

THE LAW OF THE ITERATED LOGARITHM FOR RANDOM INTERVAL HOMEOMORPHISMS

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ABSTRACT. A proof of the law of the iterated logarithm for random homeomorphisms of the interval is given.

In this short note we prove that admissible iterated function systems considered in [1] satisfy, besides the central limit theorem, the law of the iterated logarithm. Our argument is based on the criterion from the paper by O. Zhao and M. Woodroffe [3] and some computations provided in [1].

We start by recalling the definition of an admissible iterated function system. Let f_1, \dots, f_N be increasing homeomorphisms of the interval $[0, 1]$ such that for every $x \in (0, 1)$ there exist $i, j \in \{1, \dots, N\}$ with $f_i(x) < x < f_j(x)$. It is assumed that all the homeomorphisms are differentiable at 0 and 1 with nonzero derivatives. Let (p_1, \dots, p_N) be a probability vector such that

$$\sum_{i=1}^N p_i \log f'_i(0) > 0 \text{ and } \sum_{i=1}^N p_i \log f'_i(1) > 0.$$

The family $(f_1, \dots, f_N; p_1, \dots, p_N)$ is then called an *admissible iterated function system*.

By $\mathcal{M}([0, 1])$ we denote the set of all finite measures on the σ -algebra $\mathcal{B}([0, 1])$ of all Borel subsets of $[0, 1]$, and by $\mathcal{M}_1([0, 1]) \subseteq \mathcal{M}([0, 1])$ we denote the subset of all probability measures on $[0, 1]$. By $B([0, 1])$ we denote the family of bounded Borel functions on $[0, 1]$.

From now on we assume that an admissible iterated function system $(f_1, \dots, f_N; p_1, \dots, p_N)$ is given. It generates a Markov operator $P : \mathcal{M}([0, 1]) \rightarrow \mathcal{M}([0, 1])$ of the form

$$(1) \quad P\mu(A) = \sum_{i=1}^N p_i \mu(f_i^{-1}(A)) \quad \text{for } \mu \in \mathcal{M}([0, 1]) \text{ and } A \in \mathcal{B}([0, 1]).$$

By continuity of the f_i , P is a Feller operator, and its predual operator $U : B([0, 1]) \rightarrow B([0, 1])$ is given by the formula

$$U\psi(x) = \sum_{i=1}^N p_i \psi(f_i(x)) \text{ for } \psi \in B([0, 1]) \text{ and } x \in [0, 1].$$

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It has been proved in [1] that P is asymptotically stable on measures supported in $(0, 1)$. In particular, P has a unique invariant measure $\mu_* \in \mathcal{M}_1([0, 1])$ satisfying $\mu_*((0, 1)) = 1$, by Theorem 2 in [1].

By $(X_n)_{n \geq 0}$ we shall denote the Markov chain on $[0, 1]^{\mathbb{N}}$ corresponding to the transition function $\pi : [0, 1] \times \mathcal{B}([0, 1]) \rightarrow [0, 1]$ of the form

$$\pi(x, A) = U\mathbf{1}_A(x) = P\delta_x(A) \quad \text{for } x \in [0, 1] \text{ and } A \in \mathcal{B}([0, 1]).$$

The law of the Markov chain $(X_n)_{n \geq 0}$ with initial distribution ν is the probability measure \mathbb{P}_ν on $([0, 1]^{\mathbb{N}}, \mathcal{B}([0, 1])^{\otimes \mathbb{N}})$ such that

$$\mathbb{P}_\nu[X_{n+1} \in A | X_n = x] = \pi(x, A) \quad \text{and} \quad \mathbb{P}_\nu[X_0 \in A] = \nu(A),$$

where $x \in [0, 1]$, $A \in \mathcal{B}([0, 1])$. The existence of \mathbb{P}_ν follows from the Kolmogorov extension theorem. For $\nu = \delta_x$, that is, the Dirac measure at $x \in [0, 1]$, we write just \mathbb{P}_x . Obviously $\mathbb{P}_\nu(\cdot) = \int_{[0, 1]} \mathbb{P}_x(\cdot) \nu(dx)$. When an initial probability ν is equal to μ_* , the Markov chain $(X_n)_{n \geq 0}$ is stationary.

