Paired domination subdivision and multisubdivision numbers of graphs

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Abstract

The paired domination subdivision number $\mathrm{sd}_{pr}(G)$ of a graph G is the minimum number of edges that must be subdivided (where an edge can be subdivided at most once) in order to increase the paired domination number of G. We prove that the decision problem of the paired domination subdivision number is NP-complete even for bipartite graphs. For this reason we define the *paired domination multisubdivision number* of a nonempty graph G, denoted by $\mathrm{msd}_{pr}(G)$, as the minimum positive integer k such that there exists an edge which must be subdivided k times to increase the paired domination number of G. We show that $\mathrm{msd}_{pr}(G) \leq 4$ for any graph G with at least one edge. We also determine paired domination multisubdivision numbers for some classes of graphs. Moreover, we give a constructive characterizations of all trees with paired domination multisubdivision number equal to 4.

Keywords: Paired domination; domination subdivision number; domination multisubdivision number; computational complexity; trees.
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1 Introduction

For domination problems, multiple edges and loops are irrelevant, so we forbid them. We use $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G and denote $|V(G)| = n$, $|E(G)| = m$. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v and $N_G[v] =$

$N_G(v) \cup \{v\}$. The *private neighbourhood of a vertex* u *with respect to a set* $D \subseteq V(G)$, where $u \in D$, is the set $\mathrm{PN}_G[u, D] = N_G[u] - N_G[D - \{u\}]$. If $v \in \mathrm{PN}_G[u, D]$, then we say that v is *a private neighbour of* u *with respect to the set* D.

We say that a vertex v of a graph G is a *leaf* if $|N_G(v)| = 1$. A vertex u is called a *support vertex* if it is adjacent to a leaf. If u is adjacent to more than one leaf, then we call u a *strong support vertex*.

A subset D of $V(G)$ is a *dominating set* in G if every vertex of $V(G)-D$ has at least one neighbour in D.

A *paired dominating set* of a graph G is a set S of vertices of G such that every vertex of G is adjacent to some vertex of S and the subgraph $G[S]$ induced by S contains a perfect matching M (not necessary induced). Two vertices joined by an edge of M are said to be *paired* in S. A matching in an induced subgraph $G[S]$ we denote by M_S. Every graph without an isolated vertex has a paired dominating set since the leaves of any maximal matching form such a set. The *paired domination number* of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality among all paired dominating sets in G. A minimum paired dominating set of a graph G is called a $\gamma_{pr}(G)$-set. Paired domination was studied for example in [3], [10] and [11].

For a graph $G = (V, E)$, the subdivision of an edge $e = uv \in E$ with vertex x leads to a graph with vertex set $V \cup \{x\}$ and edge set $(E - \{uv\}) \cup \{ux, xv\}$. Let $G_{e_1,e_2,\dots,e_k}$ denote the graph G with subdivided edges $e_1, e_2, \dots, e_k$, where each edge is subdivided once. Let $G_{e,t}$, $t \in \{1, 2, \dots\}$ denote graph G with subdivided edge e with t vertices (instead of the edge $e = uv$ we have a path $(u, x_1, x_2, \dots, x_t, v)$). The influence of subdividing an edge on the domination number is studied for example in [1], [2] and [4].

The *spider* $S(\ell_1, ..., \ell_k)$, $\ell_i \geq 1$, $k \geq 2$, is a tree obtained from $K_{1,k}$ by subdividing the edge uv_i $\ell_i - 1$ times, $i = 1, ..., k$. Note that $S(2, 2) \cong P_5$.

A vertex $u \in V(G)$ is said to be $\gamma_{pr}(G)$*-critical* if removing u from G results in a graph with smaller paired domination number, i.e. $\gamma_{pr}(G-u) < \gamma_{pr}(G)$.

For a set $U \subseteq V(G)$ and a vertex $u \in V(G) - U$ let $d(u, U)$ denote the minimum distance between u and a vertex of U, that is $d(u, U) = \min\{d(u, v) : v \in U\}$.

The *paired domination subdivision number*, $\mathrm{sd}_{pr}(G)$, of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the paired domination number. The paired domination subdivision number was defined in [6] and is also studied for example in [5].

In [4] the domination multisubdivision number of a graph was introduced. In this paper we define a similar parameter, denoted $\mathrm{msd}_{pr}(uv)$, to be the minimum number of subdivisions of the edge uv such that the paired domination number of the graph with multisubdivided uv is greater than $\gamma_{pr}(G)$. Moreover, let the *paired domination multisubdivision number* of a graph G, having at least one edge, denoted by $\mathrm{msd}_{pr}(G)$, be defined as

$$\mathrm{msd}_{pr}(G) = \min\{\mathrm{msd}_{pr}(uv) : \; uv \in E(G)\}.$$

Paired domination multisubdivision number is well defined for all graphs without an isolated vertex. For any unexplained terms and symbols see [9].

2 NP-completeness of paired domination subdivision problem

The decision problem of paired domination subdivision problem in this paper is stated as follows:
PAIRED DOMINATION SUBDIVISION NUMBER (PDSN)
INSTANCE: Graph $G = (V, E)$ and the paired domination number of G, $\gamma_{pr}(G)$.
QUESTION: Is $\mathrm{sd}_{pr}(G) > 1$?

