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BLOCK GRAPHS WITH LARGE PAIRED DOMINATION MULTISUBDIVISION NUMBER

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Abstract

The paired domination multisubdivision number of a nonempty graph G, denoted by $\mathrm{msd}_{\mathrm{pr}}(G)$, is the smallest positive integer k such that there exists an edge which must be subdivided k times to increase the paired domination number of G. It is known that $\mathrm{msd}_{\mathrm{pr}}(G) \leq 4$ for all graphs G. We characterize block graphs with $\mathrm{msd}_{\mathrm{pr}}(G) = 4$.

Keywords: paired domination, domination subdivision number, domination multisubdivision number, block graph.

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1. Introduction

The study of changes that occur in domination-related parameters of a graph when its edges are subdivided[1] was initiated in [11]. If π is a domination-type parameter of G, the smallest number of edges that must be subdivided, where each edge of G can be subdivided at most once, in order to increase π is called

[1] See Section 2 for definitions of terms used in this section.

the π-*subdivision number*, denoted by $\mathrm{sd}_\pi(G)$. Subdivision numbers have been studied for the domination number [6, 11], as well as for connected [4], double [1], Roman [10], total [7, 9] and paired domination numbers [5].

Instead of subdividing multiple edges once each, one may wish to subdivide a single edge multiple times. The smallest number of times that a single edge of G must be subdivided to increase π is called the π-*multisubdivision number*, denoted by $\mathrm{msd}_\pi(G)$. Domination and paired domination multisubdivision numbers were studied in [3] and [2], respectively. In particular, it was shown in [2] that the paired domination multisubdivision number $\mathrm{msd}_{\mathrm{pr}}(G)$ of any graph G is at most four. For brevity we refer to a graph G with $\mathrm{msd}_{\mathrm{pr}}(G) = 4$ as an msd-4 graph. Msd-4 trees were characterized in [2].

We discuss methods of combining msd-4 graphs to yield new msd-4 graphs and use our results, combined with results from [2], to characterize msd-4 block graphs. Definitions and previous results are given in Section 2. We state the characterization of msd-4 block graphs in Section 3, but defer its proof to Section 6 to allow us to prove a number of results used in the proof; results that apply to general msd-4 graphs are given in Section 4, while results specific to block graphs can be found in Section 5.

2. Definitions and Previous Results

We refer the reader to [8] for domination parameters not defined here. A set S of vertices of a graph $G = (V, E)$ without isolated vertices is a *paired dominating set* of G if every vertex of G is adjacent to a vertex in S, and the subgraph $G[S]$ of G induced by S has a perfect matching. If $u, v \in S$ and there exists a perfect matching M of $G[S]$ such that $uv \in M$, we say that u and v are *paired* in S. The smallest cardinality of a paired dominating set of G is the *paired domination number* of G, denoted by $\gamma_{\mathrm{pr}}(G)$. If S is a paired dominating set of G such that $|S| = \gamma_{\mathrm{pr}}(G)$, we call S a $\gamma_{\mathrm{pr}}(G)$-*set*, or simply a γ_{pr}-*set* if the graph is clear from the context. If u is a vertex of G such that $G - u$ has no isolated vertices and $\gamma_{\mathrm{pr}}(G - u) < \gamma_{\mathrm{pr}}(G)$ (in which case $\gamma_{\mathrm{pr}}(G - u) = \gamma_{\mathrm{pr}}(G) - 2$), we say that u is a $\gamma_{\mathrm{pr}}(G)$-*critical vertex*, or simply a γ_{pr}-*critical vertex*, and define $\mathrm{Cr}(G) = \{u \in V(G) : u \text{ is a } \gamma_{\mathrm{pr}}\text{-critical vertex}\}$.

A *neighbour* of a vertex $u \in V(G)$ is a vertex adjacent to u. The (*open*) *neighbourhood* $N(u)$ of a vertex u is the set of all vertices adjacent to u, and its *closed neighbourhood* is $N[u] = N(u) \cup \{u\}$. For a set $S \subseteq V(G)$, the (*open*) *neighbourhood* of S is $N(S) = \bigcup_{u \in S} N(u)$, and its *closed neighbourhood is* $N[S] = N(S) \cup S$. For a vertex $u \in S$, the *private neighbourhood of* u *with respect to* S is the set $\mathrm{PN}(u, S) = N[u] \setminus N[S \setminus \{u\}]$. It is possible that $u \in \mathrm{PN}(u, S)$, but if S is a paired dominating set, then u is adjacent to the vertex it is paired with, so $u \notin \mathrm{PN}(u, S)$ in this case.

An edge uv of a graph G is *subdivided* if it is replaced by a path (u, x, v), where x is a new vertex, and *multisubdivided* if it is replaced by a path $(u, x_1, \ldots, x_k, v)$, $k \geq 2$, where $x_1, \ldots, x_k$ are new vertices; we also say that uv *is subdivided* k *times.* Let $G_{uv,k}$ denote the graph obtained from G by subdividing the edge uv k times. The *paired domination multisubdivision number* $\mathrm{msd}_{\mathrm{pr}}(G)$ of a graph G without isolated vertices is the smallest positive integer k such that there exists an edge uv which must be subdivided k times for $\gamma_{\mathrm{pr}}(G_{uv,k})$ to exceed $\gamma_{\mathrm{pr}}(G)$. As mentioned above, $\mathrm{msd}_{\mathrm{pr}}(G) \leq 4$ for all graphs. The three graphs in Figure 1 are all msd-4 graphs; the red vertices form γ_{pr}-sets.

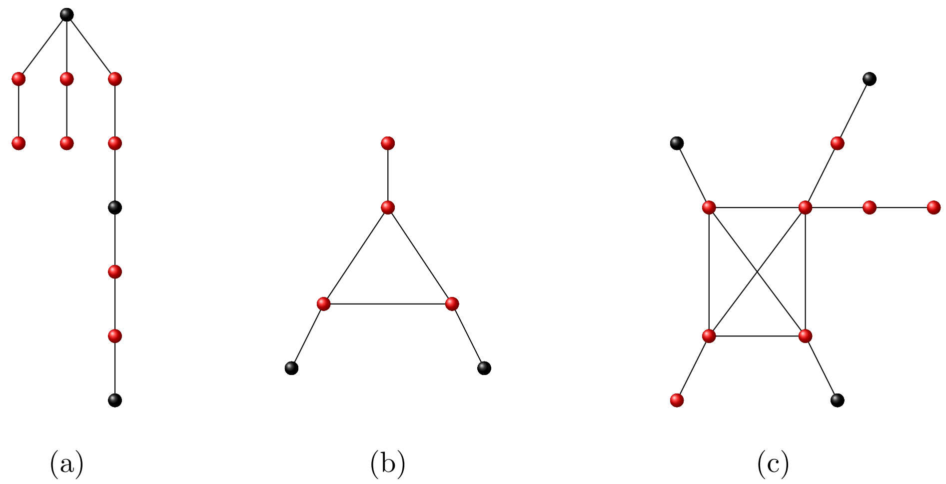

Figure 1. (a) The spider $S(2, 2, 6)$ (b) the corona $K_3 \circ K_1$ (c) a flared corona $K_4 \circ^{*2} K_1$.

A *leaf* of a graph is a vertex of degree one, and its neighbour is called a *stem.* The following properties of msd-4 graphs were proved in [2].

Theorem 1 [2]**.** *Let G be an* msd-4 *graph. Then*

(i) *each edge of G belongs to a matching of a minimum paired dominating set of G;*

(ii) *any leaf of G is a γ_{pr}-critical vertex;*

(iii) *each stem is adjacent to exactly one leaf.*

The complete bipartite graph $K_{1,k}$, $k \geq 2$, is called a *star.* Let $K_{1,k}$ have partite sets $\{u\}$ and $\{v_1, \ldots, v_k\}$. The *spider* $S(\ell_1, \ldots, \ell_k)$, $\ell_i \geq 1$, $k \geq 2$, is a tree obtained from $K_{1,k}$ by subdividing the edge uv_i $\ell_i - 1$ times, $i = 1, \ldots, k$. Note that $S(2, 2) \cong P_5$. See Figure 1(a) for $S(2, 2, 6)$. The characterization of msd-4 trees in [2] immediately gives the following result.

Proposition 2 [2]. *The spider $T = S(2,\dots,2)$ satisfies* $\mathrm{msd}_{\mathrm{pr}}(T) = 4$, *and* $\mathrm{Cr}(T)$ *consists of the leaves of* T.

The *corona* $G \circ K_1$ of a graph G is the graph obtained by joining each vertex of G to a new leaf; $K_3 \circ K_1$ is illustrated in Figure 1(b). A *flared corona* $G \circ^{*t} K_1$ of G is a graph obtained by joining each vertex of G, except one vertex w, to a new leaf, while w is joined to a single vertex of each of $t \geq 1$ copies of K_2. The flared corona $K_4 \circ^{*2} K_1$ is depicted in Figure 1(c). The following facts can be verified easily and are stated without proof.

