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# Thermal buckling of functionally graded piezomagnetic micro- and nanobeams presenting the flexomagnetic effect

Received: 7 May 2021 / Accepted: 23 June 2021  
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**Abstract** Galerkin weighted residual method (GWRM) is applied and implemented to address the axial stability and bifurcation point of a functionally graded piezomagnetic structure containing flexomagnetism in a thermal environment. The continuum specimen involves an exponential mass distributed in a heterogeneous media with a constant square cross section. The physical neutral plane is investigated to postulate functionally graded material (FGM) close to reality. Mathematical formulations concern the Timoshenko shear deformation theory. Small scale and atomic interactions are shaped as maintained by the nonlocal strain gradient elasticity approach. Since there is no bifurcation point for FGMs, whenever both boundary conditions are rotational and the neutral surface does not match the mid-plane, the clamp configuration is examined only. The fourth-order ordinary differential stability equations will be converted into the sets of algebraic ones utilizing the GWRM whose accuracy was proved before. After that, by simply solving the achieved polynomial constitutive relation, the parametric study can be started due to various predominant and overriding factors. It was found that the flexomagnetism is further visible if the ferric nanobeam is constructed by FGM technology. In addition to this, shear deformations are also efficacious to make the FM detectable.

**Keywords** Thermal stability · Functionally graded materials · Micro/nano-scale · Piezomagnetic · Flexomagnetic · Galerkin method

## Nomenclature

$\sigma_{xx}$	Stress component
$\tau_{xz}$	Shear stress
$\xi_{xxz}$	Hyper-stress
$\eta_{xxz}$	Hyper-strain
$\varepsilon_{xx}$	Strain component
$\gamma_{xz}$	Shear strain

Communicated by Andreas Öchsner.

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$E$	Elasticity modulus
$G$	Shear modulus
$u_1$	Displacement along $x$
$u_3$	Displacement along $z$
$\nu$	Poisson's ratio
$L$	Length of the beam
$b$	Width of the beam
$z$	Thickness coordinate
$h$	Thickness of the beam
$k_s$	Shear correction factor
$\Delta T$	Temperature changes
$B_z$	Magnetic flux
$\psi$	Magnetic potential
$\mu$	Nonlocal parameter
$e_0$	Nonlocal quantity
$U$	Strain energy
$N_x^0$	Buckling load
$E$	Elasticity modulus
$\omega_n$	Natural frequency
$m$	Mode number
$k$	Material property variation
$I_z$	Area moment of inertia
$u$	Axial displacement of the midplane
$w$	Transverse displacement of the midplane
$\phi$	Rotation of beam nodes around the $y$ -axis
$q_{31}$	Component of the third-order piezomagnetic tensor
$g_{31}$	Component of the sixth-order gradient elasticity tensor
$f_{31}^3$	Component of fourth-order flexomagnetic tensor
$a_{33}$	Component of the second-order magnetic permeability tensor
$A$	Area of the cross section of the beam
$N_x$	Axial stress resultant
$M_x$	Moment stress resultant
$Q_x$	Shear stress resultant
$T_{xxz}$	Hyper-stress resultant
$\alpha$	Thermal expansion coefficient
$H_z$	Magnetic field component
$z_0$	Distance between the physical neutral plane and the geometric mid-surface
$\Psi$	Magnetic potential function
$l$	Length scale parameter
$a$	Internal characteristic length
$W$	Work performed by outer forces
$T_{cr}$	Critical temperature
$t$	Time in dynamic systems
$R_m(x)$	Residuals
$X_m, Y_m$	Trial functions

## 1 Introduction

The progress in micro- and nanoelectronic technologies is directly related to the achievements in the field of materials engineering. The wide range of currently tested smart magnetic materials (SMM) provides opportunities to develop new, innovative components and devices. The physical and chemical properties will be sensitive to environmental parameters, such as temperature, pressure, electric field, and magnetic fields. With the development of science, new electromagnetic properties are found [15]. Among these, piezomagnetism is defined as a linear electromechanical reaction between two magnetic and mechanical states in insulating materials and crystals that do not have central symmetry. In fact, piezomagnetic structures are materials that,



when pressed or stressed, a magnetic charge appears on certain surfaces. This phenomenon is called the direct piezomagnetic effect, which is a reversible process, meaning that when a substance with this property is in an electric field, its dimensions change (reverse piezomagnetic effect). In the past years, the mechanical response of micro- and nanostructures, mainly electrically and magneto-electrically operated, has set off to be an intensive and significant region of investigation [6–27]. In order to confirm the novelty of this article, a thorough literature review was realized. The study of the most important published works precisely associated with the investigation in this manuscript is presented. One of the frequent discoveries is flexomagnetism (FM). Scientists are currently examining the phenomenon by considering different boundary conditions and dynamic and static terms. The flexomagnetic effect is established with a strain field gradient. In other words, it may be named the direct flexomagnetic effect. Following the attendance of a magnetic external field gradient, the flexo-effect may be distinguished in reverse impact. These new-discovered phenomena may appear in all types of materials and crystalline structures [28,29,31].

Functionally graded materials (FGMs) are new and advanced materials with a heterogeneous structure. The mechanical properties of FG materials are constantly changing from one level to another, and these changes are caused by a gradual change in the volume ratio of their constituent materials [32–43]. FG materials are typically made of both ceramic and metal. Since the structural material of ceramic has a low heat transfer coefficient and high resistance to temperature, it can withstand high heat. On the other hand, another structural material, metal, provides the required flexibility and strength. Due to the continuous changes in mechanical properties, the discontinuity problems in laminated composite structures do not arise in functional materials. These materials are widely used in thermal insulation, coatings for turbine blades, protection systems, biomedical materials, bone and dental implants, and the aerospace industry. Another implementation of FGMs can be seen in spaceship walls and engine parts, including piezoelectric, thermoelectric devices, and micro/nano-electro-mechanical systems (MEMS/NEMS).

