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THE LAW OF THE ITERATED LOGARITHM FOR RANDOM INTERVAL HOMEOMORPHISMS

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Abstract. A proof of the law of the iterated logarithm for random homeomorphisms of the interval is given.

In this short note we prove that admissible iterated function systems considered in [1] satisfy, besides the central limit theorem, the law of the iterated logarithm. Our argument is based on the criterion from the paper by O. Zhao and M. Woodroofe [3] and some computations provided in [1].

We start by recalling the definition of an admissible iterated function system. Let $f_1, \ldots, f_N$ be increasing homeomorphisms of the interval $[0,1]$ such that for every $x \in (0,1)$ there exist $i, j \in \{1, \ldots, N\}$ with $f_i(x) < x < f_j(x)$. It is assumed that all the homeomorphisms are differentiable at 0 and 1 with nonzero derivatives. Let $(p_1, \ldots, p_N)$ be a probability vector such that

$$\sum_{i=1}^{N} p_i \log f_i'(0) > 0 \text{ and } \sum_{i=1}^{N} p_i \log f_i'(1) > 0.$$

The family $(f_1, ..., f_N; p_1, ..., p_N)$ is then called an *admissible iterated function system*.

By $\mathcal{M}([0,1])$ we denote the set of all finite measures on the σ-algebra $\mathcal{B}([0,1])$ of all Borel subsets of $[0,1]$, and by $\mathcal{M}_1([0,1]) \subseteq \mathcal{M}([0,1])$ we denote the subset of all probability measures on $[0,1]$. By $B([0,1])$ we denote the family of bounded Borel functions on $[0,1]$.

From now on we assume that an admissible iterated function system $(f_1, ..., f_N; p_1, ..., p_N)$ is given. It generates a Markov operator $P : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ of the form

$$P\mu(A) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(A)) \quad \text{for } \mu \in \mathcal{M}([0,1]) \text{ and } A \in \mathcal{B}([0,1]). \tag{1}$$

By continuity of the f_i, P is a Feller operator, and its predual operator $U : B([0,1]) \to B([0,1])$ is given by the formula

$$U\psi(x) = \sum_{i=1}^{N} p_i \psi(f_i(x)) \text{ for } \psi \in B([0,1]) \text{ and } x \in [0,1].$$

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It has been proved in [1] that P is asymptotically stable on measures supported in $(0,1)$. In particular, P has a unique invariant measure $\mu_* \in \mathcal{M}_1([0,1])$ satisfying $\mu_*((0,1)) = 1$, by Theorem 2 in [1].

By $(X_n)_{n\geq 0}$ we shall denote the Markov chain on $[0,1]^{\mathbb{N}}$ corresponding to the transition function $\pi : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ of the form

$$\pi(x,A) = U\mathbf{1}_A(x) = P\delta_x(A) \quad \text{for } x \in [0,1] \text{ and } A \in \mathcal{B}([0,1]).$$

The law of the Markov chain $(X_n)_{n\geq 0}$ with initial distribution ν is the probability measure $\mathbb{P}_\nu$ on $([0,1]^{\mathbb{N}}, \mathcal{B}([0,1])^{\otimes\mathbb{N}})$ such that

$$\mathbb{P}_\nu[X_{n+1} \in A | X_n = x] = \pi(x,A) \quad \text{and} \quad \mathbb{P}_\nu[X_0 \in A] = \nu(A),$$

where $x \in [0,1]$, $A \in \mathcal{B}([0,1])$. The existence of $\mathbb{P}_\nu$ follows from the Kolmogorov extension theorem. For $\nu = \delta_x$, that is, the Dirac measure at $x \in [0,1]$, we write just $\mathbb{P}_x$. Obviously $\mathbb{P}_\nu(\cdot) = \int_{[0,1]} \mathbb{P}_x(\cdot)\nu(\mathrm{d}x)$. When an initial probability ν is equal to μ_*, the Markov chain $(X_n)_{n\geq 0}$ is stationary.

Let $\Sigma = \{1,\ldots,N\}^{\mathbb{N}}$ be equipped with the product topology induced by the discrete topology on $\{1,\ldots,N\}$, and let $f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_1} = f_{(\omega_1,\ldots,\omega_n)}$ for $\omega = (\omega_1,\omega_2,\ldots) \in \Sigma$. By $\mathbb{P}$ we denote the measure on Σ, which is the product measure of the probability vector $(p_1,\ldots,p_N)$. By abuse of notation, we shall also write $\mathbb{P}$ for the product measure of the probability vector $(p_1,\ldots,p_N)$ on $\Sigma_n = \{1,\ldots,N\}^n$ for $n \in \mathbb{N}$.