Remark 3.
(i) A corona $K_n \circ K_1,\ n \geq 2$, is an msd-4 graph if and only if n is odd.
(ii) A flared corona $K_n \circ^{*t} K_1,\ n \geq 2$, is an msd-4 graph if and only if n is even.
(iii) A vertex of $K_{2n+1} \circ K_1$ or $K_{2n} \circ^{*t} K_1$ is γ_{pr}-critical if and only if it is a leaf (see Theorem 1).

A *block* of a graph is a maximal connected subgraph with no cut-vertex, and a *block graph* is a graph, each of whose blocks is a complete graph. Thus, trees are block graphs since each block of a nontrivial tree is a K_2. Evidently, coronas and flared coronas are also block graphs. To characterize msd-4 block graphs, we use spiders $S(2,\dots,2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^{*t} K_1$, combining them by identifying vertices and edges in a prescribed way.

We begin by describing two operations, collectively known as $\oplus$-*operations*, for joining disjoint graphs; since the operations can be performed on any graphs, we state them in their most general form. (The operations are well known but we need to define our notation.)

$G_1 \oplus^{u_1u_2} G_2$: Let G_1 and G_2 be vertex disjoint graphs and $u_i \in V(G_i)$ for $i \in \{1,2\}$. We denote the graph obtained from G_1 and G_2 by identifying u_1 and u_2 into one vertex $u = u_1 = u_2$ by $G_1 \oplus_u^{u_1u_2} G_2$ (or by $G_1 \oplus^{u_1u_2} G_2$ if the label u is unimportant).

$G_1 \oplus^{e_1e_2} G_2$: Let G_1 and G_2 be vertex disjoint graphs and $e_i = u_iv_i \in E(G_i)$. We denote the graph obtained from G_1 and G_2 by identifying u_1 and u_2 into one vertex $u = u_1 = u_2$, v_1 and v_2 into one vertex $v = v_1 = v_2$, and e_1 and e_2 into one edge $e = uv$ by $G_1 \oplus_e^{e_1e_2} G_2$ (or by $G_1 \oplus^{e_1e_2} G_2$ if the label e is unimportant).

The graph $G_1 \oplus_e^{e_1e_2} G_2$, where $G_1 = S(2,2,6)$, $G_2 = K_3 \circ K_1$, and $e_i = u_iv_i$ for $i = 1,2$, is illustrated in Figure 2. Note that u_i is $\gamma_{\mathrm{pr}}(G_i)$-critical for $i = 1,2$, and $u_1 = u_2$ is γ_{pr}-critical in $G_1 \oplus_e^{e_1e_2} G_2$. The spider $S(2,2,6)$, in turn, is obtained as $H_1 \oplus^{u_1u_2} H_2$, where $H_1 = S(2,2,2)$, $H_2 = P_5 = S(2,2)$, and u_i is a leaf of H_i, $i = 1,2$.

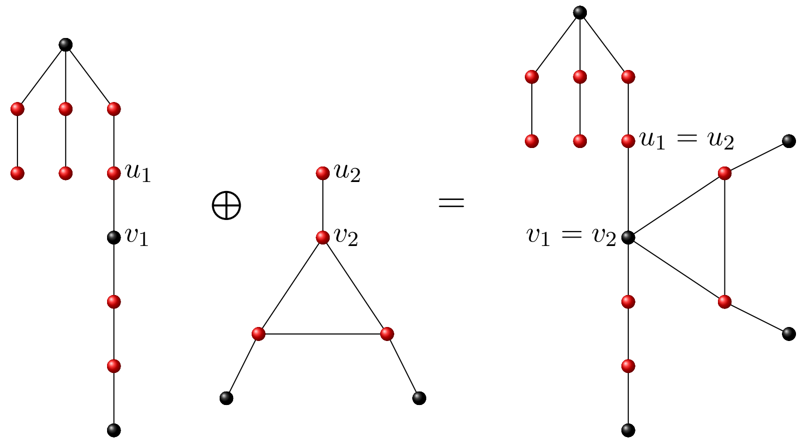

Figure 2. The graph $S(2,2,6) \oplus^{u_1v_1\ u_2v_2} K_3 \circ K_1$.

3. Characterization of msd-4 Block Graphs

We now state our main result — the characterization of msd-4 block graphs. The proof is deferred to Section 6.

Let $\mathcal{U}$ be the collection of all spiders $S(2,\ldots,2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^{*t} K_1$, $n \geq 1$. Define $\mathcal{B}$ to be the family of all block graphs G that can be obtained as a graph G_j, $j \geq 1$, from a sequence $G_1, \ldots, G_j$ of graphs, where $H_1 = G_1 \in \mathcal{U}$, and, if $j > 1$, G_{i+1} can be constructed recursively from G_i by

- adding a graph $H_{i+1} \in \mathcal{U}$,
- choosing vertices $u_1 \in \mathrm{Cr}(G_i)$, $u_2 \in \mathrm{Cr}(H_{i+1})$, and if necessary, $v_1 \in N(u_1)$, $v_2 \in N(u_2)$,
- performing the operation $G_i \oplus^{u_1u_2} H_{i+1}$ or $G_i \oplus^{u_1v_1\ u_2v_2} H_{i+1}$.

Theorem 4. *Let G be a connected block graph. Then G is an* msd-4 *graph if and only if $G \in \mathcal{B}$. Moreover, if G is an* msd-4 *graph constructed from the graphs $H_1, \ldots, H_j \in \mathcal{U}$, then* $\mathrm{Cr}(G) = \bigcup_{i=1}^{j} \mathrm{Cr}(H_i)$.

The second statement of Theorem 4 implies that any γ_{pr}-critical vertex v of an msd-4 block graph remains γ_{pr}-critical after the $\oplus$-operations have been performed any number of times, whether v was identified with another vertex or not. The following corollary of Theorem 4 was proved in [2].

Corollary 5. *A tree T is an* msd-4 *graph if and only if $T \in \mathcal{B}$, that is, if and only if T can be constructed as described, using only spiders $S(2,\ldots,2)$.*

4. General Results

In this section we discuss ways of constructing larger msd-4 graphs from smaller ones. We first prove a useful lemma.

Lemma 6. *Let G be a graph with* $\mathrm{msd}_{\mathrm{pr}}(G) = 4$. *For any edge uv of G, subdivide uv by replacing it with the path (u, x_1, x_2, x_3, v). If D is any $\gamma_{\mathrm{pr}}(G_{uv,3})$-set, then* $D \cap \{u, x_1, x_2, x_3, v\} =$

(i) $\{x_1, x_2\}$ *or* $\{x_2, x_3\}$, *or*

(ii) $\{u, x_1, v\}$ *or* $\{u, x_3, v\}$.

If the first part of (i) *holds, then u is γ_{pr}-critical, and if the second part of* (i) *holds, then v is γ_{pr}-critical.*

Proof. Let $X = \{x_1, x_2, x_3\}$. To dominate x_2, $X \cap D \neq \emptyset$. We consider three cases.

Case 1. $X \cap D = X$. Without loss of generality assume that x_1 is paired with $u \in D$, and x_2 and x_3 are paired. Then $v \notin D$, otherwise $D \backslash \{x_2, x_3\}$ is also a paired dominating set of $G_{uv,3}$, contradicting the minimality of D. But now $D' = (D \backslash X) \cup \{v\}$ is a paired dominating set of G, which is impossible because $\mathrm{msd}_{\mathrm{pr}}(G) = 4$.

Case 2. $|X \cap D| = 2$. If $X \cap D = \{x_1, x_3\}$, then $\{u, v\} \subseteq D$ with u paired with x_1, and v with x_3. However, then $D \backslash \{x_1, x_3\}$ is a paired dominating set of G, contradicting $\mathrm{msd}_{\mathrm{pr}}(G) = 4$. Suppose $X \cap D = \{x_1, x_2\}$. Then x_1 and x_2 are paired in D. If $\{u, v\} \cap D \neq \emptyset$, then $D \backslash \{x_1, x_2\}$ is a paired dominating set of G, which is a contradiction. Hence $D \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\}$. Now $D \backslash \{x_1, x_2\}$ is a paired dominating set of $G - u$, so $\gamma_{\mathrm{pr}}(G - u) < \gamma_{\mathrm{pr}}(G_{uv,3}) = \gamma_{\mathrm{pr}}(G)$. We conclude that u is γ_{pr}-critical. Arguing similarly if $X \cap D = \{x_2, x_3\}$, we conclude that (i) and the last part of the statement of the lemma hold.

Case 3. $|X \cap D| = 1$. Then $x_2 \notin D$. If $x_1 \in D$, then x_1 is paired with $u \in D$, while $v \in D$ to dominate x_3. Consequently, $D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_1, v\}$. Similarly, if $x_3 \in D$, then $D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_3, v\}$. ■

Our first result regarding the construction of msd-4 graphs from smaller graphs shows that subdividing any edge of an msd-4 graph four times produces another msd-4 graph. Repeatedly subdividing edges of an msd-4 graph thus yields, for example, msd-4 graphs of arbitrary large girth. In fact, we prove a stronger result: subdividing any edge of any graph G without isolated vertices four times produces a graph that has the same multisubdivision number as G.

Proposition 7. *For any graph G and any edge e of G,* $\mathrm{msd}_{\mathrm{pr}}(G_{e,4}) = \mathrm{msd}_{\mathrm{pr}}(G)$.