The smart micro- and nanobeams discussion is well-known, and there are many papers according to this subject. Liang et al. [44,45] investigated flexoelectricity and its impact on static bending issues, considering the Euler–Bernoulli beam model with the piezoelectric effect. Tadi Beni [46,47] analyzed the dynamic behavior and static deflection of a nanobeam exposed to electrical and mechanical loads, following the Euler–Bernoulli and Timoshenko beam model theory. Arefi et al. [48] performed a study of bending and vibration of a piezomagnetic layered nanobeam exposed to magnetic and electric potential laying on a two-parameter foundation following nonlocal Eringen’s theory. Another investigation performed by Tadi Beni et al. [49] considered Van der Waals forces and electric effect and their impact on nanobeam deflection, following the modified couple stress theory. Alibeigi et al. [50,51] performed an investigation of piezomagnetic and piezoelectric nanobeams exposed to buckling on electrical, thermal, and mechanical loads, following the Euler–Bernoulli beam theory and modified strain gradient theory. Qi et al. [52] studied bending analysis and its impact on electro-elastic nanobeams based on Euler–Bernoulli beam theory and nonlocal strain gradient theory. Li et al. [53] performed an investigation of buckling analysis of bilayered piezoelectric nanobeams with imperfections under mechanical and electrical loads, following trigonometric shear and Eringen’s nonlocal elasticity theory. Sidhardh and Ray [54] studied deflection analysis of pinned-roller-supported nanobeams, including the flexoelectric layer, using the finite element method (FEM). Baroudi et al. [55] established an analytical solution for the free vibration and transverse deflection study of a piezoelectric nanobeam exposed to an electrical load based on strain gradient theory. Mohtashami et al. [56] investigated buckling and vibration of piezoelectric nanobeams following Euler–Bernoulli beam theory.

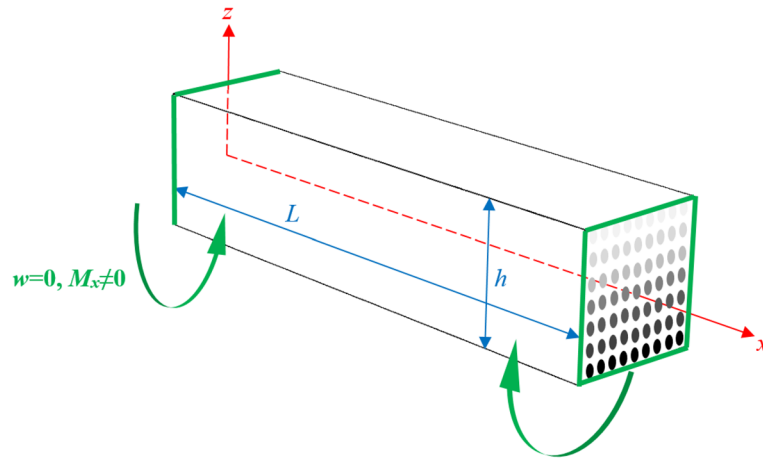
The nano-electro-mechanical devices are widely made as functionally graded nanobeams (FGN). This area is being crucially developed and investigated. Esfahani et al. [57] investigated the vibration and deflection of FGNs exposed to the external electric voltage, following the nonlocal strain gradient and Euler–Bernoulli beam theory. Zhao et al. [58] analyzed the free vibration and bending of flexoelectric FGNs with axial porous based on strain gradient and Euler–Bernoulli beam theory. Xiao et al. [59] studied elastic-electro-magneto-thermal functionally graded nanobeams with porosity, following Eringen’s nonlocal elasticity and higher-order shear deformation theory.

The literature on the study of thermal effects of FG micro- and nanobeams is rich enough. Thus, we will consider some among them. Shafiei et al. [60] studied a micro/nanobeam made of axial FG composition under thermal buckling effects. Based on the thin beam assumptions the relationships were derived. Couple stress theory and nonlocal elasticity model developed their governing equations for micro- and nanobeams, respectively. The generalized differential quadrature technique extracted their results. Tounsi et al. [61] used a higher-order shear deformation beam model to carry out the thermal buckling of nanobeams. Barretta et al. [62] developed a new study on FG nanobeams based on the novel nonlocal integral elasticity model. In the

new proposed model, both mechanical and thermal effects were involved. Sarparast et al. [63] investigated the dynamic model of an axially moving nanobeam covered by a viscoelastic medium. Thermal and hygral impacts of the environment were included as well. Fang et al. [64] expanded thermal modeling of nanobeams fabricated by FGMs by rotational movement based on the thin beam theory and Eringen's nonlocal assumptions. Ebrahimi and Barati [65] derived a mathematical model of thermal buckling of a nanobeam constructed by FGMs based on the Reddy beam theory. Karami et al. [66] inspected the thermal effects on the dynamics of an FGM tapered nanobeam by assuming functionality in two dimensions.

