting predictions by using the introduced mathematical models, it is clear that a great economy of thought is gained by starting the modelling procedure basing it on variational principles. It is remarkable that in [73, 74], the more general theory of second gradient

elasticity is considered together with couple stress one, albeit in the title only the particular case of couple stress models is evoked (and the reader is explicitly warned about this circumstance).

Another example of incomplete second gradient models, which already was mentioned above, is the Korteweg fluid with  $W = W(\rho, \nabla\rho)$ , where  $\rho$  is a mass density [8, 29]. The cited examples are not exhausting the possibilities explored up to now. Indeed, as already mentioned, it has been proven that also some materials having a beam-lattice micro-structure can be modelled using gradient incomplete models [2, 7, 22, 24, 26, 37, 62], for pantographic beam lattice microstructured materials see [30, 32]. In all aforementioned cases of incomplete second gradient models, the analysis of the well posedness of boundary value problems requires a proper modification of the standard techniques [18, 33, 39]. We believe that the correct framework for studying, in an unitary way, many (if not all) incomplete second gradient models is given by the theory of anisotropic Sobolev's spaces as formulated by Nikol'skii [55] and other nonclassical Sobolev's spaces. Most likely, Nikol'skii fundamental contributions supply a general and powerful tool for studying the mathematical problems related to the more general class of  $N$ -th gradient micro-structured continua [25].

The aim of this paper is to prove the uniqueness and existence of weak solutions when considering the particular case of incomplete second gradient model given by the dilatational strain gradient continuum. Within this specific model, the strain energy density depends on strains and the gradient of dilatation, i.e. on the gradient of the trace of the strain tensor. Clearly, the model is closely related to the model of Korteweg or Cahn–Hilliard fluids [8, 29, 67] and could be treated as a gradient extension of generalised models with one-dimensional microstructure [17] or of the damage and poroelasticity models presented in [6, 19, 41, 44, 45, 60, 60, 63]. It is also a particular case of the model [80] where the dependence on the gradient of deviatoric part of strains is neglected. Moreover, the model studied in the present paper could be useful also when modelling the behaviour of porous media undergoing small deformations including ones used for 3D printing [79].

The paper is organised as follows. After the short mathematical preliminaries needed to set the used functional spaces, which are given in Sect. 2, we consider the constitutive equation for deformation energy and external interactions, and consequently, we formulate the principle of virtual work used to determine equilibrium configurations (Sect. 3). The class of external interactions which can be applied to dilatational strain gradient continua were formally determined in [23]. In the present paper, we give the rigorous proof of the fact that dilatational strain gradient continua can support forces per unit area but also double forces per unit area at their (Lipschitz continuous) boundaries.

In Sect. 4, based on the formulated principle of virtual work, we introduce the definition of weak solution that we use, and based on our listed assumptions, we prove that such a solution exists and is unique. Moreover, we prove that it is a minimiser of the total energy functional. Also in Sect. 4, we remark that the two Euler laws of dynamics are necessary but not sufficient conditions for the equilibrium of dilatational strain gradient continua. This remark naturally implies that variational principles seem to be more suitable to formulate novel mathematical models in applied science, see, e.g. [27, 36, 70, 71].

Some conclusions are presented in the final section, together with some future research perspectives.

## 2. Mathematical preliminaries

Let  $V$  be a bounded connected domain in  $\mathbb{R}^n$ ,  $n = 2, 3$ , with  $C^1$ -regular boundary  $S \equiv \partial V$ . For other possible requirements to  $S$ , we refer to [3]. In what follows, we use Sobolev's spaces  $H^k(V)$ ,  $k = 1, 2$ , defined as follows

$$\begin{aligned} H^1(V) &= \{u \in L^2(V), \nabla u \in (L^2(V))^n\}, \\ H^2(V) &= \{u \in L^2(V), \nabla u \in (L^2(V))^n, \nabla \nabla u \in (L^2(V))^{n^2}\}, \end{aligned}$$

where  $L^2(V)$  is the Lebesgue space of square integrable functions, and  $\nabla$  is the spatial nabla operator.

We denote vectors and tensors of any order using semibold font. So for simplicity, we use the notation for vector- or tensor-valued functions  $\mathbf{u} \in \mathbf{L}^2(V)$ , etc., as an equivalent to  $(L^2(V))^n$ , that means that each Cartesian component of  $\mathbf{u} \equiv (u_1, \dots, u_n)$  belongs to  $L^2(V)$ .