Theorem 1 PAIRED DOMINATION SUBDIVISION NUMBER *is* NP-*complete even for bipartite graphs.*

Proof. The proof is by a transformation from 3-SAT, which was proven to be *NP*-complete in [7]. The problem 3-SAT is the problem of determining if there exists an interpretation that satisfies a given Boolean formula. The formula in 3-SAT is given in conjunctive normal form, where each clause contains three literals. We assume that the formula contains the instance of any literal u and its negation $\neg u$ (in the other case all clauses containing the literal u are satisfied by the true assignment of u).

Given an instance, the set of literals $U = \{u_1, u_2, \ldots, u_n\}$ and the set of clauses $C = \{c_1, c_2, \ldots, c_m\}$ of 3-SAT, we construct the following graph G. For each literal u_i construct a gadget G_i on 9 vertices, where u_i and $\neg u_i$ are the leaves (however u_i and $\neg u_i$ are not necessarily to be leaves in G), see Fig. 1.

For each clause c_j we have a clause vertex c_j, where vertex c_j is adjacent to the literal vertices that correspond to the three literals it contains. For example, if $c_j = (u_1 \vee \neg u_2 \vee u_3)$, then the clause vertex c_j is adjacent to the literal vertices u_1, $\neg u_2$ and u_3. Then add new vertices x_0, x_1 and x_2 in such a way that x_2 is adjacent to every clause vertex c_j and to x_1, and x_0 is adjacent x_1. Hence x_0 is of degree one, x_1 is of degree two and x_2 is of degree $m + 1$. Clearly we can see that G is a bipartite graph and it can be built in polynomial time (see Fig. 2).

First observe that each support vertex is contained in any minimum paired dominating set of G. Moreover, any two suport vertices are not

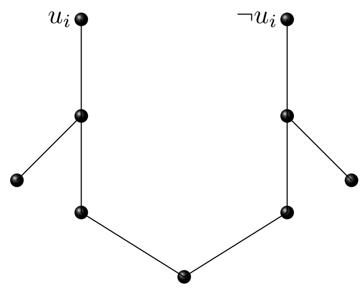

Figure 1: A gadget G_i

adjacent. There are $2n+1$ support vertices in G: two in each gadget and x_1. Thus, $\gamma_{pr}(G) \geq 4n+2$. On the other hand, it is possible to construct a paired dominating set of G of cardinality $4n+2$. Therefore, $\gamma_{pr}(G) = 4n+2$.

Denote by $G_1, G_2, \ldots, G_{m(G)}$ the graph obtained from G by subdividing once edge $e_1, e_2, \ldots, e_{m(G)}$, respectively. For a given graph G and its paired domination number $\gamma_{pr}(G)$ it is possible to verify a certificate for the PDSN problem, which are paired dominating sets of cardinality $\gamma_{pr}(G)$ in $G_1, G_2, \ldots, G_{m(G)}$, in polynomial time.

Assume first C has a satisfying truth assignment. If we subdivide any edge belonging to a gadget G_i, then we may construct a minimum paired dominating set of the resulting graph by adding to it four vertices from each gadget G_i and additionally x_1, x_2. Hence the paired domination number does not increase. The situation is similar if we subdivide any edge incident with a clause vertex. Now let x be the new vertex obtained by subdividing the edge x_0x_1 in G and denote by G_x the obtained graph. Since C has a satisfying truth assignment, the minimum paired dominating set of G_x is constructed by taking the vertices defined by the truth assignment together with three more vertices from each gadget G_i and together with x and x_1. The situation is similar if we subdivide the edge x_1x_2. Therefore we conclude that if C has a satisfying truth assignment, then $\mathrm{sd}_{pr}(G) > 1$.

Assume now C does not have a satisfying truth assignment. Then subdivide the edge x_0x_1 to obtain the graph G_x. The minimum paired dominating set of G_x must contain at least four vertices from each gadget G_i and additionally x and x_1. However, since C does not have a satisfying truth assignment, no subset of $4n$ vertices of $G_1 \cup G_2 \cup \cdots \cup G_n$ can pairwise dominate each gadget vertex and each clause vertex. Therefore if C does not have a satisfying truth assignment, then $\mathrm{sd}_{pr}(G) = 1$. ■