Following the literature review made on the piezo-flexomagnetic mechanics of structures, theoretical studies may be found. So far, several studies have been performed on nanostructures considering the flexomagnetic effect. The pioneers in this subject were Zhang et al. [67] and Sidhardh et al. [68]. In their investigation, they presented studies on piezo-flexomagnetic nanostructures subjected to linear bending. Additionally, the assumption of small displacement was made, and only the linear-elastic region was taken into account. In both investigations, the model of the structure was nanobeam, modeled following the Euler–Bernoulli beam theory. Furthermore, the two cases of magnetization influences were considered, the direct and reverse impact. In the first study, it was deemed to be different boundary conditions. However, [68] assumed a cantilever beam in his investigation. The static load acting on a beam was applied uniformly and vertically across the length of the nanobeam. In both analyses, there is a lack of consideration of the size-dependent effect. What is more, microstructure or nonlocal impact were not investigated as well. Nevertheless, they checked the surface and flexomagnetic effects on the nanobeam. Lately, Malikan and Eremeyev [69] investigated a piezo-flexomagnetic nanobeam exposed to vibrational mode, following the Euler–Bernoulli beam theory. According to the nonlocal stress-driven elasticity method, the size-dependent effect was analyzed. Furthermore, the structures were subjected to linear frequency analysis. Following the obtained results, it could be concluded that the size-dependent effect concerns the flexomagnetic feature. Furthermore, according to the nonlinear model, Malikan and Eremeyev [70] investigated the natural frequencies of piezo-flexomagnetic nanostructures. Following the nonlocal strain gradient elasticity model, they confirmed the size-dependent effect. Another study performed by Malikan et al. [71] presented piezo-flexomagnetic nanobeams subjected to large deflections, following two-step analytical and numerical solution methods. They found out that nano-electro-mechanical systems (NEMS) subjected to nonlinear bending and piezo-flexomagnetic effects are crucial in designing these systems. This article showed a significant influence of the flexomagnetic effect and its impact on reducing nanobeam deflection Malikan et al. [72] investigated magnetic nanoparticles with piezo and flexomagnetic effect. The study concerning smart nanosensors investigates the analysis of the post-buckling impact on these structures. The conclusions included in this paper are very significant in the field of nanostructures. Malikan et al. [73] presented the nanobeam with porous state and piezo-flexomagnetic effect and the impact of beam size. Obtained results show that the flexomagnetic effect of the structure is dependent on the porosity of the materials. Malikan and Eremeyev [74] investigated the composite nanoplate with the piezo-flexomagnetic effect subjected to the one-dimensional magnetic field, following the nonlocal strain gradient and classical plate theory. The conclusion earned from this literature is that the flexomagnetic response is more significant if the nanoplate's aspect ratio is less than one Malikan and Eremeyev [75] studied the effects of surface elasticity on the flexomagnetic influence of nanostructures. It was acquired that there is a direct and substantial relation between flexomagneticity in bulk and surface effects.

Many influential factors can highlight or play down the flexomagnetic effect in bulk We prepared this study to examine one of these factors, namely functionally grading composition on this phenomenon to predict that what would happen if the magnetic material is made of FGMs. This attempt is the first time addressing the FM in an FGM structure. The effort of this paper is to present a thermal buckling analysis and the thermal capacity approach to the problems of nano- and microbeam functionally graded structures with the flexomagnetic effect. The issue has been examined according to the reverse magnetic solution. The primary investigation is concerned about the functionally graded material composition and its response to the practical flexomagnetic effect. The FGMs nanosensors are frequently exposed to contact with soft tissues that may be designed with elastic structures. The problem is modeled with constitutive relations following the nonlocal strain gradient theory (NSGT). Subsequent to Gauss's law and Maxwell's equation, a relevant magnetic potential distribution is obtained. Following the clamp support boundary conditions and Timoshenko beam theory, the interaction governing equations of the structure are derived with the terms of the piezo and flexomagnetic effect. The thermal stability problem is computed with the use of discretization of the equations following Galerkin's principle. The characteristics equations are calculated straightforwardly to obtain the buckling values. The answers received from the introduced method were presented on different diagrams to certify the correctness of the current approach Concerning the static behavior of the system, the effect of various parameters is shown



**Fig. 1** A typical continuum model of FGM beam-like intelligent nano-actuator having rectangular coordinates

with some illustrations. The obtained results in the numerical section of the manuscript could be conducive to attain a significant and efficient nano-actuators/sensors design.

## 2 Mathematical modeling

To start up, an FGM piezomagnetic nanobeam is shown in Fig. 1 A square cross-section figures out the beam. Moreover, a schematic grading of grains of the cross section is drawn, and the Cartesian coordinate system is installed on the model. The dimensions of the beam are associated with  $L$  that is the length and  $h$  which introduces the thickness.

Let us assume that it is essential to take into consideration the shear deformations for the FGM nanobeam. Accordingly, to concatenate the shear deformation throughout the thickness, this study utilizes the Timoshenko beam approach as [76,77]

$$u_1(x, z) = u(x) + (z - z_0) \phi(x) \quad (1a)$$

$$u_3(x, z) = w(x) \quad (1b)$$

The neutral plan's location is a crucial issue in FGMs with a physical concept. There is a minor deviation between the physical neutral surface and the mid-plane while appraising the FGMs due to the distribution of mechanical properties. The physical neutral plan's position can be described by [76,78,79],

$$z_0 = \frac{\int_{-h/2}^{h/2} z E(z) dz}{\int_{-h/2}^{h/2} E(z) dz} \quad (2)$$

in which  $z_0$  defines the distance between the physical neutral plane and the geometric mid-surface. It is essential to say that the FG composition is assumed to be along with the thickness only, and there is no FG composition in line with the length.