In addition following Jacques Louis Lions [72, p. 100], we introduce the functional space

$$\mathbf{H}(\text{div}, V) = \{\mathbf{u} \in \mathbf{L}^2(V), \nabla \cdot \mathbf{u} \in L^2(V)\},$$

where  $\nabla \cdot \mathbf{u}$  is the divergence of  $\mathbf{u}$ . Obviously,  $\mathbf{H}^1(V) \subset \mathbf{H}(\text{div}, V)$ . Considering vector-functions  $\mathbf{u} \in \mathbf{H}^2(V)$ , we also introduce the space

$$\mathbf{H}^1(\text{div}, V) = \{\mathbf{u} \in \mathbf{H}^1(V), \nabla \cdot \mathbf{u} \in H^1(V)\}.$$

Obviously,  $\mathbf{H}^1(\text{div}, V)$  is a separable Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1(\text{div}, V)} = \int_V [\mathbf{u} \cdot \mathbf{v} + \nabla \mathbf{u} : \nabla \mathbf{v} + (\nabla \nabla \cdot \mathbf{u}) \cdot (\nabla \nabla \cdot \mathbf{v})] dV,$$

and the norm

$$\|\mathbf{u}\|_{\mathbf{H}^1(\text{div}, V)} = [(\mathbf{u}, \mathbf{u})_{\mathbf{H}^1(\text{div}, V)}]^{1/2},$$

where “ $\cdot$ ” and “ $:$ ” stands for the dot and double dot products, respectively. Using Tartar’s example [72, p. 101], we can see that  $\mathbf{u} = (f_1(x_1)g_1(x_2, x_3), f_2(x_2)g_2(x_1, x_3), f_3(x_3)g_3(x_1, x_3))$  belongs to  $\mathbf{H}^1(\text{div}, V)$  if  $f_i \in H^2(V)$  and  $g_i \in H^1(V)$ . As for Tartar’s example given for  $\mathbf{H}(\text{div}, V)$ , this example gives the possibility to understand which kinds of discontinuity are possible for functions belonging to  $\mathbf{H}^1(\text{div}, V)$ . For further properties of these spaces, we refer to [3, 18, 46, 72].

### 3. Strain energy density and the principle of virtual work

For infinitesimal deformations, we introduce a vector of displacements

$$\mathbf{u} = \mathbf{u}(\mathbf{x}), \quad \mathbf{x} \in V,$$

where  $\mathbf{x}$  is a position vector. The corresponding strain tensor and its trace, i.e. dilatation, are given by

$$\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad \theta(\mathbf{u}) \equiv \text{tr } \mathbf{e} = \nabla \cdot \mathbf{u},$$

where  $T$  stands for a transpose tensor.

Within the dilatational strain gradient elasticity, there exists a strain energy density  $W$  which depends on  $\mathbf{e}$  and  $\nabla \theta$ :

$$W = W(\mathbf{e}, \nabla \theta).$$

Considering quadratic form of  $W$ , we came to

$$W(\mathbf{u}) = \frac{1}{2} \mathbf{e} : \mathbf{C} : \mathbf{e} + \mathbf{e} : \mathbf{B} : \nabla \theta + \frac{1}{2} \nabla \theta : \mathbf{D} : \nabla \theta, \quad (3.1)$$

where  $\mathbf{C}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$  are fourth-order tensors of elastic moduli. In what follows, we restrict ourselves to isotropic materials, so  $W$  takes the form

$$W(\mathbf{u}) = \frac{1}{2} \lambda \theta^2 + \mu \mathbf{e} : \mathbf{e} + \frac{1}{2} \alpha \nabla \theta \cdot \nabla \theta, \quad (3.2)$$

where  $\lambda$  and  $\mu$  are the Lamé moduli,  $\lambda > -2/3\mu$ ,  $\mu > 0$ , and  $\alpha > 0$  is an additional elastic modulus responsible for strain gradient effects.

The equilibrium equations and the corresponding natural boundary conditions within the considered model can be derived using the Lagrange variational principle modified for the strain gradient media as in [1, 8, 29]. Let  $F(\mathbf{u})$  be the functional of the strain energy

$$F(\mathbf{u}) = \int_V W(\mathbf{u}) \, dV.$$

The first variation of  $F$  takes the form

$$\begin{aligned} \delta F(\mathbf{u}; \mathbf{v}) &= \int_V \delta W(\mathbf{u}) \, dV \\ &= \int_V [\lambda \theta(\mathbf{u}) \theta(\mathbf{v}) + 2\mu \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \alpha \nabla \theta(\mathbf{u}) \cdot \nabla \theta(\mathbf{v})] \, dV, \end{aligned} \quad (3.3)$$

where  $\mathbf{v} = \delta \mathbf{u}$  and  $\delta$  is the variation symbol. After integration by parts,  $\delta F$  can be transformed into

$$\begin{aligned} \delta F(\mathbf{u}; \mathbf{v}) &= - \int_V (\nabla \cdot \mathbf{T}) \cdot \mathbf{v} \, dV \\ &\quad + \int_S [\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} - 2H(\mathbf{n} \cdot \mathbf{m}(\mathbf{u})) \mathbf{n} + \nabla_s(\mathbf{n} \cdot \mathbf{m}(\mathbf{u}))] \cdot \mathbf{v} \, dS \\ &\quad + \int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} \, dS, \end{aligned} \quad (3.4)$$

where  $\mathbf{n}$  is the vector of unit outward normal to  $S$ ,  $\frac{\partial}{\partial n}$  means the normal derivative,  $\nabla_s = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla$  is the surface nabla-operator,  $\mathbf{I}$  is the unit tensor, “ $\otimes$ ” means dyadic product,  $H = -1/2 \nabla_s \cdot \mathbf{n}$  is the mean curvature of  $S$ , and the surface divergence theorem was used as in [31], see details in Appendix. In addition, here  $\mathbf{T}$ ,  $\boldsymbol{\sigma}$ , and  $\mathbf{m}$  are the total stress, stress tensors, and double force vector given by the relations

$$\mathbf{T} = \boldsymbol{\sigma} - (\nabla \mathbf{m})^T, \quad \boldsymbol{\sigma} \equiv \frac{\partial W}{\partial \mathbf{e}} = \lambda \theta \mathbf{I} + 2\mu \mathbf{e}, \quad \mathbf{m} \equiv \frac{\partial W}{\partial \nabla \theta} = \alpha \nabla \theta.$$