Proof. Say $\mathrm{msd}_{\mathrm{pr}}(G) = t \leq 4$ and $e = uv$ has been subdivided by replacing it with the path $(u, x_1, x_2, x_3, x_4, v)$. Then $\gamma_{\mathrm{pr}}(G_{e,4}) = \gamma_{\mathrm{pr}}(G) + 2$ and there exists an edge e' of G such that $\gamma_{\mathrm{pr}}(G_{e',t}) = \gamma_{\mathrm{pr}}(G) + 2$. If $e \neq e'$, then subdividing $e \in E(G_{e',t})$ four times yields the graph $(G_{e',t})_{e,4}$. Since $\mathrm{msd}_{\mathrm{pr}}(G_{e',t}) \leq 4$, $\gamma_{\mathrm{pr}}((G_{e',t})_{e,4}) = \gamma_{\mathrm{pr}}(G_{e',t}) + 2 = \gamma_{\mathrm{pr}}(G) + 4$. But $(G_{e',t})_{e,4} = (G_{e,4})_{e',t}$, hence $\gamma_{\mathrm{pr}}((G_{e,4})_{e',t}) = \gamma_{\mathrm{pr}}(G) + 4 = \gamma_{\mathrm{pr}}(G_{e,4}) + 2$. If $e = e'$, say uv has been subdivided, in G, by replacing it with $(u, x_1, \ldots, x_t, v)$. Subdividing (without loss of generality) the edge $x_t v$ four times by replacing it with $(x_t, x_{t+1}, \ldots, x_{t+4}, v)$, we obtain the graph $(G_{e,t})_{x_t v,4} = (G_{e,4})_{x_4 v,t}$ with $\gamma_{\mathrm{pr}}((G_{e,4})_{x_4 v,t}) = \gamma_{\mathrm{pr}}(G_{e,4}) + 2$. It follows that $\mathrm{msd}_{\mathrm{pr}}(G_{e,4}) \leq t$.

We show that $\mathrm{msd}_{\mathrm{pr}}(G_{e,4}) \geq t$. If $t = 1$, this is obvious, hence assume $t \geq 2$. Consider any $e' \in E(G)$. Suppose first that $e' \neq e$. Since $\mathrm{msd}_{\mathrm{pr}}(G) = t$, $\gamma_{\mathrm{pr}}(G_{e',t-1}) = \gamma_{\mathrm{pr}}(G)$. If D' is any $\gamma_{\mathrm{pr}}(G_{e',t-1})$-set, then $D = D' \cup \{x_1, x_4\}$ (if u and v are paired in D') or $D = D' \cup \{x_2, x_3\}$ (otherwise) is a paired dominating set of $(G_{e,4})_{e',t-1}$ of cardinality $|D| = \gamma_{\mathrm{pr}}(G_{e',t-1}) + 2 = \gamma_{\mathrm{pr}}(G) + 2 = \gamma_{\mathrm{pr}}(G_{e,4})$.

Assume $e' = e$. Without loss of generality subdivide the edge $x_4 v$ of $G_{e,4}$ $t-1$ times by replacing it with the path $(x_4, \ldots, x_{3+t}, v)$ and denote the resulting graph $(G_{e,4})_{x_4 v,t-1}$ by $G_{e,3+t}$ for simplicity. Also consider the graph $G_{e,t-1}$ obtained from G by subdividing $e = uv$ by replacing it with $(u, x_1, \ldots, x_{t-1}, v)$. Since $\mathrm{msd}_{\mathrm{pr}}(G) = t$, $\gamma_{\mathrm{pr}}(G_{e,t-1}) = \gamma_{\mathrm{pr}}(G)$. Let S' be any $\gamma_{\mathrm{pr}}(G_{e,t-1})$-set. We consider three cases. In each case we construct a paired dominating set S of $G_{e,3+t}$ such that $|S| = |S'| + 2 = \gamma_{\mathrm{pr}}(G_{e,4})$; this shows that $\mathrm{msd}_{\mathrm{pr}}(G_{e,4}) \geq t$.

Case 1. $t = 2$. If $x_1 \notin S'$, then without loss of generality $u \in S'$ to dominate x_1, and $S' \backslash \{u\}$ dominates v. Let $S = S' \cup \{x_3, x_4\}$. If $x_1 \in S'$, then again without loss of generality x_1 is paired with u. Let $S = S' \cup \{x_4, x_5\}$.

Case 2. $t = 3$. If $S' \cap \{x_1, x_2\} = \emptyset$, then u dominates x_1 while v dominates x_2; let $S = S' \cup \{x_3, x_4\}$ (so v dominates x_6). If (without loss of generality) $S' \cap \{x_1, x_2\} = \{x_1\}$, then u and x_1 are paired, and $S' \backslash \{u, x_1\}$ dominates v. Let $S = S' \cup \{x_4, x_5\}$. If $\{x_1, x_2\} \subseteq S'$, then x_1 and x_2 are paired (otherwise $S' \backslash \{x_1, x_2\}$ is a paired dominating set of G, which is not the case). Let $S = S' \cup \{x_5, x_6\}$.

Case 3. $t = 4$. By Lemma 6, without loss of generality $S' \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\}$ or $\{u, x_1, v\}$. In the former case, let $S = S' \cup \{x_5, x_6\}$, and in the latter case, let $S = S' \cup \{x_4, x_5\}$.

In all cases, S is a paired dominating set of $G_{e,3+t}$ of cardinality $\gamma_{\mathrm{pr}}(G) + 2 = \gamma_{\mathrm{pr}}(G_{e,4})$, and $\mathrm{msd}_{\mathrm{pr}}(G_{e,4}) \geq t$. It follows that $\mathrm{msd}_{\mathrm{pr}}(G_{e,4}) = t$, as required. ■

We next prove results pertaining to the $\oplus$-operations defined above that hold for general msd-4 graphs, not only block graphs. We show that the $\oplus$-operations can be used to construct new connected msd-4 graphs from smaller ones.

Our next result shows that performing the operation $G_1 \oplus_u^{u_1u_2} G_2$ on msd-4 graphs G_1 and G_2 with $\gamma_{\rm pr}$-critical vertices u_1 and u_2, respectively, results in an msd-4 graph in which each $\gamma_{\rm pr}(G_i)$-critical vertex is $\gamma_{\rm pr}(G)$-critical.

Proposition 8. *Let G_1 and G_2 be disjoint* msd-4 *graphs with $\gamma_{\rm pr}(G_i)$-critical vertices u_i, $i = 1, 2$. Then for the graph $G = G_1 \oplus_u^{u_1u_2} G_2$, $\gamma_{\rm pr}(G) = \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2$, any $\gamma_{\rm pr}(G_i)$-critical vertex (including u) is $\gamma_{\rm pr}(G)$-critical and*

$$\mathrm{msd}_{\rm pr}(G) = 4.$$

Proof. Since $u_i \in V(G_i)$ is $\gamma_{\rm pr}(G_i)$-critical, $\gamma_{\rm pr}(G_1 - u_1) + \gamma_{\rm pr}(G_2 - u_2) = \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 4$, and at most two more vertices are needed to pairwise dominate G. Therefore $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2$.

Suppose there exists a paired dominating set S of G such that $|S| < \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2$ and let $S_i = S \cap V(G_i)$. First suppose that $u \notin S$. Assume without loss of generality that S_1 dominates u. Then S_1 is a paired dominating set of G_1 and S_2 is a paired dominating set of $G_2 - u_2$. Hence $|S_1| \ge \gamma_{\rm pr}(G_1)$ and $|S_2| \ge \gamma_{\rm pr}(G_2) - 2$. But then $|S| = |S_1| + |S_2| \ge \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2$, which is not the case. Therefore we may assume that $u \in S$ (in this case $u_i \in S_i$, $i = 1, 2$) and $|S_1| + |S_2| = |S| + 1$. Without loss of generality, u is paired with $v \in V(G_1)$, hence S_1 is a paired dominating set of G_1. Therefore $|S_1| \ge \gamma_{\rm pr}(G_1)$ so that $|S_2| \le \gamma_{\rm pr}(G_2) - 3$. If $N_{G_2}(u_2) \subseteq S_2$, then $S_2 \backslash \{u_2\}$ is a paired dominating set of G_2, and if there exists $w \in N_{G_2}(u_2) \backslash S_2$, then $S_2 \cup \{w\}$ is a paired dominating set of G_2. This is impossible because $|S_2 \cup \{w\}| \le \gamma_{\rm pr}(G_2) - 2$. Hence

$$\gamma_{\rm pr}(G) = \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2.$$

If w_i is $\gamma_{\rm pr}(G_i)$-critical, then, for $j \ne i$, the union of any $\gamma_{\rm pr}(G_i - w_i)$-set and any $\gamma_{\rm pr}(G_j - u_j)$-set is a paired dominating set of $G - w_i$ (this holds for $w_i = u_i = u$ also), so

$$\gamma_{\rm pr}(G - w_i) \le \gamma_{\rm pr}(G_i - w_i) + \gamma_{\rm pr}(G_j - u_j) = \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 4 < \gamma_{\rm pr}(G).$$

Therefore w_i is $\gamma_{\rm pr}(G)$-critical.