The decision problem of paired domination multisubdivision problem

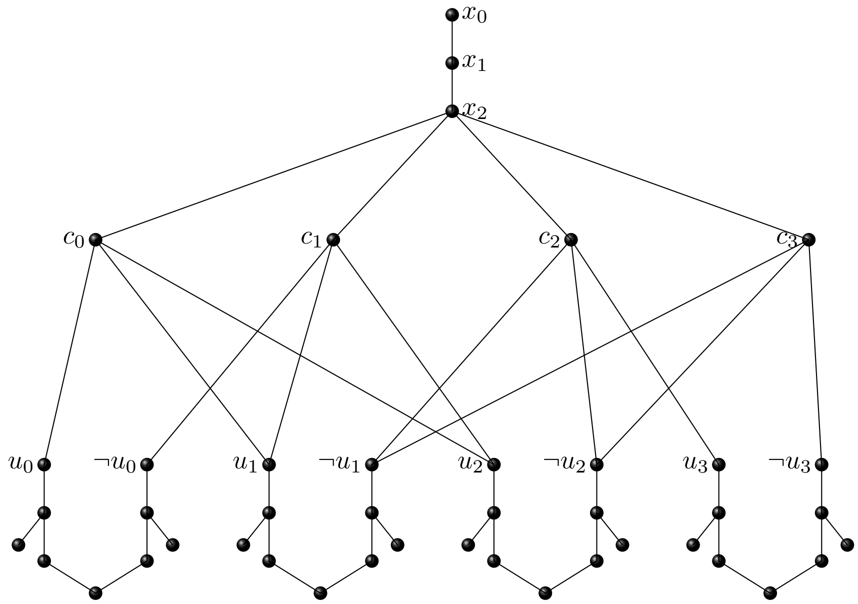

Figure 2: A construction of G for $(u_0 \vee u_1 \vee u_2) \wedge (\neg u_0 \vee u_1 \vee u_2) \wedge (\neg u_1 \vee \neg u_2 \vee u_3) \wedge (\neg u_1 \vee \neg u_2 \vee \neg u_3)$

may be stated similarly:
PAIRED DOMINATION MULTISUBDIVISION NUMBER (DMN)
INSTANCE: Graph $G = (V, E)$ and the paired domination number $\gamma_{pr}(G)$.
QUESTION: Is $\mathrm{msd}_{pr}(G) > 1$?

Observation 2 *Let G be a graph. Then*

$$\mathrm{sd}_{pr}(G) = 1 \textit{ if and only if } \mathrm{msd}_{pr}(G) = 1.$$

This observation implies that one may prove the following result in similar manner as Theorem 1.

Theorem 3 PAIRED DOMINATION MULTISUBDIVISION NUMBER *is* NP-*complete even for bipartite graphs.* ∎

3 Results and bounds for paired domination multisubdivision number

In [6] the paired domination subdivision numbers of cycles and paths are determined. Since any cycle (any path) with an edge subdivided k times is isomorphic to the cycle (path) with k edges subdivided once each, we obtain the following observation.

Observation 4 *For a cycle C_n and a path P_n, where $n \geq 3$,*

$$\mathrm{msd}_{pr}(C_n) = \mathrm{sd}_{pr}(C_n) = \mathrm{msd}_{pr}(P_n) = \mathrm{sd}_{pr}(P_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4, \\ 2 & \text{if } n \equiv 3 \pmod 4, \\ 3 & \text{if } n \equiv 2 \pmod 4, \\ 4 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Proposition 5 *For any connected graph G with at least one edge,*

$$\mathrm{msd}_{pr}(G) \leq 4.$$

Proof. We subdivide an edge $uv \in E(G)$ with vertices w, x, y, z. Denote by D' any $\gamma_{pr}(G_{uv,4})$-set. Then $x, y \in D'$ or $w, x \in D'$ or $w, z \in D'$ or $y, z \in D'$. In all these cases it is no problem to verify that $\gamma_{pr}(G) < |D'|$. Hence, $\mathrm{msd}_{pr}(G) \leq 4$. ■

Now we recall a result of Favaron et al. [6].

Theorem 6 [6] *For every connected graph G on $n \geq 3$ vertices and m edges, $\mathrm{sd}_{pr}(G) = m$ if and only if G is a spider $S(2, \ldots, 2)$.*

This result implies that the difference between $\mathrm{sd}_{pr}(G)$ and $\mathrm{msd}_{pr}(G)$ cannot be bounded from above by any integer in the general case. Moreover, the inequality $\mathrm{msd}_{pr}(G) \leq \mathrm{sd}_{pr}(G)$ is not true in general, since for a corona $C_n \circ K_1$, where C_n is an odd cycle, we have $\mathrm{sd}_{pr}(C_n \circ K_1) = 2$ and $\mathrm{msd}_{pr}(C_n \circ K_1) = 4$.

Next two results are given without proof.

Proposition 7 *For a complete graph K_n, $n \geq 3$,*

$$\mathrm{msd}_{pr}(K_n) = 2.$$

Proposition 8 *For a complete bipartite graph $K_{p,q}$, $p \leq q$,*

$$\mathrm{msd}_{pr}(K_{p,q}) = \begin{cases} 1 & \text{for } p > 1, \\ 2 & \text{for } p = 1, q > 1, \\ 3 & \text{for } p = q = 1. \end{cases}$$

Proposition 9 *If G contains a strong support vertex, then*

$$\mathrm{msd}_{pr}(G) \leq 2.$$

Proof. Let $u, v \in V(G)$ be two leaves adjacent to a support vertex w. Let us subdivide the edge uw with vertices x and y (we replace the edge uw with the path (u, x, y, w)) and let D be a $\gamma_{pr}(G_{uw,2})$-set. Obviously, $x, w \in D$. If w is paired with y in D, then x is paired with u in D and then $D - \{x, y\}$ is a paired dominating set in G. If w is paired with vertex $z \neq y$, then x can be paired with $a \in \{u, y\}$ and again $D - \{x, a\}$ is a paired dominating set in G. In all cases $|D| > \gamma_{pr}(G)$, which completes the proof. ■

Lemma 10 *Let G be a graph with $\mathrm{msd}_{pr}(G) = 4$ and let $uv \in E(G)$ be such that $d_G(u) = 1$. Then*

- *u belongs to some $\gamma_{pr}(G)$-set;*
- *$\gamma_{pr}(G - u) < \gamma_{pr}(G)$, i.e. u is $\gamma_{pr}(G)$-critical.*

Proof. Let G be a graph with $\mathrm{msd}_{pr}(G) = 4$ and let $uv \in E(G)$ be such that $d_G(u) = 1$. Denote by G' the graph obtained from G by subdividing uv with vertices x_1, x_2, x_3 (we replace the edge uv with the path (u, x_1, x_2, x_3, v)). Since $\mathrm{msd}_{pr}(G) = 4$, clearly $\gamma_{pr}(G') = \gamma_{pr}(G)$.