In what follows, we assume kinematic boundary conditions on a part  $S_0$  of  $S$

$$\mathbf{u}|_{S_0} = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{n} \Big|_{S_0} = \mathbf{0}, \quad (3.5)$$

which describe the case when  $S_0$  is fixed. So the external loads can be applied on the rest of  $S$  that is on  $S_1 = S/S_0$ .

The form of the first variation (3.4) dictates an admissible form of the work of external forces and double forces

$$\delta A(\mathbf{v}) = \int_V \mathbf{f} \cdot \mathbf{v} \, dV + \int_{S_1} \left( \mathbf{t} \cdot \mathbf{v} + c \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} \right) \, dS, \quad (3.6)$$

where  $\mathbf{f}$  is a volume force vector,  $\mathbf{t}$  is a traction vector,  $c$  is a surface normal double force.

As a result, we formulate the principle of virtual work

$$\begin{aligned} \delta F - \delta A &= \int_V [\lambda \theta(\mathbf{u}) \theta(\mathbf{v}) + 2\mu \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \alpha \nabla \theta(\mathbf{u}) \cdot \nabla \theta(\mathbf{v})] \, dV \\ &\quad - \int_V \mathbf{f} \cdot \mathbf{v} \, dV - \int_{S_1} \left( \mathbf{t} \cdot \mathbf{v} + c \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} \right) \, dS = 0, \end{aligned} \quad (3.7)$$

for all admissible functions  $\mathbf{v}$ .

Let us consider (3.7) for rigid body motion, i.e.  $\mathbf{v}$  given by

$$\mathbf{v} = \mathbf{a} + \mathbf{x} \times \mathbf{b}, \quad (3.8)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are constant vectors and  $\times$  is the cross product. Let us recall that for such  $\mathbf{v}$  from (3.7) can obtain the conditions of equilibrium of a free solid body ( $S_1 = S$ ) in classic linear elasticity [33]. These conditions say that the total force and the total torque must be zero. Substituting (3.8) into (3.7), we get

$$\begin{aligned} \delta F - \delta A = & - \left( \int_V \mathbf{f} dV + \int_S \mathbf{t} dS \right) \cdot \mathbf{a} \\ & + \left( \int_V \mathbf{x} \times \mathbf{f} dV + \int_S \mathbf{x} \times \mathbf{t} dS \right) \cdot \mathbf{b} = \mathbf{0}. \end{aligned} \quad (3.9)$$

So we have again the total force and the total torque as zero

$$\mathbf{F} \equiv \int_V \mathbf{f} dV + \int_S \mathbf{t} dS = \mathbf{0}, \quad (3.10)$$

$$\mathbf{M} \equiv \int_V \mathbf{x} \times \mathbf{f} dV + \int_S \mathbf{x} \times \mathbf{t} dS = \mathbf{0}. \quad (3.11)$$

Note that double forces are not included in these balance equations. For derivation, we used the identities

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \quad \forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \\ \frac{\partial \mathbf{x}}{\partial n} &= \mathbf{n} \cdot \nabla \mathbf{x} = \mathbf{n} \cdot \mathbf{I} = \mathbf{n}, \quad \left( \frac{\partial \mathbf{x}}{\partial n} \times \mathbf{b} \right) \cdot \mathbf{n} = (\mathbf{n} \times \mathbf{b}) \cdot \mathbf{n} = 0. \end{aligned}$$

Just calculated Eqs. (3.10) and (3.11) express the two Euler laws for dilatational strain gradient continua. They are necessary conditions for equilibrium. However, one has to remark that they cannot be sufficient to equilibrium. In facts, in the following section, we prove the existence and uniqueness of weak solutions for the equilibrium elastic problem when surface forces, volume forces and surface double forces are given. As the field of double forces  $c$  does not appear in found Euler laws, for a given system of balanced external forces, there exists a different equilibrium solution for every different choice of the field  $c$ . An important consequence of this circumstance seems now evident: the balance laws of forces and torques (couples) are not sufficient to characterise the equilibrium configurations of considered continua. Therefore, postulation schemes based *only* on these two basic laws cannot give the right conceptual schemes for studying them.

