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# To description of nonlinear interactions of wave and non-wave modes in a non-adiabatic plasma flow

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## Abstract

The method of derivation of nonlinear equations for interacting modes is explained and applied to a plasma's flow affected by a magnetic field. It bases on the linear projecting of the total perturbation field into specific variations of variables in individual modes of a flow. The method may be applied in many examples of fluid flows with different mechanisms of non-adiabaticity. It is of especial importance in the complex flows with the large number of various modes. A flow of an ionized gas is one of examples of such flows: it incorporates fast, slow magnetosonic modes, the Alfvén mode and the entropy mode. In the frames of this study, the wave vector may form an arbitrary angle  $\theta$  with the equilibrium straight magnetic field. Thermal conduction, dynamic viscosity and a generic heating -cooling function which is responsible for an energy inflow and radiative losses, are taken into account. Variable  $\theta$  and plasma- $\beta$ , diversity of the wave modes and a balance of different mechanisms of non-adiabaticity makes the description of nonlinear interaction of modes especial. The nonlinear excitation of the magnetosonic mode by the entropy mode is considered as an example.

## Keywords

Nonlinear magnetohydrodynamics, Acoustic activity, Projecting operators

## 1 Introduction

The nonlinear character of processes in a fluid flow not only results in distortion of waveforms in the course of propagation, but to the nonlinear effects associated with interaction of various wave and non-wave modes. In particular, excitation of non-wave modes by an intense wave, scattering sound by sound or sound by non-wave modes attract special attention. The entropy mode is the non-wave mode which is responsible for isobaric variations of the medium's equilibrium temperature. If the entropy mode is excited in the field of intense sound, it is associated with acoustic heating. The induced entropy perturbations are not longer stationary but slowly vary in time forming a new background for propagation of the waves [1, 2]. The nonlinear

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2 distortion of an intense mode waveform is also some kind of the nonlinear self-interaction. All  
3 variety of weakly nonlinear interactions may be described by means of projecting. Projecting  
4 in the context of fluid dynamics originates from the system of linear PDEs representing the  
5 conservation laws. Projection operators may be readily evaluated by use of dispersion relations  
6 which determine every mode uniquely along with the linear links of thermodynamic pertur-  
7 bations which specify all modes. The projectors (their number coincides with the number of  
8 modes of the linear flow) distinguish a specific perturbation for some mode from the total vec-  
9 tor of perturbations. The projecting is successfully applied in derivation of systems of coupling  
10 weakly nonlinear equations of the first order with respect to time. Many examples of applica-  
11 tion to various fluid flows may be found in Ref.[3]. **These applications are not connected with a**  
12 **plasma's flow and refer to flows of newtonian and non-newtonian fluids (such as the Bingham**  
13 **plastic and this with non-zero yield stress). Nonlinear flows of the non-uniform in equilibrium**  
14 **media affected by the forces and bubbly liquids have been successfully studied by means of the**  
15 **method. The important application of the method concerns non-equilibrium flows such as gases**  
16 **with excited internal degrees of molecules freedom and gases where a chemical reaction occurs.**  
17 **The nolinear phenomena in these flows may be unusual. This happens to acoustically active**  
18 **flows. The procedure is represented by algorithmic set of actions and hence is especially useful**  
19 **in the complex flows with large amount of modes, including flows with different mechanisms**  
20 **of disturbed adiabaticity. The advantage of the method is individual tracking of the modes in**  
21 **the course of all possible nonlinear interactions in a flow. In particular, distortion of any wave**  
22 **mode due to nonlinear interaction with wave and non-wave modes may be described in the**  
23 **frames of the method. The acoustic streaming and heating (that is, excitation of the vortex**  
24 **and entropy mode) are covered by the method. The method deals with the instantaneous fields**  
25 **and does not require averaging perturbations. Thus, it allows to determine detailed temporal**  
26 **evolution due to impulsive and non-periodic excitors. The only limitation of the method is**  
27 **weak nonlinearity, that is, comparatively small perturbations of the thermodynamic variables.**

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34 Close attention to the wave processes and related nonlinear phenomena in a plasma con-  
35 stantly grows (e.g. [4, 5, 6]). The variety of magnetohydrodynamic (MHD) modes provides  
36 diversity of their nonlinear interactions in a plasma's flow. In particular, the properties of a  
37 flow depend on plasma- $\beta$ , geometry of a flow and direction and intensity of the magnetic field.  
38 In view of difficulty in general analytical description, we focus on the planar flow of an ideal  
39 ionized gas affected by the straight equilibrium magnetic field forming constant angle  $\theta$  with  
40 the wave vector. A deviation from a flow's adiabaticity plays the key role in MHD dynamics.  
41 We consider three reasons of the non-adiabaticity:

- 42 1) some kind of heating-cooling function which describes inflow of energy into a flow and radia-  
43 tive losses;
- 44 2) thermal conduction of a medium;
- 45 3) dynamic viscosity of a plasma.

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48 While thermal conduction and dynamic viscosity always result in damping of wave perturba-  
49 tions, the heating-cooling function may enhance them, and eventual scenario of a wave process  
50 depends on the balance of these mechanisms. The nonlinear phenomena may occur unusually  
51 due to external inflow of energy. **This concerns all fluid flows with disturbed adiabaticity, not**  
52 **only plasma flows** [5, 6, 7, 8, 9]. In particular, a perturbation in temperature specifying the  
53 entropy mode which is induced in the wave field, may take negative value which corresponds  
54 to the cooling of a medium [10]. An accurate description of various nonlinear interactions of  
55 modes, and in particular, of self-interaction of intense wave and the excited secondary modes  
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are of especial importance in plasma physics. They may indicate physical processes in a plasma and its properties in remote observations and point a way to explain and control them. The projecting yields all variety of coupling nonlinear equations which describe interaction of modes. **The system may be simplified if one mode is dominant. In this case, only this mode contributes to quadratic "sources" exciting secondary modes.** The text is organized as follows. Sec.2 reminds the initial system of PDEs describing a flow of an ideal plasma including contribution of generic heating-cooling function and mechanical and thermal losses. It considers modes in a linear flow and conditions of isentropic and thermal instability. Projectors as a tool to obtain linear and weakly nonlinear dynamic equations are discussed and derived from the linearized conservation equations in Sec.3. The nonlinear corrections to the modes and some example of nonlinear excitation (excitation of the magnetosonic mode by the entropy one) are considered in Sec.4 and 5. Sec.6 represents concluding remarks.