Without loss of generality we may assume that x_1, x_2 belong to a $\gamma_{pr}(G')$-set denoted by D' and $u \notin D'$. If $x_3, v \in D'$, then $(D' - \{x_1, x_2, x_3\}) \cup \{u\}$ would be a smaller paired dominating set of G, which is impossible. If $x_3 \notin D'$ and $v \in D'$, then $D' - \{x_1, x_2\}$ would be a smaller paired dominating set of G, a contradiction. We conclude $(D' - \{x_1, x_2\}) \cup \{u, v\}$ is a minimum paired dominating set of G and hence u belongs to some $\gamma_{pr}(G)$-set. Moreover,

$$\gamma_{pr}(G - u) = \gamma_{pr}(G' - \{u, x_1, x_2, x_3\}) < \gamma_{pr}(G') = \gamma_{pr}(G),$$

which implies that u is $\gamma_{pr}(G)$-critical. ■

Theorem 11 *Let G be a connected graph with $\mathrm{msd}_{pr}(G) = 4$. Then each edge of G belongs to a perfect matching in a subgraph induced by a minimum paired dominating set in G.*

Proof. Suppose G is a connected graph with $\mathrm{msd}_{pr}(G) = 4$ and suppose there is an edge uv such that for any minimum paired dominating set D and for any perfect matching M_D in $G[D]$, we have $uv \notin M_D$. Then by Lemma 10, neither u nor v has degree 1. Denote by G' the graph obtained from G by subdividing uv with vertices x_1, x_2, x_3 (we replace the edge uv with the path (u, x_1, x_2, x_3, v)). Let D be a $\gamma_{pr}(G')$-set. Hence, $|D| = \gamma_{pr}(G') = \gamma_{pr}(G)$ and $D \cap \{x_1, x_2, x_3\} \neq \emptyset$. We can assume that $D \cap \{x_1, x_2, x_3\} \leq 2$, because if $|D \cap \{x_1, x_2, x_3\}| = 3$, then we could exchange one vertex from $\{x_1, x_2, x_3\}$ with u or v to obtain a minimum paired dominating set of G' such that $|D \cap \{x_1, x_2, x_3\}| = 2$.

If $x_2 \in D$, then without loss of generality we assume that $x_1 \in D$ and $x_3 \notin D$. If $u \notin D$ and $v \notin D$, then $D^* = (D - \{x_1, x_2\}) \cup \{u, v\}$ is a minimum paired dominating set of G such that $uv \in M_{D^*}$, where M_{D^*} is a matching in $G[D^*]$, a contradiction. Therefore, $u \in D$ or $v \in D$. However in each of these cases $D - \{x_1, x_2\}$ is a paired dominating set in G of cardinality smaller than $\gamma_{pr}(G)$, which is not possible.

Assume now $x_2 \notin D$. Then, without loss of generality, $\{u, x_1\} \subset D$. If $x_3 \in D$, then $v \in D$ and $D - \{x_1, x_3\}$ is paired dominating set in G of smaller cardinality then $\gamma_{pr}(G)$, which again is not possible. If $x_3 \notin D$, then $v \in D$ and let us denote by v' a vertex such that $vv' \in M_D$. Hence, either $D - \{u, x_1\}$ is paired domination set in G of cardinality less than $\gamma_{pr}(G)$, which is not possible, or the graph $G' = G_{uv,3}$ contains a vertex $u' \in \mathrm{PN}_{G'}[u, D]$. In the second case $D' = (D - \{x_1\}) \cup \{u'\}$ is a paired dominating set of G. Moreover, v' has a private neighbour, let us say v'', in relation to the set D' (otherwise $D' - \{u', v'\}$ would be a $\gamma_{pr}(G)$–set, which is not possible). Hence, we obtain that $D^* = (D' - \{u'\}) \cup \{v''\}$ is a paired dominating set in G and furthermore $|D^*| = |D'| = |D| = \gamma_{pr}(G)$ and $uv \in M_{D^*}$, a contradiction.

■

The converse statement is not true. Each edge of a cycle belongs to a matching of a minimum paired dominating set of the cycle, but the paired domination multisubdivision numbers of cycles vary from 1 to 4. However the converse of Theorem 11 is true for trees without strong support vertices, see Lemma 15.

4 Paired domination multisubdivision numbers of trees

In this section we consider paired domination multisubdivision numbers of trees. The main result of this section is a constructive characterization of all trees T with $\mathrm{msd}_{pr}(T) = 4$.

The label of a vertex v is also called its status and is denoted by $sta(v)$. Let the vertices of P_5 have labels as follows: the two leaves have status A, the two support vertices have status B and the remaining vertex has status C.

Let $\mathcal{T}$ be the family of all trees T that can be obtained from a sequence $T_1, \ldots, T_j$ ($j \geq 1$) of trees such that T_1 is the path P_5 and $T = T_j$, and, if $j > 1$, then T_{i+1} can be obtained recursively from T_i by the operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$:

Operation $\mathcal{T}_1$. Assume $sta(v) = A$. Then add a path (w, x, y, z) and the edge vw. Let $sta(x) = C$, $sta(w) = sta(y) = B$ and $sta(z) = A$.