Without loss of generality consider $e \in E(G_1)$ and subdivide e three times. Then, since $\mathrm{msd}_{\rm pr}(G_1) = 4$ and u_2 is $\gamma_{\rm pr}(G_2)$-critical, we obtain

$$\gamma_{\rm pr}(G_{e,3}) \le \gamma_{\rm pr}(G_{1_{e,3}}) + \gamma_{\rm pr}(G_2 - u_2) = \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2 = \gamma_{\rm pr}(G).$$

Therefore $\mathrm{msd}_{\rm pr}(G) = 4$. ∎

We show next that performing the operation $G_1 \oplus^{e_1e_2} G_2$ on msd-4 graphs G_i, $i = 1, 2$, with edges $e_i = x_iy_i$, where x_i is a $\gamma_{\rm pr}(G_i)$-critical vertex, results in an msd-4 graph in which each $\gamma_{\rm pr}(G_i)$-critical vertex is $\gamma_{\rm pr}(G)$-critical.

Proposition 9. *Let G_i, $i = 1, 2$, be disjoint* msd-4 *graphs with $e_i = x_i y_i \in E(G_i)$, where $x_i \in \mathrm{Cr}(G_i)$. Then for the graph $G = G_1 \oplus^{e_1 e_2} G_2$, $\gamma_{\mathrm{pr}}(G) = \gamma_{\mathrm{pr}}(G_1) + \gamma_{\mathrm{pr}}(G_2) - 2$, any $\gamma_{\mathrm{pr}}(G_i)$-critical vertex (including $x = x_1 = x_2$) is $\gamma_{\mathrm{pr}}(G)$-critical and* $\mathrm{msd}_{\mathrm{pr}}(G) = 4$.

Proof. By Theorem 1, there exists a $\gamma_{\mathrm{pr}}(G_i)$-set in which x_i and y_i are matched. Therefore

$$\gamma_{\mathrm{pr}}(G) \leq \gamma_{\mathrm{pr}}(G_1) + \gamma_{\mathrm{pr}}(G_2) - 2. \tag{1}$$

On the other hand, it suffices to add two vertices to a $\gamma_{\mathrm{pr}}(G)$-set when splitting it into paired dominating sets of G_1 and G_2. Hence we have equality in (1). As in the proof of Proposition 8, any $\gamma_{\mathrm{pr}}(G_i)$-critical vertex is $\gamma_{\mathrm{pr}}(G)$-critical.

Let $e \in E(G)$ be any edge. If $e \in E(G_1) \backslash \{e_1\}$, then

$$\gamma_{\mathrm{pr}}(G_{e,3}) \leq \gamma_{\mathrm{pr}}(G_{1_{e,3}}) + \gamma_{\mathrm{pr}}(G_2 - x_2) = \gamma_{\mathrm{pr}}(G_1) + \gamma_{\mathrm{pr}}(G_2) - 2 = \gamma_{\mathrm{pr}}(G).$$

The case when $e \in E(G_2) \backslash \{e_2\}$ is analogous. Thus assume $e = xy$ and subdivide e by replacing it with the path (x, u, v, w, y). Let S be any $\gamma_{\mathrm{pr}}(G - x)$-set. As shown above, $|S| = \gamma_{\mathrm{pr}}(G) - 2$. Now $S \cup \{u, v\}$ is a paired dominating set of $G_{e,3}$ of cardinality $\gamma_{\mathrm{pr}}(G)$. It follows that G is an msd-4 graph. ■

We now describe a type of "reverse" operation, called a *split operation*, for each of the $\oplus$-operations.

$G \ominus u$. Let G be a connected graph with a cut-vertex u. Denote the components of $G - u$ by $F_1, F_2, \ldots, F_k$. For each i, let G_i be the graph obtained from F_i by adding a new vertex u_i, joining u_i to $v_i \in V(F_i)$ if and only if $uv_i \in E(G)$. Denote the disjoint union $G_1 + \cdots + G_k$ by $G \ominus u$.

$G \ominus xy$. Let G be a connected graph containing a vertex-cut $\{x, y\}$, where $xy \in E(G)$. Denote the components of $G - \{x, y\}$ by $F_1, F_2, \ldots, F_k$. For each i, let G_i be the graph obtained from F_i by adding the edge $x_i y_i$, joining x_i (y_i, respectively) to $v_i \in V(F_i)$ if and only if $xv_i \in E(G)$ ($yv_i \in E(G)$, respectively). Denote the disjoint union $G_1 + \cdots + G_k$ by $G \ominus xy$.

The next proposition shows that if an msd-4 graph G is split at a γ_{pr}-critical cut-vertex u, the components of $G \ominus u$ are msd-4 graphs having the copies of u as γ_{pr}-critical vertices.

Proposition 10. *Let G be an* msd-4 *graph with a γ_{pr}-critical cut-vertex u. Denote the components of $G \ominus u$ by $G_1, \ldots, G_k$. Then for each $i = 1, \ldots, k$, u_i is a $\gamma_{\mathrm{pr}}(G_i)$-critical vertex and* $\mathrm{msd}_{\mathrm{pr}}(G_i) = 4$.

Proof. Since u is $\gamma_{\rm pr}(G)$-critical and $G-u$ is the disjoint union of G_i-u_i, $i=1,\ldots,k$,

$$\gamma_{\rm pr}(G)-2=\gamma_{\rm pr}(G-u)=\sum_{i=1}^{k}\gamma_{\rm pr}(G_i-u_i).$$

Suppose $\gamma_{\rm pr}(G_1-u_1)\geq\gamma_{\rm pr}(G_1)$. Let R_1 be a $\gamma_{\rm pr}(G_1)$-set and, for $i\geq 2$, let R_i be a $\gamma_{\rm pr}(G_i-u_i)$-set. Since R_1 dominates u_1, $R=\bigcup_{i=1}^{k}R_i$ is a paired dominating set of G. But then

$$\gamma_{\rm pr}(G)\leq|R|\leq\gamma_{\rm pr}(G_1)+\sum_{i=2}^{k}\gamma_{\rm pr}(G_i-u_i)\leq\sum_{i=1}^{k}\gamma_{\rm pr}(G_i-u_i)=\gamma_{\rm pr}(G)-2,$$

which is impossible. Thus u_1 is $\gamma_{\rm pr}(G_1)$-critical. The same argument works for each $i\in\{2,\ldots,k\}$.

Consider an arbitrary edge $e\in E(G_1)$ and subdivide e three times. Then

$$\gamma_{\rm pr}(G_{e,3})\leq\gamma_{\rm pr}(G_{1_{e,3}})+\sum_{i=2}^{k}\gamma_{\rm pr}(G_i-u_i). \tag{2}$$

We show that equality holds in (2). Let S be any $\gamma_{\rm pr}(G_{e,3})$-set and define $S_1=S\cap V(G_{1_{e,3}})$ and $S_i=S\cap V(G_i)$ for $i=2,\ldots,k$ (if $u\in S$, then $u_i\in S_i$ for each i). First suppose that $u\notin S$. If S_1 dominates u, then S_1 is a paired dominating set of $G_{1_{e,3}}$ and S_i, $i\geq 2$, is a paired dominating set of G_i-u_i. Hence $|S_1|\geq\gamma_{\rm pr}(G_{1_{e,3}})$ and $|S_i|\geq\gamma_{\rm pr}(G_i-u_i)$, so that $\gamma_{\rm pr}(G_{e,3})=|S|=\sum_{i=1}^{k}|S_i|\geq\gamma_{\rm pr}(G_{1_{e,3}})+\sum_{i=2}^{k}\gamma_{\rm pr}(G_i-u_i)$ as required. On the other hand, if S_1 does not dominate u, then S_j is a paired dominating set of G_j for some $j\geq 2$, so that $|S_j|\geq\gamma_{\rm pr}(G_j)=\gamma_{\rm pr}(G_j-u_j)+2$ (since u_j is $\gamma_{\rm pr}(G_j)$-critical). Let S_j' be a $\gamma_{\rm pr}(G_j-u_j)$-set, $S_1'=S_1\cup\{u,u'\}$ for some $u'\in N_{G_1}(u)$, and $S'=(S\backslash S_1\backslash S_j)\cup S_1'\cup S_j'$. Then $|S'|=|S|$, S_1' is a paired dominating set of $G_{1_{e,3}}$ and the result follows as before.