The Voigt estimate or rule of mixture interprets reasonably and continuously distribution of mechanical properties for FGMs in a heterogeneous schema. Pursuant to the volume fraction of the FGMs as composed materials including functions of sigmoid, exponential, or power-law, the change in the material properties of FGMs is postulated to vary continuously along with the thickness. In this study, to describe the volume fraction an exponential function is employed. Let us depict  $P(z)$  which is a variable to define any property in the class of exponential functionality as [80],

$$P(z) = P_0 e^{kz} \quad (3)$$

An index for the property of the material is shown by  $k$  and  $P_0$  represents any property related to mid-plane ( $z=0$ ). It is requisite to remind that all these properties are varied along with the thickness. The shift from the heterogeneous beam to a homogeneous one corresponds to  $k=0$ . It is germane to note that Poisson's ratio is constant and independent of thickness because the difference in the values is negligible.

Lagrangian linear strain's relation layouts the following axial, shear, and hyper strains with the help of Eq. (1) as,

$$\varepsilon_{xx} = \frac{du}{dx} + (z - z_0) \frac{d\phi}{dx} - \alpha(z) \Delta T \quad (4)$$

$$\gamma_{xz} = \phi + \frac{dw}{dx} \quad (5)$$

$$\eta_{xxz} = \frac{d\varepsilon_{xx}}{dz} = \frac{d\phi}{dx} \quad (6)$$

Here, by expressing the principle of Lagrange, one can present the energy relation as,

$$\delta W + \delta U = 0 \quad (7)$$

The letters  $U$  and  $\delta W$  are dedicated to defining the internal strain energy and thermodynamic work of external forces.

It is necessary to expand the strain energy based on the description of variational calculus as,

$$\delta U = \int_V (\sigma_{xx} \delta \varepsilon_{xx} + \tau_{xz} \delta \gamma_{xz} + \xi_{xxz} \delta \eta_{xxz} - B_z \delta H_z) dV \quad (8)$$

By doing Eq. (9), we obtain single and double integrals which respectively correlate with non-classical end conditions and governing equations as follows:

$$\delta U = \delta U^{Mech} + \delta U^{Mag} = 0 \quad (9)$$

$$\begin{aligned} & - \int_0^L \left( \frac{dN_x}{dx} \delta u + \frac{dQ_x}{dx} \delta w - Q_x \delta \phi + \frac{dM_x}{dx} \delta \phi + \frac{dT_{xxz}}{dx} \delta \phi \right) dx \\ & + (N_x \delta u + Q_x \delta w + M_x \delta \phi + T_{xxz} \delta \phi) \Big|_0^L \\ & - \int_0^L \int_{-h/2}^{h/2} \frac{dB_z}{dz} \delta \Psi dz dx + \int_0^L (B_z \delta \Psi) \Big|_{-h/2}^{h/2} dx = 0 \end{aligned} \quad (10)$$

in which

$$(N_x, M_x, Q_x, T_{xxz}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{xxz}, k_s \tau_{xz}, \xi_{xxz}) dz \quad (11)$$

Let us here attend to the work of external forces, which can be formulated as

$$W = \frac{1}{2} \int_0^L N_x^0 \left( \frac{dw}{dx} \right)^2 dx \quad (12)$$

Then, the variational method approximates the following relation:

$$\delta W = \int_0^L \left( N_x^0 \frac{d\delta w}{dx} \frac{dw}{dx} \right) dx \quad (13)$$

where  $N_x^0$  denotes the total external compressive force. In the following, it plays a role of a critical (buckling) parameter.



It is postulated that a transverse magnetic field exists which lets us retain the electrodes on the bottom and top surfaces of the thickness, while a poling direction is kept transverse. Thus, the relation of the magnetic field's lateral component corresponds to,

$$H_z + \frac{d\Psi}{dz} = 0 \quad (14)$$

The starting point of our study and the motivation is assessing flexomagnetism in smart functionally graded structures. The converse flexomagnetism is generated because of the magnetic field's strain gradient. Let us deem a closed-circuit through the thickness for which the topmost surface of the thickness contains the maximum potential and the lowest surface involves the null of potential. To this, one can write

$$\Psi\left(+\frac{h}{2}\right) = \psi, \quad \Psi\left(-\frac{h}{2}\right) = 0 \quad (15a,b)$$

The transverse magnetic field's formulas and the magnetic potential across the beam can be achieved by joining Eqs. (6, 10, 14, and 15a,b) with one another and the more-or-less mathematical efforts,

$$\Psi = \frac{q_{31}(z)}{2a_{33}(z)} \left( (z - z_0)^2 - \frac{h^2}{4} \right) \frac{d\phi}{dx} + \frac{\psi}{h} \left( (z - z_0) + \frac{h}{2} \right) \quad (16)$$

$$H_z = - (z - z_0) \frac{q_{31}(z)}{a_{33}(z)} \frac{d\phi}{dx} - \frac{\psi}{h} \quad (17)$$

According to Eringen's nonlocal theory, the stress at a reference point inside the body, such as  $x$ , depends not only on the strain of the point  $x$  but also on the strains of all points inside the body [81,82]. This theory is consistent with predictions derived from the atomic theory of molecular lattice dynamics and observations of molecular dispersion and long-range interactions. In the limit, the classical theory of elasticity will be derived when strain effects are ignored at points other than point  $x$ . For homogeneous and isotropic objects, the linear theory of nonlocal elasticity leads to a set of partial integro-differential equations for the displacement field that is generally difficult to solve. For certain classes of integral, these equations are reduced to a group of single partial differential equations. On the other hand, atoms consist of a large strain gradient on a small scale. This is mathematically simulated by the strain gradient elasticity theory of Mindlin [73]. These two phenomena are unified, called nonlocal strain gradient elasticity theory [83], by which a small scale is transferred into a continuum media based on a differential model. The relation of NSGT is available below:

$$\left(1 - \mu \frac{d^2}{dx^2}\right) \sigma_{ij} = C_{ijkl} \left(1 - l^2 \frac{d^2}{dx^2}\right) \varepsilon_{ij} \quad (18)$$

in which the additional and higher-order parameters demonstrated by  $\mu$  (nm)<sup>2</sup> =  $(e_0 a)^2$  and  $l$  (nm), respectively, exhibit a nonlocal parameter and a strain gradient length scale parameter (SGLS). Amounts of these non-classical parameters are already determined for some classes of materials only.