Operation $\mathcal{T}_2$. Assume $sta(v) = B$. Then add a path (x, y, z) and the edge vx. Let $sta(x) = C$, $sta(y) = B$ and $sta(z) = A$.

Operation $\mathcal{T}_3$. Assume $sta(v) = C$. Then add a path (y, z) and the edge vy. Let $sta(y) = B$ and $sta(z) = A$.

Fig. 3 and 4 show examples of trees belonging to the family $\mathcal{T}$.

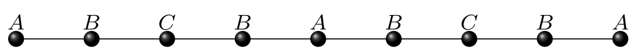

Figure 3: Tree T_2 obtained from P_5 by operation $\mathcal{T}_1$

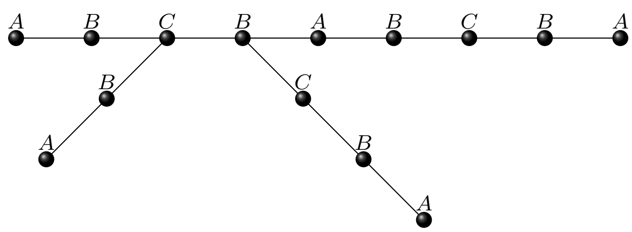

Figure 4: T_4 obtained from T_2 by operation $\mathcal{T}_2$ and operation $\mathcal{T}_3$

The next result follows immediately from the way in which each tree in the family $\mathcal{T}$ is constructed. Denote by $\Omega(T)$ the set of all leaves of T and by $A(T)$ the set of all vertices of T with status A.

Observation 12 *If a tree T belongs to the family $\mathcal{T}$ and $x, y \in V(T)$, then:*

1. *Every leaf of T has status A and every support vertex has status B.*
2. *If $d(x, \Omega(T)) = 2$, then $sta(x) = C$.*
3. *If $d(x, \Omega(T)) = 3$, then $sta(x) = B$.*

4. *If $d(x, \Omega(T)) = 4$, then $sta(x) \in \{A, C\}$.*

5. *If $sta(x) = A$ or $sta(x) = C$, then each neighbour of x has status B.*

6. *No two vertices with the same status are adjacent.*

7. *If $sta(x) = B$, then x is adjacent to exactly one vertex with status A and at least one vertex of status C.*

8. *If $sta(x) = sta(y) = A$, then $d(x, y) \geq 4$. Thus no paired vertices of a minimum paired dominating set of T can dominate both x and y. Therefore, $\gamma_{pr}(T) \geq 2|A(T)|$.*

The following lemmas give some less obvious properties of trees in the family $\mathcal{T}$.

Lemma 13 *Let T be a tree belonging to the family $\mathcal{T}$. Then*

$$\gamma_{pr}(T) = 2|A(T)|.$$

Proof. Let T be a tree belonging to the family $\mathcal{T}$. We proceed by induction on the number $s(T)$ of operations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ required to construct the tree T. If $s(T) = 0$, then clearly the statement is true.

Now let T be a tree with $s(T) = k$ for some positive integer k and assume that for each tree T' belonging to the family $\mathcal{T}$ with $s(T') < k$, is $\gamma_{pr}(T') = 2|A(T')|$. Then T can be obtained from a tree T' belonging to $\mathcal{T}$ by operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.

Clearly $|A(T)| = |A(T')| + 1$. Thus, by Observation 12(8) and the induction hypothesis,

$$\gamma_{pr}(T) \geq 2|A(T)| = 2|A(T')| + 2 = \gamma_{pr}(T') + 2.$$

On the other hand, it is possible to add to a minimum paired dominating set of T' two vertices belonging to $V(T) - V(T')$ in order to obtain a paired dominating set of T. This implies that $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$. Therefore, $\gamma_{pr}(T) = 2|A(T)|$. ■

Lemma 14 *Let T be a tree belonging to the family $\mathcal{T}$ and let $u \in V(T)$. Then*

1. *$\gamma_{pr}(T - u) < \gamma_{pr}(T)$ (or u is $\gamma_{pr}(T)$-critical) if $sta(u) = A$.*

2. *$\gamma_{pr}(T - u) = \gamma_{pr}(T)$ (or u is $\gamma_{pr}(T)$-stable) if $sta(u) = C$.*

3. $\mathrm{msd}_{pr}(T) = 4$.

Proof. Let T be a tree belonging to the family $\mathcal{T}$. We proceed by induction on the number $s(T)$ of operations required to construct the tree T. If $s(T) = 0$, then $T = P_5$ and clearly all statements are true.

Now let T be a tree with $s(T) = k$ for some positive integer k and assume that for each tree T' belonging to the family $\mathcal{T}$ with $s(T') < k$, the statements 1.-3. are true. Then T can be obtained from a tree T' belonging to $\mathcal{T}$ by operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.

Assume first that T is obtained from T' by operation $\mathcal{T}_1$, e.g. T is obtained from T', where $v \in V(T')$ and $sta(v) = A$, by adding a path (w, x, y, z) and the edge vw, with $sta(x) = C$, $sta(w) = sta(y) = B$ and $sta(z) = A$.