#### 4. Weak solutions and their properties

From (3.7), one can derive the equilibrium equations and the natural boundary conditions as given in Appendix, see also, e.g. [1, 8, 29] for more details. Here, instead, we use (3.7) as the principal equation for determination of weak solutions of the considered problem without using its strong formulation. It has to be remarked that numerical integration schemes calculate approximations of weak solutions and that strong formulation does not play any role in the context of numerical computations. The importance of the integration by part process leading to strong equilibrium conditions resides in the need of determining which externally applied interactions (i.e. forces, moments, double forces, etc.) can be sustained by a continuum whose internal structure is characterised by a certain deformation energy.

Following the classic approach to the analysis of weak solutions, we introduce the natural functional space for equation (3.7) called energy space. For  $C^2(V)$  functions, we introduce the inner product as follows

$$(\mathbf{u}, \mathbf{v})_E = \int_V [\lambda\theta(\mathbf{u})\theta(\mathbf{v}) + 2\mu\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \alpha\nabla\theta(\mathbf{u}) \cdot \nabla\theta(\mathbf{v})] dV. \quad (4.1)$$

Let us note that we treat all quantities as dimensionless ones. Equation (4.1) produces the energy norm

$$\|\mathbf{u}\|_E = (\mathbf{u}, \mathbf{v})_E^{1/2}. \quad (4.2)$$

Now, we introduce

**Definition 4.1.** The energy space  $E$  is the closure of  $C^2(V)$  functions satisfying (3.5) in the norm  $\|\cdot\|_E$ .

Obviously,  $E$  is a separable Hilbert space. Moreover, using the Korn inequality, we can prove that  $E$  is identical to  $\mathbf{H}^1(\text{div}, V)$ . Indeed, we have the Korn inequality [18]

$$\int_V \mathbf{e} : \mathbf{e} dV \geq C\|\mathbf{u}\|_{\mathbf{H}^1(V)}^2$$

with a positive constant  $C$ . So the norm  $\|\cdot\|_E$  is equivalent to  $\|\cdot\|_{\mathbf{H}^1(\text{div}, V)}$  as these norms coincide up to positive factors. Thus, we can use properties of  $\mathbf{H}^1(\text{div}, V)$  for characterisation of weak solutions.

**Definition 4.2.** We call  $\mathbf{u}_0 \in \mathbf{H}^1(\text{div}, V)$  a weak solution of the boundary value problem under consideration if it satisfies boundary conditions (3.5) and the relation

$$(\mathbf{u}_0, \mathbf{v})_E = L(\mathbf{v}), \quad (4.3)$$

where

$$L(\mathbf{v}) = \int_V \mathbf{f} \cdot \mathbf{v} dV + \int_{S_1} \left( \mathbf{t} \cdot \mathbf{v} + c \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} \right) dS$$

for all  $\mathbf{v} \in \mathbf{H}^1(\text{div}, V)$  also satisfying (3.5).

With this definition, we formulate the main theorem.

**Theorem 4.3.** Let  $V \subset \mathbb{R}^3$  be a bounded connected domain with  $C^1$ -regular boundary  $S = S_0 \cup S_1$ ,  $\mathbf{f} \in \mathbf{L}^{6/5}(V)$ ,  $\mathbf{t} \in \mathbf{L}^{4/3}(S_1)$ , and  $c \in H^1(S_1)$ . Then, there exists a weak solution  $\mathbf{u}_0 \in \mathbf{H}^1(\text{div}, V)$ . It is unique. Furthermore,

$$J(\mathbf{u}_0) = \inf_{\mathbf{u} \in E} J(\mathbf{u}),$$

where

$$J(\mathbf{u}) = \frac{1}{2}(\mathbf{u}, \mathbf{u})_E - L(\mathbf{u}).$$

*Proof.* The key point of the proof is to show that  $L(\mathbf{v})$  is a linear bounded functional in  $E$ . Then, using the Riesz representation theorem, it follows that there is an unique element  $\boldsymbol{\ell} \in E$  such that

$$L(\mathbf{v}) = (\boldsymbol{\ell}, \mathbf{v})_E.$$

As a result, Eq. (4.3) takes the form

$$(\mathbf{u}_0, \mathbf{v})_E = (\boldsymbol{\ell}, \mathbf{v})_E, \quad \forall \mathbf{v} \in E, \quad (4.4)$$

so we get the unique solution  $\mathbf{u}_0 = \boldsymbol{\ell}$ .



As  $C^2(V)$  is dense in  $E$  in what follows, we can use functions in  $E \cap C^2(V)$ . The necessary properties of  $L(\mathbf{v})$  follow from the Sobolev imbedding theorems. First, let us consider

$$L_1(\mathbf{v}) = \int_V \mathbf{f} \cdot \mathbf{v} \, dV + \int_{S_1} \mathbf{t} \cdot \mathbf{v} \, dS.$$

As  $\mathbf{H}^1(V) \subset E$ , and the imbedding operators from  $H^1(V)$  to  $L^6(V)$  and  $L^4(S_1)$  are continuous, we find that

$$|L_1(\mathbf{v})| \leq \|\mathbf{f}\|_{\mathbf{L}^{6/5}(V)} \|\mathbf{v}\|_{\mathbf{L}^6(V)} + \|\mathbf{t}\|_{\mathbf{L}^{4/3}(S_1)} \|\mathbf{v}\|_{\mathbf{L}^4(S_1)} \leq C \|\mathbf{v}\|_E,$$

where Hölder’s inequalities were used, and  $C$  stands for a positive constant depending on  $\mathbf{f}$  and  $\mathbf{t}$ .