Now suppose that $u\in S$. Then $|S_1|+\sum_{i=2}^{k}|S_i|=|S|+k-1$ and u is paired with a vertex in exactly one of the graphs $G_{1_{e,3}}$ or G_i, $i\geq 2$. For each of the $k-1$ other graphs, either $S_i\cup\{w_i\}$, for some neighbour $w_i\notin S_i$ of u_i, or $S_i\backslash\{u_i\}$ (if all neighbours of u_i in G_i belong to S_i) is a paired dominating set. Hence

$$\gamma_{\rm pr}(G_{1_{e,3}})+\sum_{i=2}^{k}\gamma_{\rm pr}(G_i)\leq|S|+2(k-1).$$

Since u_i is $\gamma_{\rm pr}(G_i)$-critical for each $i=2,3\ldots,k$, $\gamma_{\rm pr}(G_i-u_i)=\gamma_{\rm pr}(G_i)-2$. This gives

$$\gamma_{\rm pr}(G_{1_{e,3}})+\sum_{i=2}^{k}\gamma_{\rm pr}(G_i-u_i)\leq|S|=\gamma_{\rm pr}(G_{e,3}).$$

Therefore we have equality (2). Now

$$\gamma_{\rm pr}(G_{1_{e,3}}) = \gamma_{\rm pr}(G_{e,3}) - \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - u_i) = \gamma_{\rm pr}(G) - \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - u_i)$$
$$= \gamma_{\rm pr}(G_1) + \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - u_i) - \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - u_i) = \gamma_{\rm pr}(G_1).$$

Hence, for any edge $e \in E(G_1)$, $\gamma_{\rm pr}(G_{1_{e,3}}) = \gamma_{\rm pr}(G)$. Thus $\mathrm{msd}_{\rm pr}(G_1) = 4$. Similar reasoning may be applied to G_i for $i \in \{2, 3, \ldots, k\}$. ■

5. Msd-4 Block Graphs

The last three results we need for the proof of Theorem 4 concern block graphs. In the first result we prove that every non-leaf vertex of an msd-4 block graph is a cut-vertex.

Theorem 11. *Let G be a graph containing a block $B \cong K_n$, where $n \geq 3$, such that some vertex of B is not adjacent to any vertex of $G - B$. Then*

$$\mathrm{msd}_{\rm pr}(G) < 4.$$

Proof. Suppose the hypothesis of the theorem holds but $\mathrm{msd}_{\rm pr}(G) = 4$. Let $V(B) = \{v_0, \ldots, v_{n-1}\}$ and say $u = v_0$ is not adjacent to any vertex of $G - B$. Subdivide the edge uv_2 by replacing it with the path (u, x_3, x_2, x_1, v_2) (see Figure 3). Denote $X = \{x_1, x_2, x_3\}$ and let D be a $\gamma_{\rm pr}$-set of $G_{uv_2,3}$. By Lemma 6 we only have to consider the cases $D \cap \{u, x_1, x_2, x_3, v_2\} \in \{\{u, x_1, v_2\}, \{u, x_3, v_2\}, \{x_1, x_2\}, \{x_2, x_3\}\}$.

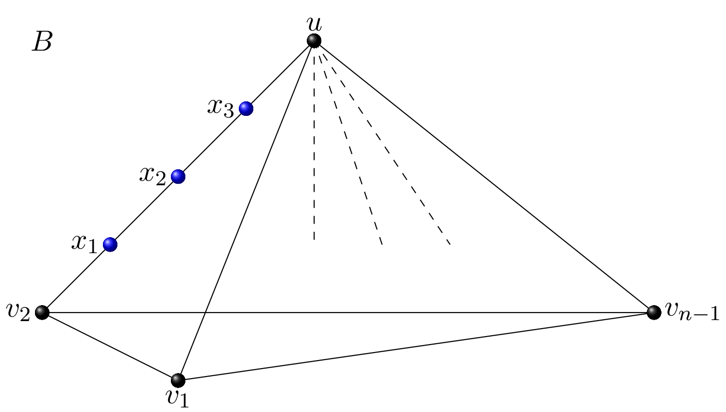

Figure 3. The block B with the edge uv_2 subdivided with vertices x_1, x_2, x_3.

Case 1. $|X \cap D| = 1$. If $D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_1, v_2\}$, then x_1 and v_2 are paired in D, while u is paired with v_i for some $i \neq 0, 2$. However, then $D \backslash \{x_1, u\}$, with v_2 and v_i paired, is a smaller paired dominating set of G. If $D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_3, v_2\}$, then $D \backslash \{x_3, u\}$ is a smaller paired dominating set of G. In either case $\text{msd}_{\text{pr}}(G) < 4$, contrary to our assumption.

Case 2. $|X \cap D| = 2$. If $D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_1, x_2\}$, then x_1 and x_2 are paired in D. To pairwise dominate u, $v_i \in D$ for some $i \neq 0, 2$. But then $D \backslash \{x_1, x_2\}$ is a paired dominating set of G (with v_i paired as in D) and $\text{msd}_{\text{pr}}(G) < 4$, contrary to our assumption. Hence assume $D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_2, x_3\}$. Then x_2 and x_3 are matched in D. If $v_i \in D$ for some i, then $D \backslash \{x_2, x_3\}$ is a paired dominating set of G (again with v_i paired as in D), a contradiction.

We therefore assume henceforth that

(i) D contains x_2 and x_3, but neither x_1 nor any $v_0, \ldots, v_{n-1}$.

By Lemma 6, u is γ_{pr}-critical, that is,

(ii) $\gamma_{\text{pr}}(G - u) = \gamma_{\text{pr}}(G) - 2$.

For each $i = 1, \ldots, n-1$, let G_i be the component of $G - E(B)$ that contains v_i. Since B is a block of G, the subgraphs G_i are distinct and pairwise vertex-disjoint. Let $D_i = D \cap V(G_i)$. Then $\left|\bigcup_{i=1}^{n-1} D_i\right| = |D \backslash \{x_2, x_3\}| = \gamma_{\text{pr}}(G) - 2$. By (i), each D_i is a $\gamma_{\text{pr}}(G_i)$-set that does not contain v_i.

We next show that

(iii) no $\gamma_{\text{pr}}(G)$-set contains $u = v_0$ and at least two v_i, $i \geq 1$.

Suppose there exists such a set Z; assume without loss of generality that $\{u, v_1, v_2, \ldots, v_k\} \subseteq Z$, $k \geq 2$. Necessarily, u is paired with some v_i, $i = 1, \ldots, k$, in Z. Assume (again without loss of generality) u is paired with v_1. Let $Z_1 = Z \cap V(G_1) \backslash \{v_1\}$ and, for $i \geq 2$, let $Z_i = Z \cap V(G_i)$. Then $\bigcup_{i=1}^{n-1} Z_i \subseteq V(G - u)$ and $\left|\bigcup_{i=1}^{n-1} Z_i\right| = |Z| - 2 = \gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G)$, by (ii). Since v_1 and u are paired, $G_1[Z_1]$ contains a perfect matching, as does $G\left[\bigcup_{i=2}^{n-1} Z_i\right]$. Since v_1 is not adjacent to any vertex of $G_i - v_i$, $i \geq 2$, and v_2 dominates B in G, $\bigcup_{i=2}^{n-1} Z_i$ is a paired dominating set of $G - G_1$.

Suppose $|Z_1| < |D_1|$. Since both Z_1 and D_1 have even cardinality, $|Z_1| \leq |D_1| - 2$. Then Z_1 does not dominate $G_1 - v_1$, otherwise $\bigcup_{i=1}^{n-1} Z_i$ is a paired dominating set of G of cardinality less than $\gamma_{\text{pr}}(G)$, which is impossible. Since $Z_1 \cup \{v_1\}$ dominates G_1, there exists a vertex $w \in N_{G_1}(v_1)$ that is undominated by Z_1. Then $W_1 = Z_1 \cup \{w, v_1\}$ is a paired dominating set of G_1 of cardinality at most $|D_1|$ that contains v_1. But now $W_1 \cup D_2 \cup D_3 \cup \cdots \cup D_{n-1}$ is a paired dominating set of G of cardinality at most $|D \backslash \{x_2, x_3\}| = \gamma_{\text{pr}}(G) - 2$, which is impossible. We conclude that $|Z_1| = |D_1|$.

Let $Z' = D_1 \cup \left(\bigcup_{i=2}^{n-1} Z_i\right)$. Since $\bigcup_{i=2}^{n-1} Z_i$ is a paired dominating set of $G - G_1$ and D_1 is a paired dominating set of G_1, Z' is a paired dominating set of G.

Moreover,

$$|Z'| = \left|\bigcup_{i=2}^{n-1} Z_i\right| + |D_1| = \left|\bigcup_{i=1}^{n-1} Z_i\right| = |Z| - 2 = \gamma_{\rm pr}(G-u) < \gamma_{\rm pr}(G),$$

which is impossible. This concludes the proof of (iii).

Subdivide the edge v_1v_2 with vertices y_1, y_2, y_3, where y_1 is adjacent to v_1 and y_3 is adjacent to v_2 (see Figure 4). Denote $Y = \{y_1, y_2, y_3\}$ and let Q be a $\gamma_{\rm pr}$-set of $G_{v_1v_2,3}$. Without loss of generality, by Lemma 6 we only have to consider the cases $Q \cap \{v_1, v_2, y_1, y_2, y_3\} \in \{\{y_1, y_2\}, \{v_1, v_2, y_1\}\}$.

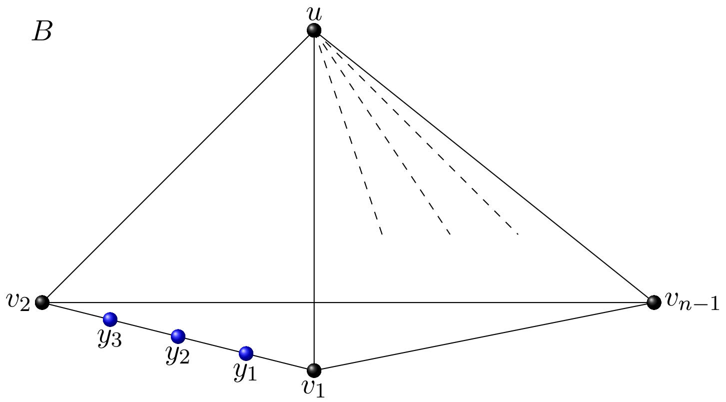

Figure 4. The block B with the edge v_1v_2 subdivided with vertices y_1, y_2, y_3.