Let $u \in V(T)$ be such that $sta(u) = A$. If u belongs also to $V(T')$, then by the inductive hypothesis, Lemma 13 and the construction of T from T',

$$\gamma_{pr}(T-u)-2 \leq \gamma_{pr}(T-\{w,x,y,z,u\}) = \gamma_{pr}(T'-u) < \gamma_{pr}(T') = \gamma_{pr}(T)-2,$$

which implies that $\gamma_{pr}(T-u) < \gamma_{pr}(T)$. If $u \notin V(T')$, then $u = z$ and since v is $\gamma_{pr}(T')$-critical, we obtain

$$\gamma_{pr}(T-u)-2 \leq \gamma_{pr}(T-\{w,x,y,u,v\}) = \gamma_{pr}(T'-v) < \gamma_{pr}(T') = \gamma_{pr}(T)-2.$$

Therefore, if $u \in V(T)$ and $sta(u) = A$, then u is $\gamma_{pr}(T)$-critical.

Let $u \in V(T)$ be such that $sta(u) = C$. If u belongs also to T', then by the inductive hypothesis, Lemma 13 and the construction of T from T',

$$\gamma_{pr}(T-u)-2 \leq \gamma_{pr}(T-\{w,x,y,z,u\}) = \gamma_{pr}(T'-u) = \gamma_{pr}(T') = \gamma_{pr}(T)-2,$$

which implies that $\gamma_{pr}(T-u) \leq \gamma_{pr}(T)$. Since T and $T-u$ have the same number of vertices with status A and each neighbour of a vertex of status A is of status B, $\gamma_{pr}(T-u) = \gamma_{pr}(T)$. If $u \notin V(T')$, then $u = x$ and since u is $\gamma_{pr}(T')$-critical, we have

$$\gamma_{pr}(T-u)-4 \leq \gamma_{pr}(T-\{w,x,y,z,u\}) = \gamma_{pr}(T'-u) < \gamma_{pr}(T') = \gamma_{pr}(T)-2.$$

Hence $\gamma_{pr}(T-u) \leq \gamma_{pr}(T)$. Since T and $T-u$ have the same number of vertices with status A and each neighbour of a vertex of status A is of status B again we obtain that $\gamma_{pr}(T-u) = \gamma_{pr}(T)$. Therefore, if $u \in V(T)$ and $sta(u) = C$, then u is $\gamma_{pr}(T)$-stable.

Let ss' be an edge of the tree T. If $ss' \in E(T')$, then by the induction hypothesis applied to T' and the construction of T, we see that subdividing ss' three times will not increase the paired domination number of T. If $ss' \notin E(T')$, then the tree T'' obtained from T by subdividing ss' three times is isomorphic to the tree obtained from T' by attaching to $v \in V(T')$ a path $P_7 = (x_1, x_2, \ldots, x_7)$. Since v is $\gamma_{pr}(T')$-critical, we conclude that any $\gamma_{pr}(T'-v)$-set may be extended to a paired dominating set of T'' by adding x_1, x_2, x_5, x_6. Thus $\gamma_{pr}(T'') \leq \gamma_{pr}(T)$. Therefore $\mathrm{msd}_{pr}(T) = 4$.

Assume next that T is obtained from T' by operation $\mathcal{T}_2$, e.g. T is obtained from T', where $v \in V(T')$ and $sta(v) = B$, by adding a path (x, y, z) and the edge vx, with $sta(x) = C$, $sta(y) = B$ and $sta(z) = A$.

Let $u \in V(T)$ be such that $sta(u) = A$. If u belongs also to T', then by the inductive hypothesis, Lemma 13 and the construction of T from T',

$$\gamma_{pr}(T-u)-2 \leq \gamma_{pr}(T-\{x,y,z,u\}) = \gamma_{pr}(T'-u) < \gamma_{pr}(T') = \gamma_{pr}(T)-2,$$

which implies that $\gamma_{pr}(T-u) < \gamma_{pr}(T)$. If $u \notin V(T')$, then $u = z$ and since $sta(v) = B$, v is adjacent in T' to a vertex with status A, say v'. Then v' is $\gamma_{pr}(T')$-critical, so we obtain

$$\gamma_{pr}(T'-v') < \gamma_{pr}(T') = \gamma_{pr}(T) - 2.$$

Since v' is $\gamma_{pr}(T')$-critical, v does not belong to any $\gamma_{pr}(T'-v')$-set. Thus any $\gamma_{pr}(T'-v')$-set may be extended to a paired dominating set of $T-u$ by adding to it x, v. Therefore,

$$\gamma_{pr}(T-u) \leq \gamma_{pr}(T'-v') + 2 < \gamma_{pr}(T).$$

We conclude that if $u \in V(T)$ and $sta(u) = A$, then u is $\gamma_{pr}(T)$-critical.