Thereupon, infliction of Eq. (19) on Eq. (12) causes Eqs. (19–21),

$$\left(1 - \mu \frac{d^2}{dx^2}\right) \xi_{xxz} = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ \left( g_{31}(z) + \frac{q_{31}(z) f_{31}(z) (z - z_0)}{a_{33}(z)} \right) \frac{d\phi}{dx} + \frac{f_{31}(z) \psi}{h} \right] \quad (19)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) \sigma_{xx} = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ E(z) \frac{du}{dx} + (z - z_0) \left( E(z) + \frac{q_{31}^2(z)}{a_{33}(z)} \right) \frac{d\phi}{dx} + \frac{q_{31}(z)\psi}{h} - \alpha(z) E(z) \Delta T \right] \quad (20)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) \tau_{xz} = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ G(z) A \left( \phi + \frac{dw}{dx} \right) \right] \quad (21)$$

Later, Eq. (12) will be written in the framework of Eq. (19). Thus, one can obtain

$$\left(1 - \mu \frac{d^2}{dx^2}\right) N_x = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ I_1 \frac{du}{dx} + (I_2 + I_3) \frac{d\phi}{dx} + I_4 - N^T \right] \quad (22)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) M_x = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ I_5 \frac{du}{dx} + (I_6 + I_7) \frac{d\phi}{dx} + I_8 \right] \quad (23)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) Q_x = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ H_{44} \left( \phi + \frac{dw}{dx} \right) \right] \quad (24)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) T_{xxz} = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left[ (I_9 + I_{10}) \frac{d\phi}{dx} + I_{11} \right] \quad (25)$$

in which the established variables are expanded as follows:

$$\begin{aligned} \{I_1, I_2\} &= \int_{-h/2}^{h/2} E(z) \{1, (z - z_0)\} dz, \quad I_3 = \int_{-h/2}^{h/2} (z - z_0) \frac{q_{31}^2(z)}{a_{33}(z)} dz, \quad I_4 = \int_{-h/2}^{h/2} \frac{\psi q_{31}(z)}{h} dz, \\ \{I_5, I_6\} &= \int_{-h/2}^{h/2} E(z) \{(z - z_0), (z - z_0)^2\} dz, \quad I_7 = \int_{-h/2}^{h/2} (z - z_0)^2 \frac{q_{31}^2(z)}{a_{33}(z)} dz, \quad I_8 = \int_{-h/2}^{h/2} (z - z_0) \frac{\psi q_{31}(z)}{h} dz, \\ I_9 &= \int_{-h/2}^{h/2} g_{31}(z) dz, \quad I_{10} = \int_{-h/2}^{h/2} (z - z_0) \frac{q_{31}(z) f_{31}(z)}{a_{33}(z)} dz, \quad I_{11} = \int_{-h/2}^{h/2} \frac{\psi f_{31}(z)}{h} dz, \quad H_{44} = k_s \int_{-h/2}^{h/2} G(z) Adz \end{aligned} \quad (26)$$

It should be reminded that the pyromagnetic effect, i.e., a coupling between temperature and magnetic field, has been neglected in this work. The equilibrium equations can be pulled out from Eqs. (9, 10), moreover, mixed with Eq. (13),

$$\frac{dN_x}{dx} = 0 \quad (27)$$

$$\frac{dQ_x}{dx} + N_x^0 \frac{d^2 w}{dx^2} = 0 \quad (28)$$

$$\frac{dM_x}{dx} + \frac{dT_{xxz}}{dx} - Q_x = 0 \quad (29)$$

Ensuingly, Eqs. (22-25) can be taken out of complexity based on Eqs. (27-29) as follows:

$$N_x = \left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ I_1 \frac{du}{dx} + (I_2 + I_3) \frac{d\phi}{dx} + I_4 - N^T \right\} \quad (30)$$

$$M_x = -\mu \left[ (I_9 + I_{10}) \frac{d^3 \phi}{dx^3} + N_x^0 \frac{d^2 w}{dx^2} \right] + \left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ I_5 \frac{du}{dx} + (I_6 + I_7) \frac{d\phi}{dx} + I_8 \right\} \quad (31)$$

$$Q_x = -\mu N_x^0 \frac{d^3 w}{dx^3} + \left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ H_{44} \left( \phi + \frac{dw}{dx} \right) \right\} \quad (32)$$

$$T_{xxz} = \mu \frac{d^2 T_{xxz}}{dx^2} + \left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ (I_9 + I_{10}) \frac{d\phi}{dx} + I_{11} \right\} \quad (33)$$

Eqs. (27-29) can be recast with the aid of Eqs. (30-33) as,

$$\left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ I_1 \frac{d^2 u}{dx^2} + (I_2 + I_3) \frac{d^2 \phi}{dx^2} \right\} = 0 \quad (34)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) \left\{ N_x^0 \frac{d^2 w}{dx^2} \right\} + \left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ H_{44} \left( \frac{d\phi}{dx} + \frac{d^2 w}{dx^2} \right) \right\} = 0 \quad (35)$$

$$\left(1 - \mu \frac{d^2}{dx^2}\right) \left\{ (I_9 + I_{10}) \frac{d^2 \phi}{dx^2} \right\} + \left(1 - l^2 \frac{d^2}{dx^2}\right) \left\{ I_5 \frac{d^2 u}{dx^2} + (I_6 + I_7) \frac{d^2 \phi}{dx^2} - H_{44} \left( \phi + \frac{dw}{dx} \right) \right\} = 0 \quad (36)$$

The abovecoupled equations should be solved toward determining the thermal stability capacity of the postulated structure.