Let $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ be equipped with the product topology induced by the discrete topology on $\{1, \dots, N\}$, and let $f_\omega^n = f_{\omega_n} \circ \dots \circ f_{\omega_1} = f_{(\omega_1, \dots, \omega_n)}$ for $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$. By \mathbb{P} we denote the measure on Σ , which is the product measure of the probability vector (p_1, \dots, p_N) . By abuse of notation, we shall also write \mathbb{P} for the product measure of the probability vector (p_1, \dots, p_N) on $\Sigma_n = \{1, \dots, N\}^n$ for $n \in \mathbb{N}$.

Note that for $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathcal{B}([0, 1])$ we have

$$\begin{aligned} & \mathbb{P}_x((X_1, \dots, X_n) \in A_1 \times \dots \times A_n) \\ &= \sum_{(\omega_1, \dots, \omega_n) \in \Sigma_n} \mathbf{1}_{A_1 \times \dots \times A_n}(f_{\omega_1}(x), \dots, f_{(\omega_1, \dots, \omega_n)}(x)) p_{\omega_1} \dots p_{\omega_n} \\ &= \int_{\Sigma_n} \mathbf{1}_{A_1 \times \dots \times A_n}(f_{\omega_1}(x), \dots, f_{(\omega_1, \dots, \omega_n)}(x)) \mathbb{P}(d\omega_1 \times \dots \times d\omega_n) \\ &= \int_{\Sigma} \mathbf{1}_{A_1 \times \dots \times A_n}(f_\omega^1(x), \dots, f_\omega^n(x)) \mathbb{P}(d\omega) \\ &= (\delta_x \otimes \mathbb{P})(\{(y, \omega) \in [0, 1] \times \Sigma : (f_\omega^1(y), \dots, f_\omega^n(y)) \in A_1 \times \dots \times A_n\}). \end{aligned}$$

Since $\mathbb{P}_\nu(\cdot) = \int_{[0, 1]} \mathbb{P}_x(\cdot) \nu(dx)$ for $\nu \in \mathcal{M}_1([0, 1])$, for $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathcal{B}([0, 1])$ we obtain

$$(2) \quad \begin{aligned} & \mathbb{P}_\nu((X_1, \dots, X_n) \in A_1 \times \dots \times A_n) \\ &= (\nu \otimes \mathbb{P})(\{(y, \omega) \in [0, 1] \times \Sigma : (f_\omega^1(y), \dots, f_\omega^n(y)) \in A_1 \times \dots \times A_n\}). \end{aligned}$$

This note is aimed at proving the following theorem.

Theorem. *If φ is a Lipschitz function satisfying the condition $\int_{[0, 1]} \varphi d\mu_* = 0$, then there exists a constant $\sigma \in [0, \infty)$ such that for every $x \in (0, 1)$ we have*

$$(3) \quad \limsup_{n \rightarrow \infty} \frac{\varphi(f_\omega^1(x)) + \dots + \varphi(f_\omega^n(x))}{\sqrt{2n \log \log n}} = \sigma \quad \mathbb{P} \text{ a.e.}$$

We start with the proof of the annealed law of the iterated logarithm.

Proposition. *If φ is a Lipschitz function satisfying the condition $\int_{[0,1]} \varphi d\mu_* = 0$, then there exists a constant $\sigma \in [0, \infty)$ such that*

$$(4) \quad \limsup_{n \rightarrow \infty} \frac{\varphi(X_1) + \cdots + \varphi(X_n)}{\sqrt{2n \log \log n}} = \sigma \quad \mathbb{P}_{\mu_*} \text{ a.e.}$$

Proof. Let φ be a Lipschitz function satisfying the condition $\int_{[0,1]} \varphi d\mu_* = 0$, and let $(\tilde{X}_n)_{n \in \mathbb{Z}}$ be a stationary ergodic Markov chain (with the law μ_*) on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ that corresponds to the given transition probability U . The existence of this chain follows from the Kolmogorov extension theorem. Set $Y_n = \varphi(\tilde{X}_n)$, $n \in \mathbb{Z}$, and observe that $(Y_n)_{n \in \mathbb{Z}}$ is again a stationary ergodic chain. Set $S_n = Y_n + \cdots + Y_1$ for $n \in \mathbb{N}$, and let $\mathcal{F}_0 = \sigma(\dots, \tilde{X}_{-n}, \tilde{X}_{-n+1}, \dots, \tilde{X}_{-1}, \tilde{X}_0)$.