Note that for $n \in \mathbb{N}$ and $A_1,\ldots,A_n \in \mathcal{B}([0,1])$ we have

$$\begin{aligned}
&\mathbb{P}_x((X_1,\ldots,X_n) \in A_1 \times \cdots \times A_n)) \\
&= \sum_{(\omega_1,\ldots,\omega_n)\in\Sigma_n} \mathbf{1}_{A_1\times\cdots\times A_n}(f_{\omega_1}(x),\ldots,f_{(\omega_1,\ldots,\omega_n)}(x))p_{\omega_1}\cdots p_{\omega_n} \\
&= \int_{\Sigma_n} \mathbf{1}_{A_1\times\cdots\times A_n}(f_{\omega_1}(x),\ldots,f_{(\omega_1,\ldots,\omega_n)}(x))\mathbb{P}(\mathrm{d}\omega_1 \times \cdots \times d\omega_n) \\
&= \int_{\Sigma} \mathbf{1}_{A_1\times\cdots\times A_n}(f_\omega^1(x),\ldots,f_\omega^n(x))\mathbb{P}(\mathrm{d}\omega) \\
&= (\delta_x \otimes \mathbb{P})(\{(y,\omega) \in [0,1]\times\Sigma : (f_\omega^1(y),\ldots,f_\omega^n(y)) \in A_1 \times \cdots \times A_n\}).
\end{aligned}$$

Since $\mathbb{P}_\nu(\cdot) = \int_{[0,1]} \mathbb{P}_x(\cdot)\nu(\mathrm{d}x)$ for $\nu \in \mathcal{M}_1([0,1])$, for $n \in \mathbb{N}$ and $A_1,\ldots,A_n \in \mathcal{B}([0,1])$ we obtain

$$\begin{aligned}
&\mathbb{P}_\nu((X_1,\ldots,X_n) \in A_1 \times \cdots \times A_n)) \\
&= (\nu \otimes \mathbb{P})(\{(y,\omega) \in [0,1]\times\Sigma : (f_\omega^1(y),\ldots,f_\omega^n(y)) \in A_1 \times \cdots \times A_n\}).
\end{aligned} \tag{2}$$

This note is aimed at proving the following theorem.

Theorem. *If φ is a Lipschitz function satisfying the condition $\int_{[0,1]} \varphi \mathrm{d}\mu_* = 0$, then there exists a constant $\sigma \in [0,\infty)$ such that for every $x \in (0,1)$ we have*

$$\limsup_{n\to\infty} \frac{\varphi(f_\omega^1(x)) + \cdots + \varphi(f_\omega^n(x))}{\sqrt{2n\log\log n}} = \sigma \quad \mathbb{P} \text{ a.e.} \tag{3}$$

We start with the proof of the annealed law of the iterated logarithm.

Proposition. *If φ is a Lipschitz function satisfying the condition $\int_{[0,1]} \varphi \mathrm{d}\mu_* = 0$, then there exists a constant $\sigma \in [0, \infty)$ such that*

$$\limsup_{n\to\infty} \frac{\varphi(X_1) + \cdots + \varphi(X_n)}{\sqrt{2n \log\log n}} = \sigma \quad \mathbb{P}_{\mu_*} \text{ a.e.} \tag{4}$$

Proof. Let φ be a Lipschitz function satisfying the condition $\int_{[0,1]} \varphi \mathrm{d}\mu_* = 0$, and let $(\tilde{X}_n)_{n\in\mathbb{Z}}$ be a stationary ergodic Markov chain (with the law μ_*) on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ that corresponds to the given transition probability U. The existence of this chain follows from the Kolmogorov extension theorem. Set $Y_n = \varphi(\tilde{X}_n)$, $n \in \mathbb{Z}$, and observe that $(Y_n)_{n\in\mathbb{Z}}$ is again a stationary ergodic chain. Set $S_n = Y_n + \cdots + Y_1$ for $n \in \mathbb{N}$, and let $\mathcal{F}_0 = \sigma(\ldots, \tilde{X}_{-n}, \tilde{X}_{-n+1}, \ldots, \tilde{X}_{-1}, \tilde{X}_0)$.