## 2 Modes in the linear MHD flow

Ideal magnetohydrodynamics is a reasonably good approximation in most cases of astrophysical plasmas such as neutron star magnetosphere, solar atmosphere and Earth's magnetosphere. We make use of a set of ideal MHD equations describing perfectly electrically conducting and fully ionized gas. It includes the continuity equation, the momentum equation, the energy balance equation and electrodynamic equations in the differential form [11, 12, 13, 14, 15, 16]:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0, \\ \rho \frac{D\vec{v}}{Dt} &= -\vec{\nabla} p + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} + \eta \Delta \vec{v} + \frac{1}{3} \eta \vec{\nabla} (\vec{\nabla} \cdot \vec{v}), \\ \frac{Dp}{Dt} - \gamma \frac{p}{\rho} \frac{D\rho}{Dt} &= (\gamma - 1) \left[ L(p, \rho) + \vec{\nabla} \cdot (\chi \vec{\nabla} T) + \frac{\eta}{2} \sum_{i,j=1,2,3} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{i,k} \vec{\nabla} \cdot \vec{v} \right)^2 \right], \\ \frac{\partial \vec{B}}{\partial t} &= \vec{\nabla} \times (\vec{v} \times \vec{B}), \\ \vec{\nabla} \cdot \vec{B} &= 0, \end{aligned} \quad (2.1)$$

where  $p$ ,  $\rho$ ,  $T$ ,  $\vec{v}$  are thermodynamic pressure, mass density and temperature of a plasma and its velocity. The irreversible losses due to mechanical friction are described by the terms proportional to the dynamic viscosity  $\eta$ . This coefficient is named as "compressional viscosity" in the Braginskii viscous tensor. This term does not reflect the physical nature of losses described by the traceless stress tensor since they are not connected with compressibility of a fluid but with the friction between layers of different velocity. It was attributed by "compressional" because it contains the term proportional to  $\vec{\nabla} \cdot \vec{v}$  (see also explanations by Ruderman et al. [17] and initial equations in Ref.[14]). Other four terms in the Braginskii's viscous tensor are not considered in this study. They are typically strongly dominated by the dynamic viscosity [14]. The second term is at least five orders of magnitude smaller in the solar corona and at least two orders of magnitude smaller in the upper chromosphere [17]. Absorption due to finite electric resistivity and the Hall effect may be also neglected (see, e.g., discussion by Ruderman et al. [17, 18]). The coefficient of thermal conduction is designated by  $\chi$ . As it follows from the

classical Braginskii transport theory, the thermal conduction parallel to the magnetic field is much larger than the perpendicular one,  $\chi_{\perp} \ll \chi_{\parallel}$  and contribution of only parallel component may be usually considered [19]. The generic heating-cooling function  $L(p, \rho)$  is responsible for the radiative cooling and inflow of energy into a system [5, 20]. The magnetic field is denoted by  $\vec{B}$ , and  $\mu_0$  is the permeability of the free space. The third equation in the set (2.1) incorporates the continuity equation and the energy balance. It is valid for an ideal gas with the ratio of specific heats under constant pressure and constant density  $\gamma$ ,  $\gamma = C_P/C_V$ . The fourth equation is the ideal induction equation, and the fifth one is the Maxwell's equation reflecting solenoidal character of  $\vec{B}$ .

We follow common conditions in regard to the geometry of a flow: we assume that the wave vector of a planar flow forms a constant angle  $\theta$  ( $0 \leq \theta \leq \pi$ ) with the constant straight equilibrium magnetic field  $\vec{B}_0$  [21, 5, 20, 22, 23]. The direction of the wave vector is pointed by axis  $z$ . The  $y$ -component of  $\vec{B}_0$  equals zero, so as

$$B_{0,x} = B_0 \sin(\theta), \quad B_{0,z} = B_0 \cos(\theta), \quad B_{0,y} = 0.$$

The ideal induction equation and the Maxwell's equation ensure zero perturbation of  $B_z$ . Hence, the number of unknowns reduces from eight to seven. The system (2.1) is nonlinear. In the frames of weakly nonlinear flows, its leading-order form considered, valid with accuracy up to quadratic nonlinear terms [20, 10, 24]. A flow with infinitely-small perturbations of thermodynamic variables (that is, a linear flow) is described by the linearized version of the system (2.1). All thermodynamic quantities are expanded around the equilibrium thermodynamic state (designated by lower index 0) as  $f(z, t) = f_0 + f'(z, t)$ . The bulk flows are absent, so as  $\vec{v}_0 = \vec{0}$ . Seven equations form the leading-order system which includes first partial derivatives of variables  $\rho'$ ,  $v_x$ ,  $v_y$ ,  $v_z$ ,  $p'$ ,  $B_x$ ,  $B_y$  with respect to time:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_z}{\partial z} &= -\rho' \frac{\partial v_z}{\partial z} - v_z \frac{\partial \rho'}{\partial z}, \\ \frac{\partial v_x}{\partial t} - \frac{B_{0,z}}{\rho_0 \mu_0} \frac{\partial B_x}{\partial z} - \frac{\eta}{\rho_0} \frac{\partial^2 v_x}{\partial z^2} &= -v_z \frac{\partial v_x}{\partial z} - \frac{B_{0,z}}{\rho_0^2 \mu_0} \rho' \frac{\partial B_x}{\partial z} - \frac{\eta}{\rho_0} \frac{\rho'}{\rho_0^2} \frac{\partial^2 v_x}{\partial z^2}, \\ \frac{\partial v_y}{\partial t} - \frac{B_{0,z}}{\rho_0 \mu_0} \frac{\partial B_y}{\partial z} - \frac{\eta}{\rho_0} \frac{\partial^2 v_y}{\partial z^2} &= -v_z \frac{\partial v_y}{\partial z} - \frac{B_{0,z}}{\rho_0^2 \mu_0} \rho' \frac{\partial B_y}{\partial z} - \frac{\eta}{\rho_0} \frac{\rho'}{\rho_0^2} \frac{\partial^2 v_y}{\partial z^2}, \\ \frac{\partial v_z}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{B_{0,x}}{\rho_0 \mu_0} \frac{\partial B_x}{\partial z} - \frac{4\eta}{3\rho_0} \frac{\partial^2 v_z}{\partial z^2} &= \frac{\rho'}{\rho_0^2} \frac{\partial p'}{\partial z} + \frac{B_{0,x}}{\rho_0^2 \mu_0} \rho' \frac{\partial B_x}{\partial z} - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{B_x^2 + B_y^2}{2\mu_0} \right) - v_z \frac{\partial v_z}{\partial z} \\ &\quad - \frac{4\eta}{3\rho_0} \frac{\rho'}{\rho_0^2} \frac{\partial^2 v_z}{\partial z^2}, \\ \frac{\partial p'}{\partial t} + c^2 \rho_0 \frac{\partial v_z}{\partial z} - (\gamma - 1)(L_p p' + L_\rho \rho') - \frac{\chi}{\rho_0 C_P} \frac{\partial^2 \gamma p'}{\partial z^2} + \frac{\chi c_0^2}{\rho_0 C_P} \frac{\partial^2 \rho'}{\partial z^2} &= \\ (\gamma - 1)(0.5 L_{pp} p'^2 + 0.5 L_{\rho\rho} \rho'^2 + L_{p\rho} p' \rho') - \gamma p' \frac{\partial v_z}{\partial z} - v_z \frac{\partial p'}{\partial z} + (\gamma - 1) \frac{4\eta}{3\rho_0} \left( \frac{\partial v_z}{\partial z} \right)^2 + \\ (\gamma - 1) \frac{\eta}{\rho_0} \left[ \left( \frac{\partial v_x}{\partial z} \right)^2 + \left( \frac{\partial v_y}{\partial z} \right)^2 \right] - \frac{\chi}{\rho_0^2 C_P} \frac{\partial^2 (\gamma p' \rho' - c_0^2 \rho'^2)}{\partial z^2}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial B_x}{\partial t} + \frac{\partial}{\partial z} (B_{0,x}v_z - B_{0,z}v_x) &= -B_x \frac{\partial v_z}{\partial z} - v_z \frac{\partial B_x}{\partial z}, \\ \frac{\partial B_y}{\partial t} - \frac{\partial}{\partial z} (B_{0,z}v_y) &= -B_y \frac{\partial v_z}{\partial z} - v_z \frac{\partial B_y}{\partial z}, \end{aligned}$$