Now, let us consider

$$L_2(\mathbf{v}) = \int_{S_1} c \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} \, dS.$$

By definition of  $\mathbf{H}^1(\text{div}, V)$ , we have that if  $\mathbf{v} \in \mathbf{H}^1(\text{div}, V)$ , then  $\theta(\mathbf{v}) = \nabla \cdot \mathbf{v}$  belongs to  $H^1(V)$ . So  $\theta$  has a trace on  $S_1$  and  $\theta|_{S_1} \in L^4(S_1)$  and  $\theta|_{S_1} \in L^2(S_1)$ . Obviously,  $\mathbf{v}|_{S_1} \in \mathbf{L}^2(S_1)$ . As  $\theta = \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} + \nabla_s \cdot \mathbf{v}$ , we can transform  $L_2(\mathbf{v})$  using integration by parts as follows

$$L_2(\mathbf{v}) = \int_{S_1} c [\theta(\mathbf{v}) - \nabla_s \cdot \mathbf{v}] \, dS = \int_{S_1} [c\theta(\mathbf{v}) + (\nabla_s c) \cdot \mathbf{v} + 2Hc\mathbf{n} \cdot \mathbf{v}] \, dS.$$

Here, we take into account that  $\mathbf{v}|_{\partial S_1} = \mathbf{0}$  along  $\partial S_1$  as this contour is an interface between  $S_0$  and  $S_1$ , whereas  $\mathbf{v}|_{S_0} = \mathbf{0}$ . As a result, there is no a contour integral in  $L_2(\mathbf{v})$ .

As a result, with Hölder’s inequality, we get

$$|L_2(\mathbf{v})| \leq \|c\|_{L^2(S_1)} \|\theta(\mathbf{v})\|_{L^2(S_1)} + \|c\|_{H^1(S_1)} \|\mathbf{v}\|_{\mathbf{L}^2(S_1)} + 2 \left( \sup_{\mathbf{x} \in S_1} |H| \right) \|c\|_{L^2(S_1)} \|\mathbf{n} \cdot \mathbf{v}\|_{L^2(S_1)} \leq C \|\mathbf{v}\|_E.$$

Thus,  $L(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v})$  is a linear and bounded functional.

The minimisation property of  $\mathbf{u}_0$  follows from the fact that  $J(\mathbf{u})$  is a quadratic functional and Eq. (4.3) states that the first variation of  $J(\mathbf{u})$  vanishes at  $\mathbf{u} = \mathbf{u}_0$ , see, e.g. [33] for more details.  $\square$

From the proof, it follows that the weak solution is bounded

$$\|\mathbf{u}_0\| \leq C (\|\mathbf{f}\|_{\mathbf{L}^{6/5}(V)} + \|\mathbf{t}\|_{\mathbf{L}^{4/3}(S_1)} + \|c\|_{H^1(S_1)})$$

with a positive constant  $C$  which depends on  $V$  and  $S_1$ . As a solution is unique,  $\mathbf{u}_0 = \mathbf{0}$  if and only if  $\mathbf{f} = \mathbf{0}$ ,  $\mathbf{t} = \mathbf{0}$ , and  $c = 0$ . In particular, if  $\mathbf{f} = \mathbf{0}$ ,  $\mathbf{t} = \mathbf{0}$ , but  $c \neq 0$  we have  $\mathbf{u}_0 \neq \mathbf{0}$ .

**Remark 1.** If  $V \subset \mathbb{R}^2$ , then the theorem is also valid for  $\mathbf{f} \in \mathbf{L}^p(V)$  and  $\mathbf{t} \in \mathbf{L}^q(V)$  for  $1 < p, q < \infty$ .

**Remark 2.** In the proof, we use the same assumptions on  $\mathbf{f}$  and  $\mathbf{t}$  as for the linear elasticity, see [18, 33], and even more strong assumption was applied to  $c$ . In fact, using dual spaces, one can consider more weak assumptions, such as  $\mathbf{t} \in \mathbf{H}^{-1/2}(S_1)$ ,  $c \in H^{1/2}(S_1)$ , see [14, 46] for trace properties in  $H(\text{div}, V)$ . We leave this analysis to forthcoming papers.

**Remark 3.** Instead of  $C^1$ -regular boundary, one can consider less regular surface for which we have required Sobolev’s imbedding theorems [3]. We leave this again to forthcoming papers.