In [1] (see Theorem 4) we have proved that there exists a positive constant C such that

$$\Big\| \sum_{j=1}^{n} U^j \varphi \Big\|_{L^2(\mu_*)} \le C n^{\frac{3}{8}} \qquad \text{for all } n \in \mathbb{N}.$$

On the other hand, we have

$$\begin{aligned} \|\mathbb{E}(S_n|\mathcal{F}_0)\|^2_{L^2(\mu_*)} &= \int_{[0,1]} |\mathbb{E}(\varphi(\tilde{X}_n) + \cdots + \varphi(\tilde{X}_1)|X_0 = x)|^2 \mu_*(\mathrm{d}x) \\ &= \int_{[0,1]} |U^n\varphi(x) + \cdots + U\varphi(x))|^2 \mu_*(\mathrm{d}x) = \|\sum_{j=1}^{n} U^j\varphi\|^2_{L^2(\mu_*)}, \end{aligned}$$

and consequently

$$\sum_{n=1}^{\infty} \left(\frac{\log n}{n}\right)^{\frac{3}{2}} \big\|\mathbb{E}(S_n|\mathcal{F}_0)\big\|_{L^2(\mu_*)} < \infty.$$

Now Corollary 1 in [3] implies that there exists a constant $\sigma \in [0, \infty)$ such that

$$\limsup_{n\to\infty} \frac{\varphi(\tilde{X}_1) + \cdots + \varphi(\tilde{X}_n)}{\sqrt{2n \log\log n}} = \sigma \qquad \tilde{\mathbb{P}} \text{ a.e.}$$

Since the chain $(\tilde{X}_n)_{n\ge 0}$ and the stationary chain $(X_n)_{n\ge 0}$ have the same law, we obtain that

$$\limsup_{n\to\infty} \frac{\varphi(X_1) + \cdots + \varphi(X_n)}{\sqrt{2n \log\log n}} = \sigma \qquad \mathbb{P}_{\mu_*} \text{ a.e.}$$

This completes the proof. □

Proof of the Theorem. Choose $a \in (0, 1/2)$ such that $\mu_*((a, 1-a)) > 3/4$. From Lemma 3 in [1] it follows that there exists $\gamma > 0$ and $\Sigma_a \subset \Sigma$ with $\mathbb{P}(\Sigma_a) \ge \gamma$ such that

$$\sum_{n=1}^{\infty} |f^n_\omega((a, 1-a))| < \infty \qquad \text{for } \omega \in \Sigma_a. \tag{5}$$

Set $\beta := \gamma/2$. We are going to show that for any $u, v \in (0,1)$, $u < v$, we may find a set $\Sigma_{u,v} \subset \Sigma$ with $\mathbb{P}(\Sigma_{u,v}) \ge \beta$ such that

$$\sum_{n=1}^{\infty} |f^n_\omega(u) - f^n_\omega(v)| < \infty \qquad \text{for } \omega \in \Sigma_{u,v}. \tag{6}$$

Fix $u, v \in (0,1)$, $u < v$. Since the system is asymptotically stable on measures supported in $(0,1)$ by Theorem 2 in [1], we may find $n \in \mathbb{N}$ such that $P^n\delta_u((a, 1-$

$a)) > 3/4$ and $P^n\delta_v((a, 1-a)) > 3/4$, by the Portmanteau theorem. Hence there exists $\tilde{\Sigma}_{u,v} \subset \{1, \ldots, N\}^n$ with $\mathbb{P}(\tilde{\Sigma}_{u,v}) \geq 1/2$ such that $f_{\omega_n} \circ \cdots \circ f_{\omega_1}(u), f_{\omega_n} \circ \cdots \circ f_{\omega_1}(v) \in (a, 1-a)$ for $(\omega_1, \ldots, \omega_n) \in \tilde{\Sigma}_{u,v}$. Set $\Sigma_{u,v} = \tilde{\Sigma}_{u,v} \times \Sigma_a$, and note that $\mathbb{P}(\Sigma_{u,v}) \geq \beta$. Moreover, from (5) it follows that (6) holds.