The total axial force is divided into the thermal load and a longitudinal magnetic load originated from the magnetic field,

$$N_x^0 = N^T + N^{Mag} \quad (37)$$

where

$$N^T = \frac{1}{1-\nu} \int_{-h/2}^{h/2} \alpha_0 E_0 \Delta T dz \quad (38)$$

$$N^{Mag} = \int_{-h/2}^{h/2} q_{31} \psi dz \quad (39)$$

where

$$\Delta T = T_{cr} - 273.15K \quad (40)$$

In this examination, it is supposed that the temperature is distributed constantly and different kinds of temperature distributions are not explored.

### 3 Solution process

A structure can be mathematically analyzed by implementing different edge/end conditions. However, [84] endorsed that if an FGM structure is considered, taking the physics of the structure into account incorporating a shift of neutral surface ( $z$ ), the bifurcation cannot occur while some supports particularly simply supported end conditions are modeled. In this manner, the specimen tends to bend instead of buckling.

Galerkin weighted residual procedure technique (GWRM) which is an estimated method is postulated to give the answer of Eqs. (34–36). In this technique, the boundary conditions can be satisfied based on some trial functions. Based on an average value and integration over the domain, integral formulations diminish the error in the method. The GWRM can be a relatively straightforward method compared to the numerical solution techniques. Given Eqs. (34–36) in the form  $H[y(x), x] = 0$  over  $0 < x < L$  exposed to homogeneous boundary conditions ( $y(0) = y(L) = 0$ )

Thereby, in continue, the approximate solutions for fully fixed end conditions are sought as

$$u(x) = \sum_{m=1}^{\infty} U_m \frac{d}{dx} X_m(x) \exp(i\omega_n t) \quad (41)$$

$$w(x) = \sum_{m=1}^{\infty} W_m X_m(x) \exp(i\omega_n t) \quad (42)$$

$$\phi(x) = \sum_{m=1}^{\infty} \Phi_m Y_m(x) \exp(i\omega_n t) \quad (43)$$

where the clamped end conditions can be satisfied by the preceding series including the following trial functions

$$X_m = \sin^2\left(\frac{m\pi}{L}x\right) \quad (44)$$

$$Y_m = \cos^2\left(\frac{m\pi}{L}x\right) \quad (45)$$

The primary necessity is that the trial functions should be admissible functions. It means these functions should be able to satisfy the desired boundary conditions and should be continuous over the domain. Regarding these conditions, Eqs. (41–43) are unlikely exact solutions. Thus, implementing Eqs. (41–43) into Eqs. (34–36) gives residual errors as

$$R_m(x) = H[c_m(x), x] \neq 0 \quad (46)$$



where

$$c_m(x) = \begin{Bmatrix} u(x) \\ w(x) \\ \phi(x) \end{Bmatrix} \quad (47)$$

in which  $R_m(x)$  ( $m = u, w, \phi$ ) illustrated the residuals and means the residuals are the function of unknown displacements as well.

The integration of the weighted residual error over the domain of the problem is zero. To this aim, the following integrals assist GWRM in calculating the unknown parameters as

$$\int_0^L R_m^u(x) u(x) dx = 0 \quad (48)$$

$$\int_0^L R_m^w(x) w(x) dx = 0 \quad (49)$$

$$\int_0^L R_m^\phi(x) \phi(x) dx = 0 \quad (50)$$

These integrations result in three algebraic equations. Note that the GWRM can capture the exact solution under certain conditions (e.g., a linear problem with simply-supported boundary conditions).

Arranging the terms of Eqs. (48–50) with respecting unknown parameters  $u$ ,  $w$ , and  $\phi$  in a matrix form leads to a generalized eigenvalue problem as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} u \\ w \\ \phi \end{Bmatrix} = T_{cr} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{Bmatrix} u \\ w \\ \phi \end{Bmatrix} \quad (51)$$

where  $[K]$  and  $[S]$  are the stiffness matrix coefficients without and with  $T_{cr}$  terms, respectively.

After that the coefficient matrix's determinant gives a polynomial characteristic equation of axial stability of the smart FGM structure consisting of flexomagneticity,

$$\det(K_{ij} - T_{cr} S_{ij}) = 0 \quad (52)$$

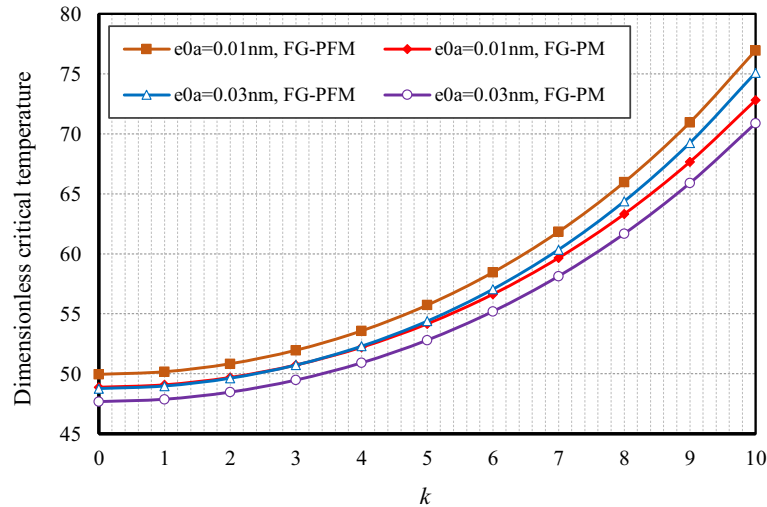
Subsequently, solving the attained polynomial equation results in having values of the critical buckling temperature ( $T_{cr}$ ).