In [1] (see Theorem 4) we have proved that there exists a positive constant C such that

$$\left\| \sum_{j=1}^n U^j \varphi \right\|_{L^2(\mu_*)} \leq C n^{\frac{3}{8}} \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, we have

$$\begin{aligned} \|\mathbb{E}(S_n | \mathcal{F}_0)\|_{L^2(\mu_*)}^2 &= \int_{[0,1]} |\mathbb{E}(\varphi(\tilde{X}_n) + \cdots + \varphi(\tilde{X}_1) | X_0 = x)|^2 \mu_*(dx) \\ &= \int_{[0,1]} |U^n \varphi(x) + \cdots + U \varphi(x)|^2 \mu_*(dx) = \left\| \sum_{j=1}^n U^j \varphi \right\|_{L^2(\mu_*)}^2, \end{aligned}$$

and consequently

$$\sum_{n=1}^{\infty} \left(\frac{\log n}{n} \right)^{\frac{3}{2}} \|\mathbb{E}(S_n | \mathcal{F}_0)\|_{L^2(\mu_*)} < \infty.$$

Now Corollary 1 in [3] implies that there exists a constant $\sigma \in [0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} \frac{\varphi(\tilde{X}_1) + \cdots + \varphi(\tilde{X}_n)}{\sqrt{2n \log \log n}} = \sigma \quad \tilde{\mathbb{P}} \text{ a.e.}$$

Since the chain $(\tilde{X}_n)_{n \geq 0}$ and the stationary chain $(X_n)_{n \geq 0}$ have the same law, we obtain that

$$\limsup_{n \rightarrow \infty} \frac{\varphi(X_1) + \cdots + \varphi(X_n)}{\sqrt{2n \log \log n}} = \sigma \quad \mathbb{P}_{\mu_*} \text{ a.e.}$$

This completes the proof. \square

Proof of the Theorem. Choose $a \in (0, 1/2)$ such that $\mu_*((a, 1-a)) > 3/4$. From Lemma 3 in [1] it follows that there exists $\gamma > 0$ and $\Sigma_a \subset \Sigma$ with $\mathbb{P}(\Sigma_a) \geq \gamma$ such that

$$(5) \quad \sum_{n=1}^{\infty} |f_{\omega}^n((a, 1-a))| < \infty \quad \text{for } \omega \in \Sigma_a.$$

Set $\beta := \gamma/2$. We are going to show that for any $u, v \in (0, 1)$, $u < v$, we may find a set $\Sigma_{u,v} \subset \Sigma$ with $\mathbb{P}(\Sigma_{u,v}) \geq \beta$ such that

$$(6) \quad \sum_{n=1}^{\infty} |f_{\omega}^n(u) - f_{\omega}^n(v)| < \infty \quad \text{for } \omega \in \Sigma_{u,v}.$$

Fix $u, v \in (0, 1)$, $u < v$. Since the system is asymptotically stable on measures supported in $(0, 1)$ by Theorem 2 in [1], we may find $n \in \mathbb{N}$ such that $P^n \delta_u((a, 1-a)$

$a)) > 3/4$ and $P^n \delta_v((a, 1 - a)) > 3/4$, by the Portmanteau theorem. Hence there exists $\tilde{\Sigma}_{u,v} \subset \{1, \dots, N\}^n$ with $\mathbb{P}(\tilde{\Sigma}_{u,v}) \geq 1/2$ such that $f_{\omega_n} \circ \dots \circ f_{\omega_1}(u), f_{\omega_n} \circ \dots \circ f_{\omega_1}(v) \in (a, 1 - a)$ for $(\omega_1, \dots, \omega_n) \in \tilde{\Sigma}_{u,v}$. Set $\Sigma_{u,v} = \tilde{\Sigma}_{u,v} \times \Sigma_a$, and note that $\mathbb{P}(\Sigma_{u,v}) \geq \beta$. Moreover, from (5) it follows that (6) holds.