where

$$L_p = \frac{\partial L}{\partial p}, \quad L_\rho = \frac{\partial L}{\partial \rho}, \quad L_{pp} = \frac{\partial^2 L}{\partial p^2}, \quad L_{\rho\rho} = \frac{\partial^2 L}{\partial \rho^2}, \quad L_{p\rho} = \frac{\partial^2 L}{\partial p \partial \rho}$$

designate partial derivatives of the heating-cooling function  $L(p, \rho)$  with respect to its variables evaluated at the equilibrium state  $(p_0, \rho_0)$  ( $L(p_0, \rho_0) = 0$ ). The dispersion relations are usual starting point. We look for solutions to the linearized equations (2.1) in the form of a sum of planar waves proportional to  $\exp(i\omega(k)t - ikz)$  ( $k$  designates the wave number). All evaluations in this study (dispersion relations, links between specific perturbations, dynamic equations) are leading-order, that is, they contain terms up to the first powers of  $L_p, L_\rho, L_{pp}, L_{\rho\rho}, L_{p\rho}, \chi, \eta$ . This concerns also quadratic nonlinear terms. Two Alfvén branches (A), four magnetosonic branches (ms) and one entropy mode are inherent to the flow. They are determined by the dispersion relations which reflect the solvability of the linearized equations (2.1):

$$\omega_A = \pm C_{A,z}k + i\frac{\eta}{2\rho_0}k^2, \quad \omega_{ms} = Ck - iCD + i\frac{\alpha}{2}k^2, \quad \omega_{ent} = i\frac{\chi}{C_P\rho_0}k^2 + i\frac{(\gamma-1)}{c_0^2}L_\rho, \quad (2.3)$$

where  $c_0$  and  $C_A$

$$c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad C_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$$

designate the acoustic speed in non-magnetized gas and the Alfvén speed in equilibrium,

$$C_{A,z} = C_A \cos(\theta) = \frac{B_{0,z}}{\sqrt{\mu_0 \rho_0}},$$

and  $C$  is one from four magnetosonic speeds satisfying the equation [5, 20]

$$C^4 - C^2(c_0^2 + C_A^2) + c_0^2 C_{A,z}^2 = 0. \quad (2.4)$$

The magnetosonic speeds depend on  $\theta$  and plasma- $\beta$ ,

$$\beta = \frac{2}{\gamma} \frac{c_0^2}{C_A^2}.$$

They represent two branches (fast and slow) propagating in the positive direction of axis  $z$  and two branches propagating in the negative direction of axis  $z$  with the same speeds.  $D$  reflects the linear impact of the heating-cooling function on the magnetosonic waves

$$D = \frac{C(C^2 - C_A^2)(\gamma - 1)}{2c_0^2(C^4 - c_0^2 C_{A,z}^2)} (c_0^2 L_p + L_\rho),$$

and  $\alpha$  is the total damping coefficient due to dynamic viscosity and thermal conduction [5, 25]:

$$\alpha = \alpha_\eta \frac{4\eta}{3\rho_0} + \alpha_\chi \frac{\chi \left( \frac{1}{C_V} - \frac{1}{C_P} \right)}{\rho_0}, \quad (2.5)$$

where

$$\alpha_\eta = \frac{C^4 + C^2(6c_0^2 - C_A^2) - 3c_0^2(c_0^2 + C_A^2)}{4c_0^2(C^4 - c_0^2 C_{A,z}^2)} C^2, \quad \alpha_\chi = \frac{C^2(C^2 - C_A^2)}{C^4 - c_0^2 C_{A,z}^2}.$$

The magnetosonic perturbations may enhance if a linear flow is adiabatically unstable [26, 27], that is, if

$$c_0^2 L_p + L_\rho > 0 \quad (2.6)$$

and the total damping is weak:

$$DC > \frac{\alpha}{2} k^2. \quad (2.7)$$

The condition of thermal instability supplemented by weakness of thermal conduction

$$\frac{(\gamma - 1)}{c_0^2} L_\rho > \frac{\chi}{C_P \rho_0} k^2$$

ensures enlargement of perturbations specifying the entropy mode in a linear flow [23]. Links of perturbations in any individual mode are determined by the corresponding dispersion relation. In particular, the Alfvén modes are specified by the relations

$$\psi_A = \begin{pmatrix} \rho' \\ v_x \\ v_y \\ v_z \\ p' \\ B_x \\ B_y \end{pmatrix}_A = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \mp \frac{B_0}{C_A} - \eta \frac{B_0}{2\rho_0 C_A C_{A,z}} \frac{\partial}{\partial z} \end{pmatrix} v_{A,y}. \quad (2.8)$$

Four magnetosonic branches are established by the relations:

$$\psi_{ms} = \begin{pmatrix} \rho' \\ v_x \\ v_y \\ v_z \\ p' \\ B_x \\ B_y \end{pmatrix}_{ms} = \quad (2.9)$$



$$\begin{pmatrix}
1 \\
-\frac{C_{A,z}(C^2-c_0^2)}{C_{A,x}C\rho_0} + \frac{(\gamma-1)C_{A,z}(C^2-c_0^2)(C^2-2c_0^2-C_A^2)}{2c_0^2C_{A,x}(2C^2-c_0^2-C_A^2)\rho_0} \left[ \frac{c_0^2}{C^2} \frac{\chi}{C_P\rho_0} \frac{\partial}{\partial z} + \frac{(c_0^2L_p+L_\rho)}{C^2} \int dz \right] \frac{\partial}{\partial z} + \\
\frac{C_{A,z}(C^2-c_0^2)(C^4+C^2(4c_0^2-C_A^2)-3c_0^2(c_0^2+C_A^2))}{\eta 6C_{A,x}C^2c_0^2(2C^2-c_0^2-C_A^2)\rho_0} \frac{\partial}{\partial z} \\
0 \\
\frac{C}{\rho_0} + (\gamma-1) \left[ \frac{c_0^2}{C^2} \frac{\chi}{C_P\rho_0} \frac{\partial}{\partial z} + \frac{(c_0^2L_p+L_\rho)}{C^2} \int dz \right] - \frac{\eta}{\rho_0} \frac{C^4+C^2(6c_0^2-C_A^2)-3c_0^2(c_0^2+C_A^2)}{6c_0^2(2C^2-c_0^2-C_A^2)\rho_0} \frac{\partial}{\partial z} \\
c_0^2 - (\gamma-1) \left[ \frac{c_0^2}{C} \frac{\chi}{C_P\rho_0} \frac{\partial}{\partial z} + \frac{(c_0^2L_p+L_\rho)}{C} \int dz \right] \\
\frac{(C^2-c_0^2)\mu_0}{B_{0,x}} - \frac{C(C^2-c_0^2)(C^2-c_0^2-C_A^2)\mu_0}{B_{0,x}c_0^2(2C^2-c_0^2-C_A^2)} \left[ \left( \frac{\eta}{3\rho_0} - (\gamma-1) \frac{c_0^2}{C^2} \frac{\chi}{C_P\rho_0} \right) \frac{\partial}{\partial z} + \frac{(\gamma-1)(c_0^2L_p+L_\rho)}{C^2} \int dz \right] \\
0
\end{pmatrix} \rho_{ms}.$$