#### 4 Results and discussions

To evaluate the flexomagnetic response inside the proposed piezomagnetic functionally graded nanobeam (FG-PFM), it is deemed that an exponential functionality fabricates the FG-PFM structure. Hence, Table 1 [85–87] in detail indicates the structural properties needed for this section. Let us here see that the FGM can emboss the role of flexomagneticity or not. Therefore, a scientific interpretation is observed based on Figs. 2, 3, 4 and 5. It is urgent to note that two states are considered in the parametric study: a piezomagnetic FGM which holds the flexomagnetic effect (FG-PFM) and a piezomagnetic FGM which excludes the flexomagnetic influence (FG-PM). Let us note that in all figures, the horizontal axis is the index  $k$ , and the vertical axis is the amount of heat stability capacity of the nanobeam. We must also confirm that the results were calculated only for the first buckling mode based on the dimensionless critical temperature  $T^* = 10^2 \alpha_0 T_{cr}$ . The values of the other variables are included next to the titles of the figures.

We start presenting the results by changing the nonlocal coefficient in Fig. 2. The most apparent possible consequence from the diagram is that increasing the coefficient  $k$  will increase the thermal stability of the nanobeams, and more importantly, the higher the value of  $k$ , the more significant the difference between the curve of the FG-PFM nanobeam and the FG-PM one. Of course, this distance between the results is crucial in



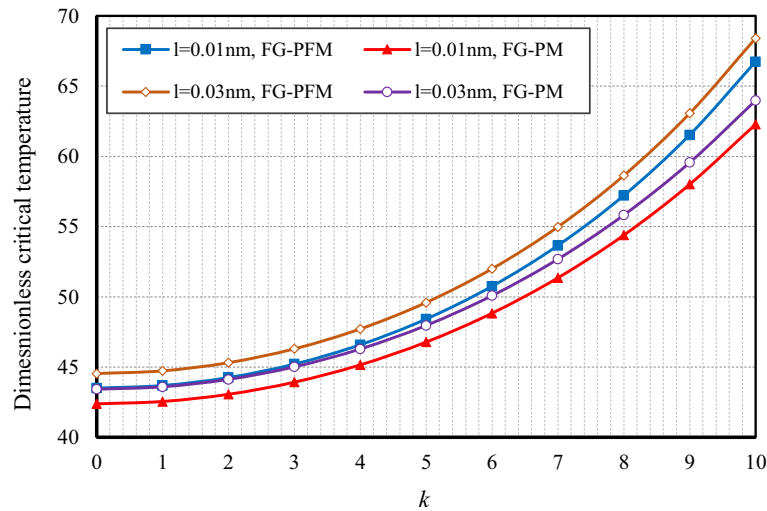
**Table 1** Available material propertiesCoFe<sub>2</sub>O<sub>4</sub> $E = 286 \text{ GPa}$  $f_{31} = 10^{-9} \text{ N/A}$  $q_{31} = 580.3 \text{ N/A}\cdot\text{m}$  $a_{33} = 1.57 \times 10^{-4} \text{ N/A}^2$  $\alpha = 11.80 \times 10^{-6} \text{ 1/K (room temperature)}$ **Fig. 2** FGM variation index versus critical temperature for various cases ( $\Psi = 1 \text{ mA}$ ,  $l = 0.05 \text{ nm}$ ,  $L/h = 1$ )

the immense value of the nonlocal coefficient. Therefore, a significant outcome that can be deduced from this figure is that producing a piezomagnetic material in the skeleton of an FGM structure, while the cross-sectional FG properties follow an exponential function, increase the flexo-effect, and this will be even more momentous when the value of nonlocality parameter is larger.

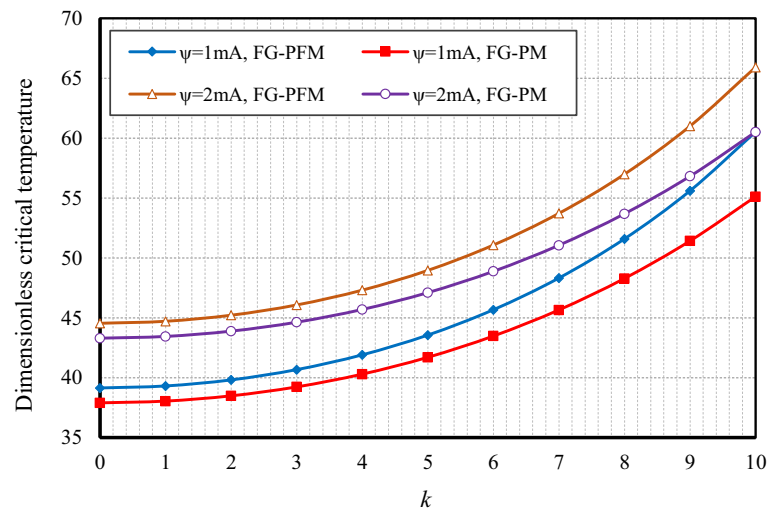
Many references have shown that the nonlocal coefficient in the relationships and modeling of the small-scale problems leads to a reduction in material stiffness. The length scale strain gradient parameter (SGLS) that exists in the couple stress relations and the first and second Mindlin gradients helps to increase the material stiffness. Therefore, regardless of Fig. 3, we must conclude that the results of Fig. 2 in Fig. 3 will be obtained when the SGLS parameter has smaller values. That is, here, a smaller value of the SGLS enhances the importance of flexo. Of course, the most substantial result remains in place, which means in Fig. 3, we again find that increasing  $k$  has raised the importance of the flexo-effect. Let us examine and compare Figs. 2 and f3 more carefully. We will come to the important conclusion that the increase in the difference between the FG-PFM and FG-PM results for  $l = 0.01 \text{ nm}$  in Fig. 3 is greater than  $e_0a = 0.03 \text{ nm}$  in Fig. 2. Therefore, it can be said that the SGLS parameter is more effective than the nonlocal parameter in this part.