The proposition and condition (2) for $\nu = \mu_*$ imply that condition (3) holds for μ_* almost every $x \in (0, 1)$. To complete the proof it is enough to show that for any $x, y \in (0, 1)$ we have

$$\mathbb{P}(\{\omega \in \Sigma : \sum_{n=1}^{\infty} |f_\omega^n(x) - f_\omega^n(y)| < \infty\}) = 1.$$

To do this fix $x, y \in (0, 1)$. Set

$$A := \{\omega \in \Sigma : \sum_{n=1}^{\infty} |f_\omega^n(x) - f_\omega^n(y)| < \infty\},$$

and assume, contrary to our claim, that $\mathbb{P}(A) < 1$. Choose a compact subset $A' \subset \Sigma \setminus A$ such that $\alpha := \mathbb{P}(A') > 0$. Let $\Sigma_1, \ldots, \Sigma_M$, $M \in \mathbb{N}$, be disjoint cylinders such that $A' \subset \bigcup_{i=1}^M \Sigma_i$ and $\mathbb{P}(\bigcup_{i=1}^M \Sigma_i \setminus A') < \beta\alpha$. Let $\Sigma_i = (\omega_1^i, \ldots, \omega_{n_i}^i) \times \Sigma$ for $i \in \{1, \ldots, M\}$. We set $u_i := f_{\omega_{n_i}^i} \circ \cdots \circ f_{\omega_1^i}(x)$ and $v_i := f_{\omega_{n_i}^i} \circ \cdots \circ f_{\omega_1^i}(y)$, and define $\hat{\Sigma}_i = (\omega_1^i, \ldots, \omega_{n_i}^i) \times \Sigma_{u_i, v_i} \subset \Sigma_i$. Obviously, $\sum_{n=1}^{\infty} |f_\omega^n(x) - f_\omega^n(y)| < \infty$ for $\omega \in \hat{\Sigma}_i$. Moreover, $\mathbb{P}(\hat{\Sigma}_i) \geq \beta\mathbb{P}(\Sigma_i)$, and consequently

$$\mathbb{P}(\bigcup_{i=1}^M \hat{\Sigma}_i) \geq \beta\mathbb{P}(\bigcup_{i=1}^M \Sigma_i) \geq \beta\mathbb{P}(A') \geq \beta\alpha.$$

Since $\mathbb{P}(\bigcup_{i=1}^M \hat{\Sigma}_i \setminus A') \leq \mathbb{P}(\bigcup_{i=1}^M \Sigma_i \setminus A') < \beta\alpha$, we finally obtain that $\mathbb{P}(\bigcup_{i=1}^M \hat{\Sigma}_i \cap A') > 0$, which is impossible due to the fact that $\sum_{n=1}^{\infty} |f_\omega^n(x) - f_\omega^n(y)| < \infty$ for $\omega \in \bigcup_{i=1}^M \hat{\Sigma}_i$. Hence $\mathbb{P}(A) = 1$, and the proof is complete. ☐

Remark. *In view of (2) the Theorem is equivalent to (4) holding $\mathbb{P}_x$ a.e. for every $x \in (0, 1)$.*

Finally, let us compare the result in this note with the one provided in [2]. Actually, the above–mentioned paper is concerned with the law of the iterated logarithm for Markov chains corresponding to the stochastically perturbed dynamical system of the form

$$x_{n+1} = S(x_n, t_{n+1}) + H_{n+1} \quad \text{for } n \geq 0,$$

where $S : H \times [0, T] \to H$ is a continuous function on some separable Banach space H, and $(t_n)_{n\geq1}$, $(H_n)_{n\geq1}$ are independent random variables with values in $[0, T]$, H respectively. Such a system may serve to describe some cell cycle models, and it seems to be more general than our admissible iterated function system. However, the assumptions made in [2] are far too restrictive. In particular, it is demanded in [2] that the system is contractive on average. But no contracting condition may hold in the case when each of the f_i has a fixed point at 0 and at 1. For the same reason the Markov chain corresponding to an admissible iterated function system may not converge exponentially to equilibrium. Therefore the techniques developed in [2] are completely useless in the present note.

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References

[1] K. Czudek, T. Szarek, *Ergodicity and central limit theorem for random interval homeomorphisms*, Israel J. Math. **239** (2020), 75–98. https://doi.org/10.1007/s11856-020-2046-4.

[2] S. Hille, K. Horbacz, T. Szarek, H. Wojewódka, *Law of the iterated logarithm for some Markov operators*, Asymptot. Anal. **97** (2016), no. 1-2, 91–112.

[3] O. Zhao, M. Woodroofe, *Law of the iterated logarithm for stationary processes*, Ann. Probab. **36** (2008), no. 1, 127–142.

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