The proposition and condition (2) for $\nu = \mu_*$ imply that condition (3) holds for μ_* almost every $x \in (0, 1)$. To complete the proof it is enough to show that for any $x, y \in (0, 1)$ we have

$$\mathbb{P}(\{\omega \in \Sigma : \sum_{n=1}^{\infty} |f_{\omega}^n(x) - f_{\omega}^n(y)| < \infty\}) = 1.$$

To do this fix $x, y \in (0, 1)$. Set

$$A := \{\omega \in \Sigma : \sum_{n=1}^{\infty} |f_{\omega}^n(x) - f_{\omega}^n(y)| < \infty\},$$

and assume, contrary to our claim, that $\mathbb{P}(A) < 1$. Choose a compact subset $A' \subset \Sigma \setminus A$ such that $\alpha := \mathbb{P}(A') > 0$. Let $\Sigma_1, \dots, \Sigma_M$, $M \in \mathbb{N}$, be disjoint cylinders such that $A' \subset \bigcup_{i=1}^M \Sigma_i$ and $\mathbb{P}(\bigcup_{i=1}^M \Sigma_i \setminus A') < \beta\alpha$. Let $\Sigma_i = (\omega_1^i, \dots, \omega_{n_i}^i) \times \Sigma$ for $i \in \{1, \dots, M\}$. We set $u_i := f_{\omega_{n_i}^i} \circ \dots \circ f_{\omega_1^i}(x)$ and $v_i := f_{\omega_{n_i}^i} \circ \dots \circ f_{\omega_1^i}(y)$, and define $\hat{\Sigma}_i = (\omega_1^i, \dots, \omega_{n_i}^i) \times \Sigma_{u_i, v_i} \subset \Sigma_i$. Obviously, $\sum_{n=1}^{\infty} |f_{\omega}^n(x) - f_{\omega}^n(y)| < \infty$ for $\omega \in \hat{\Sigma}_i$. Moreover, $\mathbb{P}(\hat{\Sigma}_i) \geq \beta\mathbb{P}(\Sigma_i)$, and consequently

$$\mathbb{P}(\bigcup_{i=1}^M \hat{\Sigma}_i) \geq \beta\mathbb{P}(\bigcup_{i=1}^M \Sigma_i) \geq \beta\mathbb{P}(A') \geq \beta\alpha.$$

Since $\mathbb{P}(\bigcup_{i=1}^M \hat{\Sigma}_i \setminus A') \leq \mathbb{P}(\bigcup_{i=1}^M \Sigma_i \setminus A') < \beta\alpha$, we finally obtain that $\mathbb{P}(\bigcup_{i=1}^M \hat{\Sigma}_i \cap A') > 0$, which is impossible due to the fact that $\sum_{n=1}^{\infty} |f_{\omega}^n(x) - f_{\omega}^n(y)| < \infty$ for $\omega \in \bigcup_{i=1}^M \hat{\Sigma}_i$. Hence $\mathbb{P}(A) = 1$, and the proof is complete. \square

Remark. In view of (2) the Theorem is equivalent to (4) holding \mathbb{P}_x a.e. for every $x \in (0, 1)$.

Finally, let us compare the result in this note with the one provided in [2]. Actually, the above-mentioned paper is concerned with the law of the iterated logarithm for Markov chains corresponding to the stochastically perturbed dynamical system of the form

$$x_{n+1} = S(x_n, t_{n+1}) + H_{n+1} \quad \text{for } n \geq 0,$$

where $S : H \times [0, T] \rightarrow H$ is a continuous function on some separable Banach space H , and $(t_n)_{n \geq 1}, (H_n)_{n \geq 1}$ are independent random variables with values in $[0, T], H$ respectively. Such a system may serve to describe some cell cycle models, and it seems to be more general than our admissible iterated function system. However, the assumptions made in [2] are far too restrictive. In particular, it is demanded in [2] that the system is contractive on average. But no contracting condition may hold in the case when each of the f_i has a fixed point at 0 and at 1. For the same reason the Markov chain corresponding to an admissible iterated function system may not converge exponentially to equilibrium. Therefore the techniques developed in [2] are completely useless in the present note.

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