## 5. Conclusions

We proved the existence and uniqueness of a weak solution of a specific class of equilibrium problems within the framework of newly proposed strain gradient elasticity model specified with the adjective “*dilatational*”. The external interactions applied to the considered dilatational strain gradient continuum are dead loads constituted by (i) forces per unit area and (ii) purely normal double forces per unit area applied at the boundary of the considered continuum, which is occupying a regular domain, and (iii) volume forces applied at every material point belonging to the same continuum. This result has a conceptual impact as it proves that a consistent mathematical problem can be formulated in which applied external dead loads cannot be reduced to forces per unit area and couples per unit area, reduction that is assumed, instead, in [76]. This means that the postulation scheme described there does not include all conceivable and logically consistent continuum models and that there are continuum models, based on variational principles, which cannot be obtained by postulating the balance of force and couples for every subbody of considered continuum. The statement of the theorems presented in Sect. 4 gives a solid mathematical ground to the discussion presented by Sedov [64–66] and should be sufficient to conclude the debate between the variational and “balancist” schools in continuum mechanics by establishing the superiority of variational principle postulation schemes. One has to remark explicitly here that the concept of stress, as developed by “balancists”, is specific for first gradient continua and cannot be easily generalised to higher gradient continua, see, e.g. [23]. In fact, assuming only balance of forces and couples for any sub-body, we necessarily come to the Cosserat continuum [34, 56] or to the Cauchy continuum if we neglect couples as independent on forces. For media with microstructure, one has to introduce additional balance equations, see [17, 35]. Instead, for hyperelastic, the concept of deformation energy can be more easily generalised and leads easily, via a process based on stationarity conditions and integration by parts, to the strong formulations of mathematically consistent equilibrium problems which naturally include well-posed natural boundary conditions. Moreover, the concept of deformation energy is based on the fundamental concept of deformation, which is pure kinematical and is strongly linked to experimental evidence. Instead, the concept of stress has a complex mathematical nature: in fact, it can be defined as the linear and continuous functional mapping displacement fields and their gradients into the expended work, see Eq. (3.3). It is rather difficult to measure stress without having developed a dynamical theory, and these measurements are always based on the indirect determination based on the direct measure of deformations and the use of ad hoc postulated constitutive equations. It seems to us rather difficult to attribute a “more intuitive” physical nature to stress than to deformation.

In the specific considered modelling instance, we consider a very specific case of the energy space adapted to the postulated deformation energy. We prove that weak solutions to the equilibrium problem belong to  $\mathbf{H}^1(\text{div}, V)$ . This is an intermediate functional space between  $\mathbf{H}^1(V)$  and  $\mathbf{H}^2(V)$  and belongs to the wider class of anisotropic Sobolev spaces introduced by Nikol’skii [55], see also [10–12, 57, 75] for a study about traces of functions in these spaces.

This characterisation of the solutions is essential if one wants to solve with numerical methods the problem of the deformation of dilatational strain gradient continua under the specified class of externally applied dead loads. In facts, in order to get a reasonable accurate approximation and a convergent integration scheme, one has to suppose that the applied loads have a regularity compatible with formula (3.7) and that the used mixed finite elements method exploits a discretisation based on a set of test functions dense in  $\mathbf{H}^1(\text{div}, V)$ .

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## Appendix: First variation

Let us discuss the derivation of the first variation given by (3.4). In what follows, for simplicity, we consider  $C^4(V)$  functions. Integrating by parts in (3.3), we get

$$\begin{aligned}\delta F(\mathbf{u}; \mathbf{v}) &= \int_V [\lambda\theta(\mathbf{u})\theta(\mathbf{v}) + 2\mu\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \alpha\nabla\theta(\mathbf{u}) \cdot \nabla\theta(\mathbf{v})] dV \\ &= - \int_V [\nabla(\lambda\theta(\mathbf{u})) \cdot \mathbf{v} + 2\nabla \cdot (\mu\mathbf{e}(\mathbf{u})) \cdot \mathbf{v} + \alpha\nabla \cdot \nabla\theta(\mathbf{u})\theta(\mathbf{v})] dV \\ &\quad + \int_S [\lambda\theta(\mathbf{u})\mathbf{n} \cdot \mathbf{v} + 2\mu\mathbf{n} \cdot \mathbf{e}(\mathbf{u}) \cdot \mathbf{v} + \alpha\mathbf{n} \cdot \nabla\theta(\mathbf{u})\theta(\mathbf{v})] dS,\end{aligned}\quad (5.1)$$

where  $S = \partial V$  and  $\mathbf{n}$  is a unit vector of outward normal to  $S$ . Here, we also used that  $\mathbf{e}$  is a symmetric tensor, so  $\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) = \mathbf{e}(\mathbf{u}) : \nabla\mathbf{v}$ . Introducing stress tensor  $\boldsymbol{\sigma}$  and double force vector  $\mathbf{m}$  by

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda\theta(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u}), \quad \mathbf{m}(\mathbf{u}) = \alpha\nabla\theta(\mathbf{u}),$$

so the strain energy density takes the form

$$W(\mathbf{u}) = \frac{1}{2}\boldsymbol{\sigma}(\mathbf{u}) : \mathbf{e}(\mathbf{u}) + \frac{1}{2}\mathbf{m}(\mathbf{u}) \cdot \nabla\theta(\mathbf{u}),$$

we re-write (5.1) in a compact form

$$\begin{aligned}\delta F(\mathbf{u}; \mathbf{v}) &= - \int_V [(\nabla \cdot \boldsymbol{\sigma}(\mathbf{u})) \cdot \mathbf{v} + (\nabla \cdot \mathbf{m}(\mathbf{u}))\theta(\mathbf{v})] dV \\ &\quad + \int_S [\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{v} + \mathbf{n} \cdot \mathbf{m}(\mathbf{u})\theta(\mathbf{v})] dS.\end{aligned}\quad (5.2)$$