The following relations are inherent to the entropy mode:

$$\psi_{ent} = \begin{pmatrix} \rho' \\ v_x \\ v_y \\ v_z \\ p' \\ B_x \\ B_y \end{pmatrix}_{ent} = \begin{pmatrix} 1 \\ \frac{(\gamma-1)C_{A,x}L_\rho}{C_{A,z}c_0^2\rho_0} \int dz - \frac{\chi C_{A,x}}{C_{A,z}C_P\rho_0^2} \frac{\partial}{\partial z} \\ 0 \\ \frac{(\gamma-1)L_\rho}{c_0^2\rho_0} \int dz - \frac{\chi}{C_P\rho_0^2} \frac{\partial}{\partial z} \\ 0 \\ 0 \\ 0 \end{pmatrix} \rho_{ent}. \quad (2.10)$$

### 3 Projecting into the specific perturbations

All perturbations in a linear flow are sums of specific ones. Four magnetosonic operators  $P_{ms}$ , two Alfvén projectors  $P_A$  and the entropy projector  $P_{ent}$  follow immediately from the linear links for all modes. They are determined by the systems

$$P_{ms} \begin{pmatrix} \rho' & v_x & v_y & v_z & p' & B_x & B_y \end{pmatrix}^T = \rho_{ms},$$

$$P_A \begin{pmatrix} \rho' & v_x & v_y & v_z & p' & B_x & B_y \end{pmatrix}^T = v_{A,y},$$

and

$$P_{ent} \begin{pmatrix} \rho' & v_x & v_y & v_z & p' & B_x & B_y \end{pmatrix}^T = \rho_{ent}$$

( $T$  designates transpose). We use a perturbation of the magnetosonic density  $\rho_{ms}$  as the referential quantities for the magnetosonic projectors, a perturbation of the entropy density  $\rho_{ent}$



for the entropy projector, and  $y$  component of Alfvén velocity  $v_{A,y}$  for the Alfvén projectors. The referential quantities must be non-zero. The magnetosonic projectors take the form:

$$\begin{aligned}
 P_{ms} = & \left( \begin{aligned}
 & -\frac{(\gamma-1)C(C^2-C_A^2)}{2c_0^2(C^4-c_0^2C_{A,z}^2)}L_\rho \int dz + \chi \frac{C(C^3-C_A^3)}{2(C^4-c_0^2C_{A,z}^2)\rho_0 C_P} \frac{\partial}{\partial z} \\
 & -\frac{C_{A,x}C_{A,z}C\rho_0}{2(C^4-c_0^2C_{A,z}^2)} - (c_0^2L_p + L_\rho) \frac{(\gamma-1)C_{A,x}C_{A,z}(2C^6-3C^4C_{A,z}^2+c_0^2C_{A,z}^4)\rho_0}{2(C^4-c_0^2C_{A,z}^2)^3} \int dz \\
 & -\eta \frac{C^6C_{A,x}C_{A,z}(2C_{A,z}^2-c_0^2-C_A^2)}{6(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} - \chi \frac{(\gamma-1)c_0^2C_{A,x}C_{A,z}(2C^6-3C^4C_{A,z}^2+c_0^2C_{A,z}^4)}{2C_P(C^2-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} \\
 & 0 \\
 & \frac{C(C^2-C_{A,z}^2)\rho_0}{2(C^4-c_0^2C_{A,z}^2)} + (c_0^2L_p + L_\rho) \frac{(\gamma-1)(C^2-C_{A,z}^2)(C^6+c_0^2C_{A,z}^2C^2-3C^4C_{A,z}^2+c_0^2C_{A,z}^4)}{2(C^4-c_0^2C_{A,z}^2)^3} \int dz + \\
 & \eta \frac{C^4C_{A,z}^2(C^2-C_{A,z}^2)(C^2-c_0^2)}{3(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} + \chi \frac{(\gamma-1)c_0^2(C^8-4C^6C_{A,z}^2-c_0^2C_{A,z}^4+C^4C_{A,z}^2(c_0^2+3C_{A,z}^2))}{2C_P(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} \\
 & \frac{C^2-C_{A,z}^2}{2(C^4-c_0^2C_{A,z}^2)} + L_\rho \frac{(\gamma-1)(C^2-C_{A,z}^2)(3C^6+c_0^2C^2C_{A,z}^2-7C^4C_{A,z}^2+3c_0^2C_{A,z}^4)}{4C(C^4-c_0^2C_{A,z}^2)^3} \int dz + \\
 & L_p \frac{(\gamma-1)(C^2-C_{A,z}^2)C^3(2c_0^4+2C^4-5C^2C_A^2+C_A^4-3c_0^2(C^2-C_A^2))}{4(C^4-c_0^2C_{A,z}^2)^3} \int dz + \\
 & \eta \frac{(C^2-C_{A,z}^2)(4C^8-3c_0^2C^4C_{A,z}^2-5C^6C_{A,z}^2+3c_0^4C_{A,z}^4+c_0^2C^2C_{A,z}^2)}{12\rho_0C(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} - \\
 & \chi \frac{(C^2-C_{A,z}^2)(c_0^4C_{A,z}^2(-C_{A,z}^2(\gamma-3)-C^2(\gamma-1))+2\gamma C^8+c_0^2C^4((3\gamma-7)C_{A,z}^2-3(\gamma-1)C^2))}{4\rho_0C_P C(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} \\
 & \frac{C^2C_{A,x}C_{A,z}^2\rho_0}{2B_0C_A(C^4-c_0^2C_{A,z}^2)} + (c_0^2L_p + L_\rho) \frac{(\gamma-1)CC_{A,x}C_A(3C^6+c_0^2C^2C_{A,z}^2-5C^4C_{A,z}^2+c_0^2C_{A,z}^4)\rho_0}{4B_0(C^4-c_0^2C_{A,z}^2)^3} \int dz + \\
 & \eta \frac{CC_A C_{A,x}(4C^8-3c_0^2C^4C_{A,z}^2-3C^6C_{A,z}^2+3c_0^4C_{A,z}^4-c_0^2C^2C_{A,z}^2)}{12B_0(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} + \\
 & \chi \frac{(\gamma-1)c_0^2CC_A C_{A,x}(3C^6+c_0^2C^2C_{A,z}^2-5C^4C_{A,z}^2+c_0^2C_{A,z}^4)}{4C_P B_0(C^4-c_0^2C_{A,z}^2)^3} \frac{\partial}{\partial z} \\
 & 0
 \end{aligned} \right)^T \quad (3.11)
 \end{aligned}$$