Figure 4 is drawn to investigate the effect of the magnetic field. Following the previous diagrams, increasing  $k$  will develop the critical temperature stability. The amount of magnetic potential produced by the magnetic field is given in milliamperes. The point to consider in this figure is that the results initially differ for two different potential values, namely one and two milliamperes, unlike the previous figures. Additionally, it is evident that the difference in the results obtained after increasing  $k$  is greater than those in the last two figures. That is, the difference between FG-PM and FG-PFM is more pronounced than in Figs. 2 and 3. This figure shows that the role of magnetic potential in magnetic FGMs will be more prominent. Also, the difference in the results of PFM and PM at  $k = 10$  is the same for both magnetic potentials, thus showing that different values of the magnetic potential have the same effect on the behavior of flexomagnetism.

Let us measure the results and the relevant discussion by showing the effectiveness of changes in length to thickness coefficient (slenderness ratio). The nanobeam is designed in two modes, relatively thick ( $L/h = 10$ ) and relatively thin ( $L/h = 20$ ). At first glance, the point requiring evaluation is that the results are increasing for three modes (FG-PM for  $L/h = 10$  and  $L/h = 20$ , and FG-PFM for  $L/h = 20$ ) with almost the same slope.



**Fig. 3** FGM variation index versus critical temperature for various cases ( $\Psi = 1$  mA,  $ea = 0.05$  nm,  $L/h = 1$ )



**Fig. 4** FGM variation index versus critical temperature for various cases ( $l = 0.05$  nm,  $ea = 0.1$  nm,  $L/h = 1$ )

However, the model FG-PFM,  $L/h=10$ , will have a greater slope and a more considerable increase in the relatively thick beam manner. At  $k = 10$ , this model distinguishes itself more than other models. This is an excellent argument to confirm that although the shear deformation is severe in relatively thick and thick beams, it will be doubly important if the flexomagnetic effect is analyzed in these beams.

## 5 Conclusions

The Galerkin weighted residual method (GWRM) warranted the numerical results in the framework of analytical solutions for fully fixed ends conditions. The beam's behavior depended on the shear deformations; therefore, the Timoshenko beam was taken into the model. The examination in nanoscale was revealed by exerting both stress nonlocality and strain gradient in the media of the nonlocal strain gradient approach. The implementation of the material composition was presumed as exponential functionality concerning the rule of mixture. Inclusive of flexomagneticity (FM) was performed in terms of reverse field effect. Under the axially compressed conditions of the system, the critical buckling temperature was explored. Notwithstanding that the functionally graded materials (FGMs) can be correctly analyzed so that the mid-plane plays the role of the

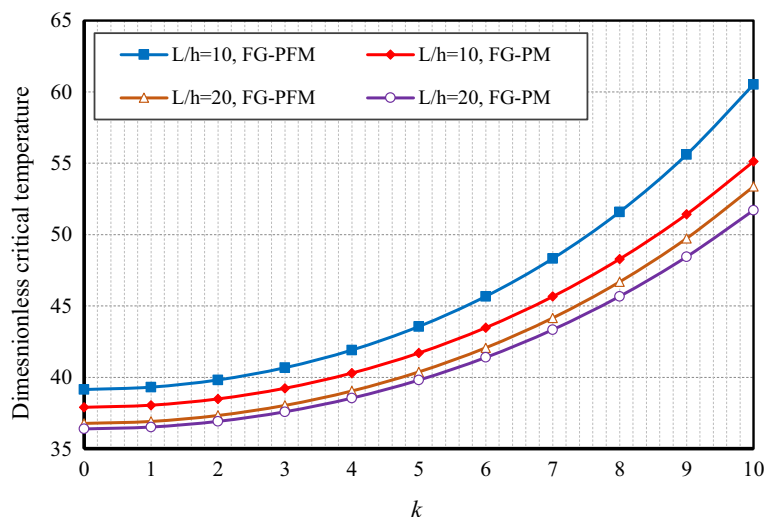


Fig. 5 FGM variation index versus critical temperature for various cases ( $\Psi = 1$  mA  $l = 0.05$  nm,  $ea = 0.1$  nm)

neutral surface, this research took into account the physical neutral plane that differs from the mid-surface. Furthermore, it was obtained.

- The flexomagnetism would be more visible in FGMs, while shear deformations exist. This study provides and offers new principal aspects that suit the designing of small-scale actuators/sensors.
- It was observed that flexomagnetism could be even more vital in a ferroic functionally graded material.

**Acknowledgements** V.A. Eremeyev acknowledges the support of the Government of the Russian Federation (Contract No. 14.Z50.31.0046).

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## Declarations

**Author contribution** Mohammad Malikan involved in conceptualization, methodology, investigation, software, visualization, validation and writing—original draft. Tomasz Wiczenbach involved in writing—original draft. Victor A. Eremeyev involved in investigation, writing—review and editing, supervision and funding acquisition.

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