Integrating by part again, we get

$$\begin{aligned}\int_V [\nabla \cdot \mathbf{m}(\mathbf{u})]\theta(\mathbf{v}) dV &\equiv \int_V [\nabla \cdot \mathbf{m}(\mathbf{u})] \nabla \cdot \mathbf{v} dV \\ &= - \int_V \nabla [\nabla \cdot \mathbf{m}(\mathbf{u})] \cdot \mathbf{v} dV + \int_S [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} \cdot \mathbf{v} dS.\end{aligned}\quad (5.3)$$

Substituting (5.3) into (5.2), we transform  $\delta F(\mathbf{u}; \mathbf{v})$  as follows

$$\begin{aligned} \delta F(\mathbf{u}; \mathbf{v}) &= \int_V [ -(\nabla \cdot \boldsymbol{\sigma}(\mathbf{u})) \cdot \mathbf{v} + \nabla(\nabla \cdot \mathbf{m}(\mathbf{u})) \cdot \mathbf{v} ] dV \\ &\quad + \int_S [\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{v} + \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \theta(\mathbf{v})] dS \\ &\quad - \int_S [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} \cdot \mathbf{v} dS. \end{aligned} \quad (5.4)$$

Using the identity  $\nabla(\nabla \cdot \mathbf{m}) = \nabla \cdot (\nabla \mathbf{m})^T$  and introducing the total stress tensor  $\mathbf{T}$  as

$$\mathbf{T}(\mathbf{u}) = \boldsymbol{\sigma}(\mathbf{u}) - (\nabla \mathbf{m}(\mathbf{u}))^T,$$

where  $T$  stands for a transpose tensor, we re-write (5.4) in more compact form

$$\begin{aligned} \delta F(\mathbf{u}; \mathbf{v}) &= - \int_V [\nabla \cdot \mathbf{T}(\mathbf{u})] \cdot \mathbf{v} dV \\ &\quad + \int_S [\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n}] \cdot \mathbf{v} dS + \int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \theta(\mathbf{v}) dS. \end{aligned} \quad (5.5)$$

Let us consider the last surface integral in (5.5). We recall that  $\theta(\mathbf{v}) = \nabla \cdot \mathbf{v}$ . We use the following representation for  $\nabla \cdot \mathbf{v}|_S$

$$\nabla \cdot \mathbf{v}|_S = \left( \nabla_s + \mathbf{n} \frac{\partial}{\partial n} \right) \cdot \mathbf{v},$$

where  $\nabla_s = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla$  and  $\frac{\partial}{\partial n}$  is the normal derivative. So we get

$$\int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \theta(\mathbf{v}) dS = \int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} dS + \int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \nabla_s \cdot \mathbf{v} dS. \quad (5.6)$$

We consider  $S$  as a smooth enough surface with a contour  $C = \partial S$ . Using the surface divergence theorem as in [31], we can integrate by parts in the last integral in (5.6) as follows

$$\begin{aligned} \int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \nabla_s \cdot \mathbf{v} dS &= \int_S [\nabla_s \cdot [(\mathbf{n} \cdot \mathbf{m}) \mathbf{v}] - \nabla_s(\mathbf{n} \cdot \mathbf{m}) \cdot \mathbf{v}] dS \\ &= - \int_S [2H(\mathbf{n} \cdot \mathbf{v})(\mathbf{n} \cdot \mathbf{m}) + \nabla_s(\mathbf{n} \cdot \mathbf{m}) \cdot \mathbf{v}] dS + \int_C (\boldsymbol{\nu} \cdot \mathbf{v})(\mathbf{n} \cdot \mathbf{m}) ds, \end{aligned} \quad (5.7)$$

where  $H$  is the mean curvature of  $S$ ,  $\boldsymbol{\nu}$  is the unit vector of the outward normal to  $C$  such that  $\boldsymbol{\nu} \cdot \mathbf{n} = 0$ .

Summarising, we get

$$\begin{aligned} \delta F(\mathbf{u}; \mathbf{v}) = & - \int_V [\nabla \cdot \mathbf{T}(\mathbf{u})] \cdot \mathbf{v} \, dV \\ & + \int_S [\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} - 2H(\mathbf{n} \cdot \mathbf{m}(\mathbf{u}))\mathbf{n} + \nabla_s(\mathbf{n} \cdot \mathbf{m}(\mathbf{u}))] \cdot \mathbf{v} \, dS \\ & + \int_S \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) \frac{\partial \mathbf{v}}{\partial n} \cdot \mathbf{n} \, dS \\ & + \int_C (\mathbf{n} \cdot \mathbf{m}(\mathbf{u}))(\boldsymbol{\nu} \cdot \mathbf{v}) \, ds. \end{aligned} \quad (5.8)$$

Note that for bounded domain  $V$ ,  $S = \partial V$ , so  $C = \partial S = \partial \partial V$  is empty, but we keep it here for the case of mixed boundary conditions when a part of  $S$  is clamped. Then,  $C$  is an interface between clamped part of the boundary and the free one. As on a clamped surface, we have  $\mathbf{v} = \mathbf{0}$ , the integral over  $C$  also vanishes.