Let $u \in V(T)$ be such that $sta(u) = C$. If u belongs also to T', then by the inductive hypothesis, Lemma 13 and the construction of T from T',

$$\gamma_{pr}(T-u)-2 \leq \gamma_{pr}(T-\{x,y,z,u\}) = \gamma_{pr}(T'-u) = \gamma_{pr}(T') = \gamma_{pr}(T)-2,$$

which implies that $\gamma_{pr}(T-u) \leq \gamma_{pr}(T)$. Since T and $T-u$ have the same number of vertices with status A, $\gamma_{pr}(T-u) = \gamma_{pr}(T)$. If $u \notin V(T')$, then $u = x$ and since $sta(v) = B$, v is adjacent in T' to a vertex with status A, say v'. Then v' is $\gamma_{pr}(T')$-critical, so we obtain

$$\gamma_{pr}(T'-v') < \gamma_{pr}(T') = \gamma_{pr}(T) - 2.$$

Thus, $\gamma_{pr}(T'-v') \leq \gamma_{pr}(T) - 4$. Since v' is $\gamma_{pr}(T')$-critical, v does not belong to any $\gamma_{pr}(T'-v')$-set. Hence, any $\gamma_{pr}(T'-v')$-set may be extended to a minimum paired dominating set of $T-u$ by adding v, v' and y, z. Therefore, $\gamma_{pr}(T-u) = \gamma_{pr}(T)$. We conclude that if $u \in V(T)$ and $sta(u) = C$, then u is $\gamma_{pr}(T)$-stable.

Let ss' be an edge of the tree T. If $ss' \in E(T')$, then by the induction hypothesis applied to T' and the construction of T, we see that the subdivision of ss' three times will not increase the paired domination number of T. Thus assume $ss' \notin E(T')$. Then the tree T'' obtained from T by subdivision of ss' three times is isomorphic to the tree obtained from T' by attaching to $v \in V(T')$ a path $P_6 = (x_1, x_2, \ldots, x_6)$. Since $sta(v) = B$, v is adjacent in T' to a vertex with status A, say v'. Moreover, $\gamma_{pr}(T' - v') \leq \gamma_{pr}(T) - 4$. Since v' is $\gamma_{pr}(T')$-critical, v does not belong to any $\gamma_{pr}(T' - v')$-set. Hence any $\gamma_{pr}(T' - v')$-set may be extended to a paired dominating set of T'' by adding v, x_1, x_4, x_5. Thus $\gamma_{pr}(T'') \leq \gamma_{pr}(T)$. Therefore $\mathrm{msd}_{pr}(T) = 4$.

Lastly, assume that T is obtained from T' by operation $\mathcal{T}_3$, e.g. T is obtained from T', where $v \in V(T')$ and $sta(v) = C$, by adding a path (y, z) and the edge vy, with $sta(y) = B$ and $sta(z) = A$.

Let $u \in V(T)$ be such that $sta(u) = A$. If u belongs also to T', then similarly as in previous cases, we obtain that $\gamma_{pr}(T - u) < \gamma_{pr}(T)$. If $u \notin V(T')$, then $u = z$ and by the induction hypothesis, each edge of T' belongs to a matching of a minimum paired dominating set of T'. Thus there exists a $\gamma_{pr}(T')$-set containing v. Since $vy \in E(T)$,

$$\gamma_{pr}(T - u) = \gamma_{pr}(T') < \gamma_{pr}(T).$$

We conclude that if $u \in V(T)$ and $sta(u) = A$, then u is $\gamma_{pr}(T)$-critical.

Let $u \in V(T)$ be such that $sta(u) = C$. Then u belongs also to T' and similarly as in previous cases, we obtain that $\gamma_{pr}(T - u) = \gamma_{pr}(T)$. Thus u is $\gamma_{pr}(T)$-stable.

Let ss' be an edge of the tree T. If $ss' \in E(T')$, then by the induction hypothesis applied to T' and the construction of T, we see that the subdivision of ss' three times will not increase the paired domination number of T. Thus assume $ss' \notin E(T')$. Then the tree T'' obtained from T by subdividing ss' three times is isomorphic to the tree obtained from T' by attaching to $v \in V(T')$ a path $P_5 = (x_1, x_2, \ldots, x_5)$. Since $sta(v) = C$, v belongs to a $\gamma_{pr}(T')$-set, say D'. Then $D = D' \cup \{x_3, x_4\}$ is a paired dominating set of T''. Then $\gamma_{pr}(T'') \leq \gamma_{pr}(T)$ and therefore $\mathrm{msd}_{pr}(T) = 4$.

■

Lemma 15 *Let T be a tree of order at least 4, without strong support vertices and such that each edge of T belongs to a matching of a minimum paired dominating set of T. Then T belongs to the family $\mathcal{T}$.*

Proof. The statement is clearly true for any tree T with 4 or 5 vertices. We proceed by induction on the number of vertices of a tree.

Let T be a tree with at least six vertices, without strong support vertices and such that each edge of T belongs to a matching of a minimum paired dominating set of T. We assume that the result is true for each tree T' with $n(T') < n(T)$. Denote by $(x_1, x_2, \ldots, x_k)$ a longest path contained in T. Then $d(x_2) = 2$.

If x_3 is a support vertex, say x_0 is of degree 1 and is adjacent to x_3, then by assumptions x_0x_3 are paired in some minimum paired dominating set D of T. But then $x_1, x_2 \in D$ and $D - \{x_0, x_1\}$ with x_2, x_3 paired is a smaller paired dominating set of T, which is impossible. Thus x_3 is not a support vertex.

If $d(x_3) > 2$, then more P_2's are attached to x_3 and, since each edge of T belongs to a matching of a minimum paired dominating set of T, each edge of $T' = T - \{x_1, x_2\}$ belongs to a matching of a minimum paired dominating set of T'. Further, by the construction of T' from T, T' also is without strong support vertices. Thus T' belongs to the family $\mathcal{T}$. Moreover, since $d_{T'}(x_3, \Omega(T')) = 2$, Observation 12 implies that $sta(x_3) = C$. Hence T may be obtained from T' by operation $\mathcal{T}_3$. Therefore $T \in \mathcal{T}$.