Projectors into the Alfvén specific velocity and perturbation in density in the entropy mode, are

$$\begin{aligned}
 P_A &= \begin{pmatrix} 0 & 0 & \frac{1}{2} \mp \frac{\eta}{4\rho_0 C_{A,z}} \frac{\partial}{\partial z} & 0 & 0 & 0 & \mp \frac{C_A}{2B_0} \end{pmatrix}, \\
 P_{ent} &= \begin{pmatrix} 1 & -\frac{(\gamma-1)C_{A,x}\rho_0}{C_{A,z}c_0^4} \left( (c_0^2L_p + L_\rho) \int dz + c_0^2 \frac{\chi}{C_P \rho_0} \frac{\partial}{\partial z} \right), \\ 0 & -\frac{(\gamma-1)\rho_0}{c_0^4} \left( (c_0^2L_p + L_\rho) \int dz + c_0^2 \frac{\chi}{C_P \rho_0} \frac{\partial}{\partial z} \right) & -\frac{1}{c_0^2} & 0 & 0 \end{pmatrix}. \quad (3.12)
 \end{aligned}$$

In particular,

$$P_{ent} + \sum P_{ms} = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0), \quad \sum P_A = (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0)$$

in accordance to usual properties of projectors, since the total perturbation of density  $\rho'$  equals  $\rho_{ent} + \sum \rho_{ms}$ , and the total perturbation of  $v_y$  equals  $\sum v_{A,y}$ . In the unmagnetized flow,  $P_{ms}$  is converted into two operators ( $C = \pm c_0$ ,  $C_A = 0$ )

$$P_s = \begin{pmatrix} \mp \frac{(\gamma-1)L_\rho}{2c_0^3} \int dz \pm \frac{\chi}{2c_0 C_P \rho_0} \frac{\partial}{\partial z} \\ 0 \\ 0 \\ \pm \frac{\rho_0}{2c_0} + \frac{(\gamma-1)\rho_0(c_0^2 L_p + L_\rho)}{2c_0^4} \int dz + \frac{(\gamma-1)\chi}{2c_0^2 C_P} \frac{\partial}{\partial z} \\ \frac{1}{2c_0^2} \pm \left( \frac{\eta}{3\rho_0 c_0^3} + \frac{(\gamma-3)\chi}{4\rho_0 c_0^3 C_P} \right) \frac{\partial}{\partial z} \pm \frac{(\gamma-1)(c_0^2 L_p + 3L_\rho)}{4c_0^5} \int dz \\ 0 \\ 0 \end{pmatrix}^T, \quad (3.13)$$

and  $P_{ent}$  takes the form

$$P_{ent} = \begin{pmatrix} 1 & 0 & 0 & -\frac{(\gamma-1)\rho_0}{c_0^4} \left( (c_0^2 L_p + L_\rho) \int dz + c_0^2 \frac{\chi}{C_P \rho_0} \frac{\partial}{\partial z} \right) & -\frac{1}{c_0^2} & 0 & 0 \end{pmatrix}. \quad (3.14)$$

which coincides with the operators derived in Ref.[28] for  $L = 0$ .

## 4 Nonlinear corrections to the linear links

Going to studies of nonlinear interactions of the modes, the linear links should be corrected in the leading order, that is, by involving quadratic nonlinear terms. This is important for the mode which is treated as dominative compared to all other modes. The mode is dominative over spatial and temporal domains where its specific referential perturbations are much larger than that of the other modes. The corrections to the magnetosonic mode support the leading-order isentropicity of the wave motion (with accuracy up to quadratic nonlinear terms). The nonlinear corrections also yield a nonlinear term in the dynamic equation for the dominative

mode which reflects its self-interaction. The nonlinear corrections to the the dominative mode make possible the proper subdivision of all nonlinear terms between various dynamic equations when projectors apply at the system (2.2). Links for magnetosonic modes has been derived by the author in terms of  $v_{ms,z}$  in Ref.[10] and in terms of  $\rho_{ms}$  in Ref.[29]. They are determined by the vector  $\psi_{ms,nonl}$  and evidently coincide with the corrections to the isentropic Riemann wave in an unmagnetized gas [30]. The same algorithm to establish corrections is valid for any other dominative wave mode.

#### 4.1 Dominative Alfvén and entropy modes

This subsection explains how to evaluate corrections inherent to the Alfvén modes (say, the first branch) and the entropy mode. We seek corrections in the form of quadratic terms containing unknowns  $A_1, \dots, A_6$  which supplement the linear links for the Alfvén mode, so as

$$\begin{aligned} \rho_A &= A_1 v_{A,y}^2, & v_{A,x} &= A_2 v_{A,y}^2, & v_{A,z} &= A_3 v_{A,y}^2, & p_A &= A_4 v_{A,y}^2, & B_{A,x} &= A_5 v_{A,y}^2, \\ B_{A,y} &= -\frac{B_0}{C_A} v_{A,y} + A_6 v_{A,y}^2 \end{aligned}$$

and substitute them in the system (2.2) treating terms belonging to the other modes zero in view of dominance of the first Alfvén mode and disregarding all mechanisms of nonadiabaticity. We arrive to the system

$$\begin{aligned} 2v_{A,y} \left( A_1 \frac{\partial v_{A,y}}{\partial t} + A_3 \rho_0 \frac{\partial v_{A,y}}{\partial z} \right) &= 0, \\ 2v_{A,y} \left( A_2 \frac{\partial v_{A,y}}{\partial t} - A_5 \frac{B_{0,z}}{\mu_0 \rho_0} \frac{\partial v_{A,y}}{\partial z} \right) &= 0, \\ \frac{\partial v_{A,y}}{\partial t} + C_{A,z} \frac{\partial v_{A,y}}{\partial z} - 2A_6 \frac{B_{0,z}}{\mu_0 \rho_0} v_{A,y} \frac{\partial v_{A,y}}{\partial z} &= 0, \\ 2v_{A,y} \left( A_3 \frac{\partial v_{A,y}}{\partial t} + \left( \frac{A_4}{\rho_0} + A_5 \frac{B_{0,x}}{\mu_0 \rho_0} + \frac{B_0^2}{C_A^2 \rho_0 \mu_0} \right) \frac{\partial v_{A,y}}{\partial z} \right) &= 0, \tag{4.15} \\ 2v_{A,y} \left( A_4 \frac{\partial v_{A,y}}{\partial t} + A_3 c_0^2 \rho_0 \frac{\partial v_{A,y}}{\partial z} \right) &= 0, \\ 2v_{A,y} \left( A_5 \frac{\partial v_{A,y}}{\partial t} + (A_3 B_{0,x} - A_2 B_{0,z}) \frac{\partial v_{A,y}}{\partial z} \right) &= 0, \\ -\frac{B_{0,z}}{C_{A,z}} \left( \frac{\partial v_{A,y}}{\partial t} + C_{A,z} \frac{\partial v_{A,y}}{\partial z} \right) + 2A_6 v_{A,y} \frac{\partial v_{A,y}}{\partial t} &\approx -\frac{B_{0,z}}{C_{A,z}} \left( \frac{\partial v_{A,y}}{\partial t} + C_{A,z} \frac{\partial v_{A,y}}{\partial z} \right) - 2A_6 C_{A,z} v_{A,y} \frac{\partial v_{A,y}}{\partial z} = 0. \end{aligned}$$