Case 3a. $Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{y_1, y_2\}$. Then these two vertices are paired in Q. To pairwise dominate u, $v_i \in Q$ for some i. It follows that $Q \backslash \{y_1, y_2\}$ is a paired dominating set of G, so ${\rm msd}_{\rm pr}(G) < 4$, contrary to our assumption.

Case 3b. $Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{v_1, v_2, y_1\}$. Then y_1 is paired with v_1. If $u \notin Q$, then $Q' = (Q \backslash \{y_1\}) \cup \{u\}$ is a paired dominating set of G containing u, v_1, v_2. By (iii), Q' is not a $\gamma_{\rm pr}$-set of G, from which it follows that $\gamma_{\rm pr}(G) < |Q|$ and ${\rm msd}_{\rm pr}(G) < 4$. Assume therefore that $u \in Q$. Then u is paired in Q with v_i for some $i > 1$. Now $Q'' = Q \backslash \{y_1, u\}$ is a paired dominating set of G in which v_1 and v_i are paired. In both cases we again have a contradiction and the proof is complete. ■

The graph in Figure 5 shows that the statement of Theorem 11 is false if the complete subgraph B is not a block of G.

The next result in this section shows that msd-4 block graphs have many $\gamma_{\rm pr}$-critical vertices.

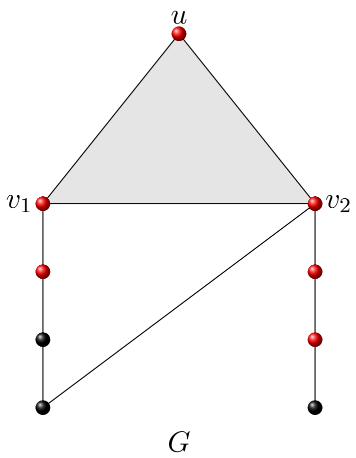

Figure 5. A graph G with $\mathrm{msd}_{\mathrm{pr}}(G) = 4$ and a subgraph K_3 that is not a block of G.

Theorem 12. *If G is a block graph with* $\mathrm{msd}_{\mathrm{pr}}(G) = 4$*, then for any edge* $uv \in E(G)$,

$$(N_G[u] \cup N_G[v]) \cap \mathrm{Cr}(G) \neq \emptyset.$$

Proof. Suppose there exists an edge $uv \in E(G)$ such that $(N_G[u] \cup N_G[v]) \cap \mathrm{Cr}(G) = \emptyset$. By Theorem 1, no vertex in $N_G[u] \cup N_G[v]$ is a leaf. We subdivide the edge uv by replacing it with the path (u, x_1, x_2, x_3, v) to obtain the graph $G_{uv,3}$. By Lemma 6, for any γ_{pr}-set S of $G_{uv,3}$, $S \cap \{u, v, x_1, x_2, x_3\} \in \{\{u, v, x_1\}, \{u, v, x_3\}\}$. Without loss of generality assume there exists such a set S such that $S \cap \{u, v, x_1, x_2, x_3\} = \{u, v, x_1\}$, and among all such sets S, let D be one for which $\mathrm{PN}(u, D)$ is as small as possible. Then x_1 and u are paired in D.

Say v is paired with v' and let B be the block of G that contains uv. If $v' \in V(G) \backslash V(B)$, let G_v be the subgraph of $G - E(B)$ that contains v, and if $v' \in V(B)$, let G_v be the subgraph of $G - (E(B) - \{vv'\})$ that contains v. In either case, $v' \in V(G_v)$. Let $D_v = D \cap V(G_v)$ and $D' = D \backslash \{x_1, u\}$. Then $G[D']$ has a perfect matching and D_v is a paired dominating set of G_v containing v and v'. In fact, D_v is a $\gamma_{\mathrm{pr}}(G_v)$-set, for if not, let D'' be a smaller paired dominating set of G_v. Consider $N_G(u) \backslash V(B)$. If $B \cong K_2$, then $N_G(u) \backslash V(B) = N_G(u) \backslash \{v\}$ is nonempty because u is not a leaf, and if $B \cong K_n$ for $n \geq 3$, then $N_G(u) \backslash V(B)$ is nonempty by Theorem 11. If $N_G(u) \backslash V(B) \subseteq D$, then D' is a paired dominating set of G, and if there exists $w \in N_G(u) \backslash V(B) \backslash D$, then $(D \backslash \{x_1\} \backslash D_v) \cup D'' \cup \{w\}$ is a smaller paired dominating set of G than D. In both cases we have a contradiction to $\mathrm{msd}_{\mathrm{pr}}(G) = 4$.

Since $\mathrm{msd}_{\mathrm{pr}}(G) = 4$, $|D'| = \gamma_{\mathrm{pr}}(G_{uv,3}) - 2 = \gamma_{\mathrm{pr}}(G) - 2$. Consequently, D' does not dominate G. Since $v \in D'$ dominates B in G, there exist vertices $w_1, \ldots, w_k \in N_G(u) \backslash N_G[v] \subseteq N_G(u) \backslash B$ that are undominated by D', that is,

$\{w_1, \ldots, w_k\} = \mathrm{PN}(u, D)$. For $i = 1, \ldots, k$, let G_i be the component of $G-u$ that contains w_i. Possibly, $G_i = G_j$ for $i \neq j$; this happens exactly when $w_iw_j \in E(G)$, and then w_i and w_j also belong to the same (complete) block of G_i. Since no w_i is adjacent to v or v', $V(G_i) \cap V(G_v) = \emptyset$ for each i. Define $D_i = D \cap V(G_i)$. Then $G_i[D_i]$ has a perfect matching, but does not dominate w_i. If it is nevertheless true that $\gamma_{\mathrm{pr}}(G_i) = |D_i|$ for some i, let Q_i be a $\gamma_{\mathrm{pr}}(G_i)$ set. Then $D^* = (D \backslash D_i) \cup Q_i$ is a $\gamma_{\mathrm{pr}}(G_{uv,3})$-set such that $\mathrm{PN}(u, D^*) \subseteq \mathrm{PN}(u, D) \backslash \{w_i\}$, contrary to the choice of D. Therefore $\gamma_{\mathrm{pr}}(G_i) \geq |D_i| + 2$ for each i.

Since each stem belongs to all paired dominating sets, no w_i is a stem, and by our initial assumption, no w_i is a leaf. Subdivide the edge uw_1 by replacing it with the path (u, y_1, y_2, y_3, w_1). Consider a $\gamma_{\mathrm{pr}}(G_{uw_1,3})$-set S. Since $u, w_1 \notin \mathrm{Cr}(G)$, Lemma 6 states that $S \cap \{u, y_1, y_2, y_3, w_1 \in \{\{u, y_1, w_1\}, \{u, y_3, w_1\}\}$.

- In the former case, y_1 is paired with u and $S_1 = S \cap V(G_1)$ is a paired dominating set of G_1; hence $|S_1| \geq \gamma_{\mathrm{pr}}(G_1) \geq |D_1| + 2$. Since w_1 is adjacent to all $w_i \in V(G_1)$, $D_1 \cup \{w_1\}$ dominates G_1 (but not pairwise). Now $S' = (S \backslash S_1) \cup D_1 \cup \{w_1, y_3\}$ is a paired dominating set of $G_{uw_1,3}$ such that $|S'| \leq |S|$, hence S' is a $\gamma_{\mathrm{pr}}(G_{uw_1,3})$-set. Moreover, $S' \cap \{u, y_1, y_2, y_3, w_1\} = \{u, y_1, y_3, w_1\}$, contrary to Lemma 6.
- In the latter case, y_3 is paired with w_1. Then $S_2 = (S \cap V(G_1)) \cup \{y_3\}$ is a paired dominating set of the graph obtained from G_1 by joining y_3 to w_1. If all neighbours of w_1 in G_1 belong to S_2, then $S_2 \backslash \{w_1, y_3\}$ is a paired dominating set of G_1. But then $S'' = S \backslash \{w_1, y_3\}$ is a paired dominating set of G such that $|S''| < |S|$, contradicting $\mathrm{msd}_{\mathrm{pr}}(G) = 4$. Assume some neighbour z of w_1 in G_1 does not belong to S_2. Then $S_3 = (S_2 \backslash \{y_3\}) \cup \{z\}$ is a paired dominating set of G_1, so that $|S_2| = |S_3| \geq |D_1| + 2$. Since $u \in S$ and $\{w_1, \ldots, w_k\} \subseteq N(u)$, $S^* = (S \backslash S_2) \cup D_1$ is a paired dominating set of G such that $|S^*| < |S|$, again a contradiction.

This completes the proof of the theorem. ■

Although the graph G in Figure 5 satisfies $\mathrm{msd}_{\mathrm{pr}}(G) = 4$ without being a block graph, Theorem 12 holds for G as well.