Using standard techniques of calculus of variations from variational equation  $\delta F(\mathbf{u}; \mathbf{v}) = 0$  for all admissible  $\mathbf{v}$  it follows the homogeneous boundary-value problem for the determination of  $\mathbf{u}(\mathbf{x}) \in C^4(V)$

$$\nabla \cdot \mathbf{T}(\mathbf{u}) = \mathbf{0}, \quad \mathbf{x} \in V, \quad (5.9)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} - 2H(\mathbf{n} \cdot \mathbf{m}(\mathbf{u}))\mathbf{n} + \nabla_s(\mathbf{n} \cdot \mathbf{m}(\mathbf{u})) = \mathbf{0}, \quad \mathbf{x} \in S, \quad (5.10)$$

$$\mathbf{n} \cdot \mathbf{m}(\mathbf{u}) = 0, \quad \mathbf{x} \in S. \quad (5.11)$$

Obviously, using (5.11), we can simplify (5.10). Indeed, substituting (5.11) into (5.10), we get

$$\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} = \mathbf{0}, \quad \mathbf{x} \in S. \quad (5.12)$$

Boundary-value problem (5.9)–(5.11) can be easily extended to the case of external loadings. Indeed, in this case, we have the equilibrium equation and the natural boundary conditions in the form

$$\nabla \cdot \mathbf{T}(\mathbf{u}) + \mathbf{f} = \mathbf{0}, \quad \mathbf{x} \in V, \quad (5.13)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} - 2H(\mathbf{n} \cdot \mathbf{m}(\mathbf{u}))\mathbf{n} + \nabla_s(\mathbf{n} \cdot \mathbf{m}(\mathbf{u})) = \mathbf{t}, \quad \mathbf{x} \in S, \quad (5.14)$$

$$\mathbf{n} \cdot \mathbf{m}(\mathbf{u}) = c, \quad \mathbf{x} \in S, \quad (5.15)$$

where  $\mathbf{f}$  is a vector of volume forces,  $\mathbf{t}$  is a traction, and  $c$  is a scalar surface double force.

Let us note that in this case can exclude  $\mathbf{n} \cdot \mathbf{m}(\mathbf{u})$  from (5.14). We have

$$\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} = \tilde{\mathbf{t}}, \quad \tilde{\mathbf{t}} = 2Hc\mathbf{n} + \nabla_s(c) + \mathbf{t}, \quad \mathbf{x} \in S. \quad (5.16)$$

At first look, we can replace boundary conditions (5.14) and (5.15) by more simple equations

$$\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}) - [\nabla \cdot \mathbf{m}(\mathbf{u})] \mathbf{n} = \mathbf{t}, \quad \mathbf{n} \cdot \mathbf{m}(\mathbf{u}) = c. \quad (5.17)$$

with two given functions  $\mathbf{t}$  and  $c$ . But here, we cannot treat  $\mathbf{t}$  and  $c$  independently. For example, we cannot apply  $\mathbf{t} = \mathbf{0}$  with  $c \neq 0$ . For example, for a curved surface ( $H \neq 0$ ), a constant double force  $c$  produces normal pressure  $\tilde{\mathbf{t}} = 2Hc\mathbf{n}$ . So one should aware of application of such “simplified” boundary conditions in the case of strain-gradient continua. This situation is similar to the case of Kirchhoff plates when at the boundary a combination of transverse forces and moments, see, e.g. [31].

As kinematical counterparts of (5.10) and (5.11), we consider the following essential boundary conditions

$$\mathbf{u}|_S = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial n} \cdot \mathbf{n} \Big|_S = 0.$$

Considering mixed boundary conditions on the base of (5.8) and (5.13)–(5.15), we come to the formulation of the virtual work principle in the form (3.7). In the case of  $C^4$ -functions, (3.7) is equivalent to (5.13)–(5.15).

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Victor A. Eremeyev  
 Faculty of Civil and Environmental Engineering  
 Gdańsk University of Technology  
 ul. Gabriela Narutowicza 11/12  
 80-233 Gdańsk  
 Poland  
 e-mail: eremeyev.victor@gmail.com



Victor A. Eremeyev  
Department of Civil and Environmental Engineering and Architecture (DICAAR)  
University of Cagliari  
Via Marengo, 2  
09123 Cagliari  
Italy

Sergey A. Lurie, Yury O. Solyaev and Francesco dell'Isola  
Institute of Applied Mechanics of Russian Academy of Sciences  
Moscow  
Russia  
e-mail: salurie@mail.ru

Yury O. Solyaev  
e-mail: yos@iam.ras.ru

Francesco dell'Isola  
e-mail: francesco.dellisola.me@gmail.com

Francesco dell'Isola  
DICEAA and International Research Center on Mathematics and Mechanics of Complex System (M&MOCS)  
Università dell'Aquila  
L'Aquila  
Italy

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