In the last equality, the leading-order dynamic equation which corresponds to the first  $\omega_A$ :

$$\frac{\partial v_{A,y}}{\partial t} + C_{A,z} \frac{\partial v_{A,y}}{\partial z} = 0 \tag{4.16}$$

is used. Comparing third and seventh equations in (4.15), we obtain  $A_6 = 0$  and make conclusion that the dynamic equation for the Alfvén velocity does not include a nonlinear term. We also arrive at the system making every equation to satisfy Eq.(4.16):

$$\rho_0 A_3 = C_{A,z} A_1, \quad \frac{B_{0,z}}{\rho_0 \mu_0} A_5 = -C_{A,z} A_2, \quad \frac{A_4}{\rho_0} + \frac{B_{0,x} A_5}{\rho_0 \mu_0} + \frac{1}{\rho_0 \mu_0} \frac{B_0^2}{C_A^2} = C_{A,z} A_3,$$

$$c_0^2 \rho_0 A_3 = C_{A,z} A_4, \quad A_3 B_{0,x} - A_2 B_{0,z} = C_{A,z} A_5$$

The unknowns take the forms

$$A_1 = A_3 = A_4 = 0, \quad A_2 = \frac{1}{C_{A,x}}, \quad A_5 = -\frac{B_0}{C_A C_{A,x}},$$

if  $C_{A,x} \neq 0$ , and

$$A_2 = A_5 = 0, \quad A_1 = \frac{\rho_0}{C_A^2 - c_0^2}, \quad A_3 = \frac{C_A}{C_A^2 - c_0^2}, \quad A_4 = -\frac{c_0^2 \rho_0}{C_A^2 - c_0^2},$$

if  $C_{A,x} = 0$ , that is, in the case  $\theta = 0$  or  $\theta = \pi$ . Hence, the dominant Alfvén mode with the corrected links is specified not only by perturbations in the magnetic field  $B_{A,y}$  and transversal velocity  $v_{A,y}$ , but by other perturbations differently in the cases  $\theta = 0$ ,  $\theta = \pi$  and other values of  $\theta$ . For all  $\theta$ , the nonlinear corrections do not bring in nonlinear term in the evolutionary equation which takes the form Eq.(4.16) supplemented by the term proportional to the dynamic viscosity

$$\frac{\partial v_{A,y}}{\partial t} + C_{A,z} \frac{\partial v_{A,y}}{\partial z} - \frac{\eta}{2\rho_0} \frac{\partial^2 v_{A,y}}{\partial z^2} = 0. \quad (4.17)$$

It may be rearranged in the leading order into the diffusion equation which has an exact solution (e.g. [1]).

The non-wave entropy mode of infinitely-small magnitude is stationary in the flow with absence of the mechanisms disturbing adiabaticity. This imposes zero quadratic corrections to the specific perturbations. The dynamic equation does not include the quadratic nonlinear term and takes the form which follows from the dispersion relation  $\omega_{ent}$ :

$$\frac{\partial \rho_{ent}}{\partial t} - \frac{\chi}{C_P \rho_0} \frac{\partial^2 \rho_{ent}}{\partial z^2} + \frac{(\gamma - 1)L_\rho}{c_0^2} \rho_{ent} = 0. \quad (4.18)$$

The conclusion is that there is not leading-order nonlinear self-interaction for the both Alfvén modes and the entropy mode.

## 5 Nonlinear excitation of the secondary modes

Perturbations of infinitely-small magnitudes (that is, linear modes), propagate independently on each other. The linear dynamic equations for different specific perturbations do not couple. Application of some projector  $P$  at the linearized system (2.2) (which is briefly represented by

$$\frac{\partial}{\partial t} E \psi_{lin} + K \psi_{lin} = 0,$$

with  $K$  being the matrix operator containing spatial derivatives,  $E$  denoting the unit matrix) yields the evolutionary equation for the specific perturbation. In particular, application of  $P_{ms}$  results in the dynamic equation:

$$P_{ms} \left( \frac{\partial}{\partial t} E \psi_{lin} + K \psi_{lin} \right) = \frac{\partial \rho_{ms}}{\partial t} + C \frac{\partial \rho_{ms}}{\partial z} - DC \rho_{ms} - \frac{\alpha}{2} \frac{\partial^2 \rho_{ms}}{\partial z^2} = 0,$$

which reflects the dispersion relation for the magnetosonic modes (2.3). Application of  $P_{ms}$  at the system (2.2) with the nonlinear vector  $\tilde{\psi}$  on the right

$$\frac{\partial}{\partial t} E\psi_{lin} + K\psi_{lin} = \tilde{\psi},$$

distinguishes the dynamic equation for the magnetosonic density on the left and results in the variety of coupling nonlinear terms on the right. In particular, considering only magnetosonic perturbations inherent to  $P_{ms}$  in  $\tilde{\psi}$ , one arrives at the nonlinear evolutionary equation

$$\frac{\partial \rho_{ms}}{\partial t} + C \frac{\partial \rho_{ms}}{\partial z} - DC\rho_{ms} + \frac{C}{\rho_0} \varepsilon \rho_{ms} \frac{\partial \rho_{ms}}{\partial z} - \frac{\alpha}{2} \frac{\partial^2 \rho_{ms}}{\partial z^2} = 0,$$

where  $\varepsilon$  is responsible for the nonlinear self-interaction of the magnetosonic wave which is treated as dominative [5, 20]

$$\varepsilon = \frac{(\gamma + 4)C^2 - 3c_0^2 - (\gamma + 1)C_{A,z}^2}{2(C^4 - c_0^2 C_{A,z}^2)} C^2.$$

The dominance may be broken due to nonlinear interaction of modes and enlargement of the secondary perturbations. The leading order nonlinear corrections of the dominant mode should be taken into account in the description of the nonlinear excitation of the secondary modal fields. The magnetosonic modes do not excite Alfvén ones. This is due to zero third and seventh elements of  $P_{ms}$  (3.11) which only could couple with perturbations in the Alfvén modes. There is also no coupling between dominant Alfvén mode and the magnetosonic modes and dominative Alfvén mode and the entropy mode. Excitation of the entropy mode by intense magnetosonic mode may lead to heating or cooling associating with an isobaric variations in density of the entropy mode (magnetosonic heating or cooling). It was considered in [10, 24].

Applying  $P_{ms}$  at the system (2.2) and treating the entropy mode as dominant, we arrive at the dynamic equation

$$\frac{\partial \rho_{ms}}{\partial t} + C \frac{\partial \rho_{ms}}{\partial z} - DC\rho_{ms} + \frac{C}{\rho_0} \varepsilon \rho_{ms} \frac{\partial \rho_{ms}}{\partial z} - \frac{\alpha}{2} \frac{\partial^2 \rho_{ms}}{\partial z^2} = \frac{(C^2 - C_{A,z}^2)(\gamma - 1)}{4(C^4 - c_0^2 C_{A,z}^2)} L_{\rho\rho} \rho_{ent}^2, \quad (5.19)$$

or, making use of the leading-order link between  $\rho_{ms}$  and  $v_{ms,z}$  given by  $\psi_{ms}$  (Eqs(2.9)), to the equation in terms of  $v_{ms}$ :

$$\frac{\partial v_{ms,z}}{\partial t} + C \frac{\partial v_{ms,z}}{\partial z} - DCv_{ms,z} + \varepsilon v_{ms,z} \frac{\partial v_{ms,z}}{\partial z} - \frac{\alpha}{2} \frac{\partial^2 v_{ms,z}}{\partial z^2} = \frac{C(C^2 - C_{A,z}^2)(\gamma - 1)}{4\rho_0(C^4 - c_0^2 C_{A,z}^2)} L_{\rho\rho} \rho_{ent}^2. \quad (5.20)$$