Our final result in this section concerns the reverse operation $G \ominus xy$ for certain msd-4 block graphs.

Proposition 13. *Let G be a connected* msd-4 *block graph such that the only $\gamma_{\mathrm{pr}}(G)$-critical vertices are leaves. Let x be a leaf adjacent to the stem y, where $\{x, y\}$ is a vertex-cut, and denote the components of $G \ominus xy$ by $G_1, \ldots, G_k$. Then for each $i = 1, \ldots, k$, G_i is an* msd-4 *graph and $x_i \in \mathrm{Cr}(G_i)$.*

Proof. If G_i is an msd-4 graph, it will follow from Theorem 1(ii) that $x_i \in \mathrm{Cr}(G_i)$. However, we need the fact that x_i is $\gamma_{\mathrm{pr}}(G_i)$-critical to show that $\mathrm{msd}_{\mathrm{pr}}(G_i) = 4$, hence this is what we prove first.

Since G is a block graph, $N_{G_i-x_i}(y_i)$ induces a clique for each $i = 1, \ldots, k$. Since x is a leaf, y belongs to every paired dominating set of G, and by Theorem 1(ii), $x \in \mathrm{Cr}(G)$. Hence y belongs to no $\gamma_{\mathrm{pr}}(G-x)$-set (for such a set would dominate x and thus G, contradicting $x \in \mathrm{Cr}(G)$).

Let D be a $\gamma_{\mathrm{pr}}(G-x)$ set such that $|D \cap N(y)|$ is maximum and let $D_i = D \cap V(G_i)$, $i = 1, \ldots, k$. Since $x \in \mathrm{Cr}(G)$ and $y \notin D$, $|D| = \sum_{i=1}^{k} |D_i| = \gamma_{\mathrm{pr}}(G) - 2$. Also, D_i is a paired dominating set of $G_i - \{x_i, y_i\}$ for each i, and a paired dominating set of $G_i - x_i$ for at least one i. We show that, in fact,

(A) D_i is a paired dominating set of $G_i - x_i$ for each i.

First suppose $|N_{G_i-x_i}(y_i)| \geq 2$; say $z_1, z_2 \in N_{G_i-x_i}(y_i)$. Since $N_{G_i-x_i}(y_i)$ induces a clique, $z_1z_2 \in E(G)$. By Theorem 12, $(N_G[z_1] \cup N_G[z_2]) \cap \mathrm{Cr}(G) \neq \emptyset$. Since $N_G[z_i] = N_{G_i-x_i}[z_i]$ and z_i is not a leaf (and thus, by the hypothesis, not $\gamma_{\mathrm{pr}}(G)$-critical), z_1 or z_2 is adjacent to a $\gamma_{\mathrm{pr}}(G)$-critical vertex, i.e., a leaf. Say z_1 is adjacent to a leaf z'. Then z_1 belongs to any paired dominating set of any subgraph of G containing both z_1 and z', so $z_1 \in D$. Therefore D_i dominates y_i and (A) holds.

Assume therefore that $|N_{G_i-x_i}(y_i)| = 1$, say $N_{G_i-x_i}(y_i) = \{z\}$. If $z \in D$, we are done, hence assume $z \notin D$. By Theorem 1(iii), z is not a leaf, hence there exists a vertex $z' \in N_{G_i-x_i}(z) \backslash \{y_i\}$. By Theorem 1(i), G has a γ_{pr}-set X such that zz' belongs to a matching of $G[X]$. Now $y \in X$, but y is not paired with any vertex of $G_i - x_i$, since $N_{G_i-x_i}(y_i) = \{z\}$. Therefore $X_i = (X \backslash \{x, y\}) \cap V(G_i)$ is a paired dominating set of $G_i - x_i$. Moreover, $|X_i| \leq |D_i|$, otherwise $(X - X_i) \cup D_i$ is a smaller paired dominating set of G, which is impossible. However, now $D' = (D \backslash D_i) \cup X_i$ is a paired dominating set of $G - x$, hence a $\gamma_{\mathrm{pr}}(G-x)$-set, containing more neighbours of y than D, contrary to the choice of D. Hence (A) holds in this case as well.

Therefore $\gamma_{\mathrm{pr}}(G_i - x_i) \leq |D_i|$ for each i, so that

$$\sum_{i=1}^{k} \gamma_{\mathrm{pr}}(G_i - x_i) \leq \sum_{i=1}^{k} |D_i| = |D| = \gamma_{\mathrm{pr}}(G-x). \tag{3}$$

Suppose there exists a $\gamma_{\mathrm{pr}}(G_i - x_i)$-set Y_i containing y_i. Since no D_j contains y_j, $D' = (D \backslash D_i) \cup Y_i$ is a paired dominating set of $G - x$ such that $|D'| \leq |D| = \gamma_{\mathrm{pr}}(G) - 2$ and D' dominates x. Then D' is a paired dominating set of G, which is impossible. Therefore no $\gamma_{\mathrm{pr}}(G_i - x_i)$-set contains y_i. Similarly, if $\gamma_{\mathrm{pr}}(G_i - x_i) < |D_i|$ for some i and Z_i is a $\gamma_{\mathrm{pr}}(G_i - x_i)$-set, then $D'' = (D \backslash D_i) \cup Z_i$ is a paired dominating set of $G - x$ such that $|D''| < |D|$, which is also impossible. From these two facts we deduce that D_i is a $\gamma_{\mathrm{pr}}(G_i - x_i)$-set, equality holds in (3) and $\gamma_{\mathrm{pr}}(G_i) = \gamma_{\mathrm{pr}}(G_i - x_i) + 2$, that is, x_i is $\gamma_{\mathrm{pr}}(G_i)$-critical for each i.

We show that $\mathrm{msd}_{\mathrm{pr}}(G_1) = 4$; it will follow similarly that $\mathrm{msd}_{\mathrm{pr}}(G_i) = 4$ for each i. Since D_1 is a $\gamma_{\mathrm{pr}}(G_1 - x_1)$-set, it is easy to see that we can pairwise dominate $G_{1_{xy,3}}$ by $|D_1| + 2 = \gamma_{\rm pr}(G_1)$ vertices. Hence consider any edge $e \in E(G_1 - x_1)$ and the graphs $G_{e,3}$ and $G_{1_{e,3}}$. Since combining any $\gamma_{\rm pr}(G_{1_{e,3}})$-set with the sets D_j, $j = 2, \ldots, k$, produces a paired dominating set of $G_{e,3}$,

$$\gamma_{\rm pr}(G_{e,3}) \leq \gamma_{\rm pr}(G_{1_{e,3}}) + \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - x_i). \tag{4}$$

We show that equality holds in (4). For convenience of notation, define $H_1 = G_{1_{e,3}}$ and $H_i = G_i$, $i \geq 2$. Let S be a $\gamma_{\rm pr}(G_{e,3})$-set and define $S_i = S \cap V(H_i)$ for $i = 1, \ldots, k$ (since $y \in S$, $y_i \in S_i$ for each i, and if $x \in S$, then $x_i \in S_i$ for each i). We consider two cases, depending on whether $x \in S$ or not.

Case 1. $x \notin S$. Then $\sum_{i=1}^{k} |S_i| = |S| + k - 1$. Note that y is paired with $w \in V(H_i) \backslash \{x_i, y_i\}$ for exactly one i. Then S_i is a paired dominating set of H_i. For $j \neq i$, $S_j \cup \{x_j\}$ is a paired dominating set of H_j. Therefore $\gamma_{\rm pr}(H_i) \leq |S_i|$ and $\gamma_{\rm pr}(H_j) \leq |S_j| + 1$ for $j \neq i$. For $\ell \geq 2$, x_ℓ is $\gamma_{\rm pr}(H_\ell)$-critical, hence $\gamma_{\rm pr}(H_\ell - x_\ell) \leq \gamma_{\rm pr}(H_\ell) - 2$. Therefore

$$\gamma_{\rm pr}(G_{1_{e,3}}) + \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - x_i) \leq \sum_{i=1}^{k} |S_i| - 2(k-1) + (k-1) = \sum_{i=1}^{k} |S_i| - (k-1) = |S|$$

and equality holds in (4).

Case 2. $\{x, y\} \subseteq S$. Then x and y are paired in S, $\{x_i, y_i\} \subseteq S_i$ for each i, and S_i is a paired dominating set of H_i. Also, $\sum_{i=2}^{k} |S_i| = |S| + 2(k-1) - |S_1|$. Since x_i is $\gamma_{\rm pr}(G_i)$-critical,

$$\gamma_{\rm pr}(G_{1_{e,3}}) + \sum_{i=2}^{k} \gamma_{\rm pr}(G_i - x_i) \leq |S_1| + \sum_{i=2}^{k} |S_i| - 2(k-1) = |S| = \gamma_{\rm pr}(G_{e,3}),$$

giving equality in (4).

It now follows as in the proof of Proposition 10 that $\mathrm{msd}(G_1) = 4$. Similarly, $\mathrm{msd}(G_i) = 4$ for $i \geq 2$. ∎

6. Proof of Theorem 4

We are now ready to prove our main theorem, the characterization of msd-4 block graphs. We restate the theorem here for convenience.