Assume $d(x_3) = 2$. If x_4 is a support vertex, then since each edge of T belongs to a matching of a minimum paired dominating set of T, each edge of $T' = T - \{x_1, x_2, x_3\}$ belongs to a matching of a minimum paired dominating set of T'. Further, by the construction of T' from T, T' also is without strong support vertices. Thus T' belongs to the family $\mathcal{T}$. Moreover, since x_4 is a support vertex, $sta(x_4) = B$. Hence T may be obtained from T' by operation $\mathcal{T}_2$. Therefore $T \in \mathcal{T}$.

Assume x_4 is not a support vertex and there exists a leaf, say x_0, such that $d_T(x_4, x_0) = 2$. Denote by x_0' the neighbor of x_4 and x_0. Consider a matching M of a paired dominating set D of T containing x_3x_4. Then $x_1x_2 \in M$ and $x_0x_0' \in M$. However then $D - \{x_1, x_0\}$ is a smaller paired dominating set of T, a contradiction. We conclude that $d_T(x_4, \Omega(T)) = 3$.

If $d_T(x_4) > 2$ and $d_T(x_4, \Omega(T)) = 3$, then since each edge of T belongs to a matching of a minimum paired dominating set of T, each edge of $T' = T - \{x_1, x_2, x_3\}$ belongs to a matching of a minimum paired dominating set of T'. Further, by the construction of T' from T, T' also is without strong support vertices. Thus T' belongs to the family $\mathcal{T}$. Moreover, since $d_{T'}(x_4, \Omega(T')) = 3$ (otherwise there is a longer path in T than $(x_1, x_2, \ldots, x_k)$), Observation 12 implies that $sta(x_4) = B$. Hence T may be obtained from T' by operation $\mathcal{T}_2$. Therefore in this case $T \in \mathcal{T}$.

Therefore assume $d_T(x_4) = 2$. Denote $T' = T - \{x_1, x_2, x_3, x_4\}$. Then any $\gamma_{pr}(T')$-set may be extended to a paired dominating set of T by adding x_2 and x_3, so $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$. On the other hand, there exists a $\gamma_{pr}(T)$-set containing x_2, x_3 and not containing x_1, x_4, so $\gamma_{pr}(T') \leq \gamma_{pr}(T) - 2$.

Since each edge of T belongs to a matching of a minimum paired dominating set of T, each edge of T' belongs to a matching of a minimum paired dominating set of T'. Suppose $d_T(x_5) = 2$ and x_6 is a support vertex. Then, by the assumptions, x_3x_4 belongs to a matching of a minimum paired dominating set of T, say D, and therefore $x_1, x_2, x_6 \in D$. However in this situation $D - \{x_1, x_2, x_3, x_4\}$ is a paired dominating set of T' of cardinality smaller than $\gamma_{pr}(T) - 2$, a contradiction. Thus T' is without strong support vertices. Hence $T' \in \mathcal{T}$. By the assumption, x_3x_4 belongs to a matching M of a minimum paired dominating set D of T. Then $x_1x_2 \in M$ and $D - \{x_1, x_2, x_3, x_4\}$ is a paired dominating set of $T' - x_5$. Hence

$$\gamma_{pr}(T' - x_5) \leq \gamma_{pr}(T) - 4 = \gamma_{pr}(T') - 2.$$

Thus x_5 is $\gamma_{pr}(T')$-critical. By Lemma 14, $sta(x_5) \neq C$. Observe that if $d_{T'}(x_5, \Omega(T')) \in \{1, 2, 3\}$, then x_5 would not be $\gamma_{pr}(T')$-critical. Thus we obtain that either x_5 is a leaf or $d_{T'}(x_5, \Omega(T')) = 4$. In both cases Observation 12 implies that $sta(x_5) = A$. Hence T may be obtained from T' by operation $\mathcal{T}_1$. Therefore in this case $T \in \mathcal{T}$. ∎

Theorem 16 *Let T be a tree with $n(T) \geq 4$. Then the following statements are equivalent:*

1. $\mathrm{msd}_{pr}(T) = 4$.
2. *T belongs to the family $\mathcal{T}$.*

Proof. If T belongs to the family $\mathcal{T}$, then by Lemma 14, $\mathrm{msd}_{pr}(T) = 4$.

If T is a tree of order at least 4 and $\mathrm{msd}_{pr}(T) = 4$, then by Lemma 10, T does not contain any strong support vertex and by Theorem 11, each edge of T belongs to some matching of a minimum paired dominating set of T. Therefore Lemma 15 implies that T belongs to the family $\mathcal{T}$. ∎

5 Open problems

We conclude with a short list of open problems for future work.

Question 1 *Characterize other classes of graphs with the paired domination multisubdivision number equal to 4.*

Question 2 *Let G be a graph with $\mathrm{msd}_{pr}(G) = 4$ and let $xy \in E(G)$ be such that both x and y are $\gamma_{pr}(G)$-critical vertices. Is $\mathrm{msd}_{pr}(G - xy) = 4$? This result is true for cycles. Is it true for all graphs?*

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