$\rho_{ent}$  is dominant, so as  $|\rho_{ent}| \gg \max|\rho_{ms}|$  in the considered temporal and spatial domains. It satisfies Eq.(4.18) which may be readily rearranged into the diffusion equation for a function  $\tilde{\rho}(z, t)$  by the substitution

$$\rho_{ent}(z, t) = \tilde{\rho}(z, t) \exp\left(-\frac{(\gamma - 1)L_{\rho} t}{c_0^2}\right)$$

which has the well-known analytical solution in terms of the Green function. We focus on the case  $\chi = 0$  with a solution

$$\rho_{ent}(z, t) = \rho_{ent,0} \exp\left(-\frac{(\gamma - 1)L_{\rho} t}{c_0^2}\right),$$

where  $\rho_{ent,0} = \rho_{ent}(z, t = 0)$  is the initial excess density which specifies the entropy mode, some function of  $z$ . Eq.(5.20) is nonlinear and difficult for analytical solution. If  $D = 0$ , it may be rearranged in the leading order into the inhomogeneous Burgers equation which may be resolved analytically [30, 1]. For a simple analysis, we discard nonlinearity and damping due to thermal conduction and dynamic viscosity and rearrange Eq.(5.20) into the leading-order equation

$$\frac{\partial V}{\partial t} + C \frac{\partial V}{\partial z} = \exp\left(-\frac{2(\gamma-1)L_\rho}{c_0^2}t - DCt\right) \frac{C(C^2 - C_{A,z}^2)(\gamma-1)}{4\rho_0(C^4 - c_0^2 C_{A,z}^2)} L_{\rho\rho} \rho_{ent,0}^2 = \Phi(z, t). \quad (5.21)$$

by means of substitution

$$V = v_{ms,z} \exp(-DCt).$$

The solution to Eq.(5.21) for zero initial condition  $v_{ms,z}(z, t = 0) = 0$ , takes the form

$$v_{ms,z} = \exp(DCt) \int_0^t \Phi(C\tau - Ct + z, \tau) d\tau = \frac{C(C^2 - C_{A,z}^2)(\gamma-1)}{4\rho_0(C^4 - c_0^2 C_{A,z}^2)} \exp(DCt) L_{\rho\rho} \int_0^t \exp\left(-\frac{2(\gamma-1)L_\rho}{c_0^2}\tau - DC\tau\right) \rho_{ent,0}^2(C\tau - Ct + z) d\tau.$$

In the dimensionless variables

$$Z = kz, \quad T = Ckt, \quad \theta = Ck\tau, \quad F = \frac{2(\gamma-1)L_\rho}{c_0^2 Ck} + \frac{D}{k},$$

where  $k$  designates the characteristic inverse length of an exciting signal, Eq.(5.21) may be readily rearranged as

$$\frac{v_{ms,z}}{c_0} = \frac{(C^2 - C_{A,z}^2)(\gamma-1)}{4c_0\rho_0 k(C^4 - c_0^2 C_{A,z}^2)} \exp(DT/k) L_{\rho\rho} \int_0^T \exp(-F\theta) \rho_{ent,0}^2(\theta - T + Z) d\theta. \quad (5.22)$$

For example, let the entropy excess density takes initially the form of a Gaussian impulse

$$\rho_{ent,0} = R_0 \exp(-Z^2). \quad (5.23)$$

The integral in the right-hand part divided by  $R_0^2$ , takes the form

$$\frac{4v_{ms,z}\rho_0 k(C^4 - c_0^2 C_{A,z}^2)}{(\gamma-1)R_0^2(C^2 - C_{A,z}^2)L_{\rho\rho}} \exp(-DT/k) = \int_0^T \exp(-F\theta) \exp(-2(\theta - T + Z)^2) d\theta \approx \frac{\sqrt{\pi}}{2\sqrt{2}} \exp(F(Z - T)) \left( \operatorname{Erf}\left(\frac{F + 4Z}{2\sqrt{2}}\right) - \operatorname{Erf}\left(\frac{F - 4T + 4Z}{2\sqrt{2}}\right) \right). \quad (5.24)$$

In view of smallness of attenuation or amplification over the wave period due to deviation from adiabaticity (for magnetosonic perturbations to be a wave process),  $|F| \ll 1$ , and  $|\frac{DT}{k}| \ll 1$ .

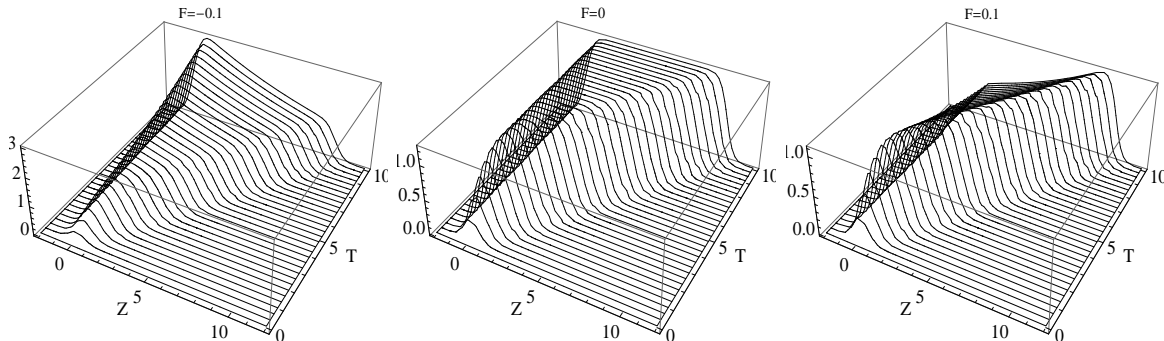


Figure 1.  $\frac{4v_{ms,z}\rho_0k(C^4 - c_0^2C_{A,z}^2)}{(\gamma-1)R_0^2(C^2 - C_{A,z}^2)L_{\rho\rho}} \exp(-DT/k)$  (Eq.(5.24)) for different  $F$  and Gaussian initial form of excess density of the entropy mode determined by Eq.(5.23).

The analysis undertaken by the author in the context of Newtonian flows has shown that the excited mode consists basically of parts propagating with the speed of dominant mode and the own linear speed of the excited mode [3, 31]. The nonlinear interaction breaks properties of "directivity" of the excited mode which is still determined by the linear links of specific perturbations. Fig.1 clearly reveals different speeds of the head and back fronts of an excited perturbation. The sum of the error functions in the samples leads to formation of some kind of plateau with variable height. The magnitude of the excited velocity varies as  $\exp\left(\frac{DT}{k} + F(Z - T)\right) = \exp\left(\frac{2(\gamma-1)L_p}{c_0^2 C k}(Z - T) + \frac{D}{k}Z\right)$ , hence, it is determined by  $L_p, L_{\rho}, \theta$  and plasma- $\beta$ . The quantity  $v_{ms,z}$  is proportional to

$$\frac{(C^2 - C_{A,z}^2)(\gamma - 1)}{4c_0\rho_0k(C^4 - c_0^2C_{A,z}^2)}L_{\rho\rho}R_0^2.$$

The sign of the excited velocity coincides with the sign  $L_{\rho\rho}$ , and its magnitude depends also on the magnitude of perturbation in density in the entropy mode and its characteristic extension, the equilibrium parameters of a plasma,  $\theta$  and plasma- $\beta$ . The dimensionless ratio  $\frac{c_0^2(C^2 - C_{A,z}^2)}{C^4 - c_0^2C_{A,z}^2}$  is positive for any  $\theta$  and plasma- $\beta$  and varies from 0 till 1. Fig.2 shows this ratio for the fast and slow magnetosonic modes. The surfaces are symmetric with respect to the plane  $\theta = \pi/2$ .