Theorem 4 (again). *Let G be a connected block graph. Then G is an* msd-4 *graph if and only if $G \in \mathcal{B}$. Moreover, if G is an* msd-4 *graph constructed from the graphs $H_1, \ldots, H_j \in \mathcal{U}$, then* $\mathrm{Cr}(G) = \bigcup_{i=1}^{j} \mathrm{Cr}(H_i)$.

Proof. If $G \in \mathcal{B}$, it follows immediately from Propositions 8 and 9 that G is an msd-4 graph and $\mathrm{Cr}(G) = \bigcup_{i=1}^{j} \mathrm{Cr}(H_i)$.

For the converse, let G be an msd-4 block graph. If G is a tree, the result follows from Corollary 5, hence we assume that $B \cong K_n$, $n \geq 3$, is a block of G. By (the contrapositive of) Theorem 11, each vertex of B is a cut-vertex, so $\deg(v) \geq n$ for each $v \in V(B)$. Since each non-leaf vertex of a K_2-block is a cut-vertex, we deduce that each vertex of G is either a leaf or a cut-vertex.

Suppose $v \in V(B)$ is γ_{pr}-critical. Applying Proposition 10 to v we obtain an msd-4 graph G_1 with $v_1 = v$ and $N_{G_1}[v_1] = B$, which contradicts Theorem 11. Thus every $\gamma_{\mathrm{pr}}(G)$-critical vertex belongs only to K_2-blocks.

We say that a vertex u is a *type-A vertex* if it is a $\gamma_{\mathrm{pr}}(G)$-critical cut-vertex, and an edge uv is a *type-A edge* if u is a leaf (hence $\gamma_{\mathrm{pr}}(G)$-critical) and $G - \{u, v\}$ is disconnected. Denote the number of type-A elements (vertices and edges together) of G by $a(G)$. First we show that

(B) if $a(G) = 0$, then $G \in \mathcal{U}$.

Suppose $a(G) = 0$. Then every $\gamma_{\mathrm{pr}}(G)$-critical vertex is a leaf. Say $V(B) = \{v_1, \ldots, v_n\}$. Since no vertex of B is $\gamma_{\mathrm{pr}}(G)$-critical, Theorem 12 implies that v_1 or v_n is adjacent to a $\gamma_{\mathrm{pr}}(G)$-critical vertex. Without loss of generality we assume that $v_1 u_1 \in E(G)$, $u_1 \notin V(B)$, and u_1 is $\gamma_{\mathrm{pr}}(G)$-critical. Similarly, without loss of generality, v_i is adjacent to a $\gamma_{\mathrm{pr}}(G)$-critical vertex $u_i \notin V(B)$ for $i = 2, \ldots, n-1$. Since $a(G) = 0$ and each vertex of G is either a leaf or a cut-vertex, $\deg_G(u_i) = 1$ for each $i = 1, \ldots, n-1$ and $G - \{v_i, u_i\}$ is connected. Thus, v_i belongs to only the two blocks B and $v_i u_i$, so $\deg_G(v_i) = n$ for each $i = 1, \ldots, n-1$.

Since v_n is a cut-vertex, $N(v_n) \backslash V(B) \neq \emptyset$. If v_n is adjacent to a $\gamma_{\mathrm{pr}}(G)$-critical vertex, say u_n, then, arguing as above, $\deg(u_n) = 1$, $\deg(v_n) = n$ and $G = K_n \circ K_1$. By Remark 3(i), n is odd, hence G belongs to the family $\mathcal{U} \subseteq \mathcal{B}$. If no vertex in $N(v_n) \backslash V(B)$ is critical, let $N(v_n) \backslash V(B) = \{w_1, \ldots, w_t\}$ for $t \geq 1$. By Theorem 12, each w_i is adjacent to a critical vertex $w_i' \neq v_n$, and since $a(G) = 0$, w_i' is a leaf. We show that

(C) $\{w_1, \ldots, w_t\}$ is an independent set of G.

Suppose (without loss of generality) that $w_1 w_2 \in E(G)$ and consider $G_{w_1 w_2, 3}$. Let w_1, x_1, x_2, x_3, w_2 be the $w_1 - w_2$ path in $G_{w_1 w_2, 3}$ and let D be a $\gamma_{\mathrm{pr}}(G_{w_1 w_2, 3})$-set. Since w_1' and w_2' are leaves, $w_1, w_2 \in D$. To dominate x_2, $\{x_1, x_2, x_3\} \cap D \neq \emptyset$. If $|\{x_1, x_2, x_3\} \cap D| = 2$, then $D \backslash \{x_1, x_2, x_3\}$ is a paired dominating set (with w_1 and w_2 paired) of G of smaller cardinality than D, contrary to $\mathrm{msd}(G) = 4$. Hence assume without loss of generality that $\{x_1, x_2, x_3\} \cap D = \{x_1\}$, so w_1 and x_1 are paired (and $w_1' \notin D$), while w_2 is paired with either w_2' or v_n. However, each vertex in $N_G(v_n)$ is adjacent to a leaf and belongs to D, thus $D \backslash \{v_n\}$ dominates G. Therefore, either $D \backslash \{x_1, w_2'\}$ or $D \backslash \{x_1, v_n\}$ is a paired dominating set of G in which w_1 and w_2 are paired, contrary to $\mathrm{msd}(G) = 4$. It follows that (C) holds.

Since G is a block graph, w_i and w_j belong to different components of $G - v_n$ for all $i \neq j$.

Consequently, if there exists a vertex $z \notin \{v_n, w_i'\}$ adjacent to w_i, then z and v_n belong to different components of $G - \{w_i, w_i'\}$. But now $w_i w_i'$ is a type-A edge, which is not the case as $a(G) = 0$. Hence $\deg(w_i) = 2$ and $G \cong K_n \circ^{*t} K_1$. Since $\mathrm{msd}(G) = 4$, n is even, by Remark 3(ii). Therefore $G \in \mathcal{U} \subseteq \mathcal{B}$. Thus (B) holds.

Now suppose $a(G) \geq 1$. If G has a type-A critical cut-vertex u, perform the operation $G \ominus u$; each resulting graph is an msd-4 graph by Proposition 10, and clearly a block graph. Moreover, the copies of u in each graph are γ_{pr}-critical. Repeat this process until no resulting msd-4 block graph has a type-A critical cut-vertex. Let $G_1, \ldots, G_k$ be the resulting graphs. Then each critical vertex of each G_i is a leaf. If any G_i has a type-A critical edge uv, where u is a leaf, perform the operation $G \ominus uv$. Each resulting graph is an msd-4 block graph by Proposition 13. Repeat this process until all resulting graphs H_j satisfy $a(H_j) = 0$. If H_j is a tree, then $H_j \cong S(2, \ldots, 2) \in \mathcal{U}$ by Corollary 5, otherwise $H_j \in \mathcal{U}$ by (B). Now G can be reconstructed by performing the $\oplus$-operations on the H_j, hence $G \in \mathcal{B}$, as required. ■

7. Open Problems

We conclude with a short list of open problems for future consideration.

Question 1. *Does Theorem* 12 *hold for all* msd-4 *graphs*?

Define another $\oplus$-operation as follows.

$\oplus_{u,Q}^{u_1Q_1,u_2Q_2}$: Let G_1 and G_2 be vertex disjoint graphs containing (not necessarily maximal) cliques Q_1 and Q_2 of equal size, and vertices $u_i \in V(Q_i)$ for $i \in \{1, 2\}$. We denote a graph obtained from G_1 and G_2 by identifying Q_1 and Q_2 into one clique Q, and u_1 and u_2 into one vertex $u = u_1 = u_2$, by $G_1 \oplus_{u,Q}^{u_1Q_1,u_2Q_2} G_2$ (or by $G_1 \oplus^{u_1Q_1,u_2Q_2} G_2$ if u and Q are unimportant).

Note that if the cliques Q_i have order at least three, then identifying the vertices of $Q_i - u_i$ in different ways may yield different graphs. Both operations $\oplus_u^{u_1u_2}$ and $\oplus_e^{e_1e_2}$ are special cases of $\oplus_{u,Q}^{u_1Q_1,u_2Q_2}$.

Question 2. *Let G_1 and G_2 be disjoint* msd-4 *graphs containing cliques Q_1 and Q_2 of equal size and $\gamma_{\mathrm{pr}}(G_i)$-critical vertices $u_i \in V(Q_i)$, $i = 1, 2$. Is it true that for any graph $G = G_1 \oplus_{u,Q}^{u_1Q_1,u_2Q_2} G_2$, u is $\gamma_{\mathrm{pr}}(G)$-critical and* $\mathrm{msd}_{\mathrm{pr}}(G) = 4$?

If G_1 and G_2 are copies of the msd-4 graph in Figure 5, with $u_i = u$, which is γ_{pr}-critical, and Q_i is the triangle containing u, then both graphs obtainable as $G_1 \oplus_{u,Q}^{u_1Q_1,u_2Q_2} G_2$ are msd-4 graphs having u as critical vertex.

Question 3. *Let G be a graph with* $\mathrm{msd}_{\mathrm{pr}}(G) = 4$. *What is the largest number of edges of G that can be subdivided three times before the paired domination number increases? If this number can be arbitrarily high, what is its ratio to the number of edges of G?*

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