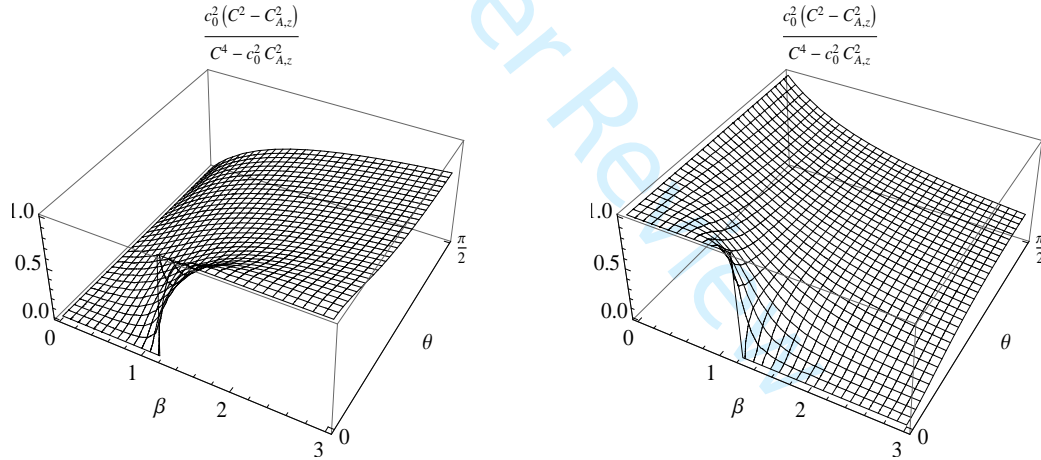


Figure 2. The ratio  $\frac{c_0^2(C^2 - C_{A,z}^2)}{C^4 - c_0^2C_{A,z}^2}$  for the fast (left panel) and slow (right panel) magnetosonic modes. The surfaces are symmetric with respect to the plane  $\theta = \pi/2$ . Evaluations for  $\gamma = 5/3$ .

The maximum absolute value  $\frac{v_{z,ms}}{c_0}$  achieves at the plateau,

$$|L_{\rho\rho}|R_0^2 \frac{(C^2 - C_{A,z}^2)(\gamma - 1)}{4c_0\rho_0k(C^4 - c_0^2C_{A,z}^2)} \frac{\sqrt{\pi}}{\sqrt{2}} \exp\left(\frac{D}{k}T + F(Z - T)\right),$$

that is less or equal

$$|L_{\rho\rho}|R_0^2 \frac{(\gamma - 1)\sqrt{\pi}}{4c_0^3\rho_0k\sqrt{2}} \exp\left(\frac{D}{k}T + F(Z - T)\right)$$

for any  $\theta$  and plasma- $\beta$  and both slow and fast magnetosonic modes. There is no excitation of the magnetosonic mode if  $L_{\rho\rho} = 0$ .



## 6 Concluding Remarks

This study explains the method which allows to obtain the nonlinear corrections to the specific modes of a flow and to derive equations describing nonlinear interaction of different modes in a fluid flow. The method bases on the definition of modes of the linear flow as relations of specific perturbations. It may be applied to the wide variety of **weakly nonlinear** fluid flows (**not necessary flows of ionized gases** [3]). A planar flow of a fully ionized plasma affected by a magnetic field, is considered **in this study**. We corrected the links specifying the Alfvén perturbations by including quadratic nonlinear terms. The set of corrected variables is different for  $\theta = 0$  or  $\theta = \pi$  ( $\rho_A, v_{A,z}, p_A$ ) and other values of  $\theta$  ( $v_{A,x}, B_{A,x}$ ). In spite of nonlinear corrections in the links, they do not have impact on the dynamic equation for the Alfvén perturbation, and  $v_{A,y}$  is still described by the linear equation (4.17). There is no leading-order nonlinear term in the dynamic equation for the dominant entropy mode. The conclusion is that both Alfvén branches are ineffective in the nonlinear excitation of the other modes (magnetosonic and the entropy mode), at least at the leading order (up to quadratic nonlinear coupling). The Alfvén wave is not excited by the other modes if they are dominant. This follows from the form of two Alfvén projectors (3.12) and the nonlinear part of Eqs(2.2) which indicate that there is no coupling with magnetosonic and entropy modes. The counterpropagating Alfvén branches also do not interact. **The conclusions refer to the planar geometry of a flow.**

The links between thermodynamic perturbations in the wave mode may be referred as polarization relations. Not to mention nonlinear corrections, the linear links between specific perturbations by themselves are undeservedly underestimated in many studies of fluid flows. In particular, in the case of mono-polar impulses the integral term in the link between  $p_{ms}$  and  $\rho_{ms}$  is responsible for the non-zero perturbation of density after an impulse passes. A Gaussian exciter

$$p_{ms} = P_0 \exp(-(\omega(t - z/C))^2),$$

gives the residual magnetosonic density in the form

$$\rho_{ms} = -P_0 \frac{2\sqrt{\pi}(\gamma - 1)(c_0^2 L_p + L_\rho)}{c_0^4 \omega}. \quad (6.25)$$

This is a constant quantity associating with the heating-cooling function (but not to the thermal conduction and dynamic viscosity) which may indicate its properties.

The method of projecting allows to derive coupling nonlinear equations for interacting modes **in a flow with different mechanisms of deviation from adiabaticity**. The generic heating-cooling function  $L(p, \rho)$  is considered. It balances with the mechanical damping and thermal conduction and introduces various scenarios of fluid dynamics and, in particular, in nonlinear interaction of modes. The dominant entropy mode may excite any magnetosonic mode if  $L_{\rho\rho}$  differs from zero. There is no excitation if  $L_{\rho\rho} = 0$ . The dynamic equation (5.20) along with (4.18) take into account mechanical and thermal attenuation, the heating-cooling function and nonlinear distortions of the excited mode. An analytic example concerns impact of the only heating-cooling function on the dominative entropy perturbations and the excited wave mode. The sign of  $L_{\rho\rho}$  coincides with the sign of velocity in the induced magnetosonic mode. The example reflects the main features of intermode interactions. The excited perturbations basically consist of parts which propagate with their own linear speed and the speed of the dominant mode. For an impulsive exciter, the head and back fronts propagate with these different speeds. In general, the damping due to dynamic viscosity and thermal conduction, the attenuation/amplification

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2 arising from the heating-cooling function  $D$  and the parameter of nonlinearity  $\varepsilon$  have impact  
3 on the excited perturbations. All these factors depend on the plasma- $\beta$  and  $\theta$ . The conclusions  
4 may be useful to specify an exciter, the equilibrium parameters of a plasma and the kind of  
5 heating-cooling function, also in remote observations.  
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#### 8 **DATA AVAILABILITY STATEMENT**

9 Data sharing is not applicable to this article as no new data were created or analyzed in  
10 this study, which is a purely theoretical one.  
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For Peer Review

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