Ergod. Th. & Dynam. Sys., (2023), **43**, 1081–1086
doi:10.1017/etds.2022.3

Distortion in the group of circle homeomorphisms

JULIUSZ BANECKI† and TOMASZ SZAREK†‡

† *Institute of Mathematics Polish Academy of Sciences, Abrahama 18, 81-967 Sopot, Poland*
(e-mail: juliusz.banecki@autonomik.pl, tszarek@impan.pl)
‡ *Faculty of Physics and Applied Mathematics, Gdańsk University of Technology, ul. Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland*

(*Received* 5 *October* 2021 *and accepted in revised form* 6 *January* 2022)

Abstract. Let G be the group $\mathrm{PAff}_+(\mathbb{R}/\mathbb{Z})$ of piecewise affine circle homeomorphisms or the group $\mathrm{Diff}^\infty(\mathbb{R}/\mathbb{Z})$ of smooth circle diffeomorphisms. A constructive proof that all irrational rotations are distorted in G is given.

Key words: homeomorphisms, distortion, rotation
2020 Mathematics Subject Classification: 37C85 (Primary); 57M60 (Secondary)

1. *Introduction*

Let G be a group with some finite generating set $\mathcal{G}$. We define the metric $d_{\mathcal{G}}$ on G by taking $d_{\mathcal{G}}(g_1, g_2)$ to be the infimum over all $k \geq 0$ such that there exist $f_1, \ldots, f_k \in \mathcal{G}$ and $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$ satisfying $g_2 = f_1^{\epsilon_1} \cdots f_k^{\epsilon_k} g_1$.

Now let H be an arbitrary group. An element $f \in H$ is called *distorted* in H if there exists a finitely generated subgroup $G \subset H$ containing f such that

$$\lim_{n\to\infty} \frac{d_{\mathcal{G}}(f^n, \mathrm{id})}{n} = 0$$

for some (and hence every) generating set $\mathcal{G}$. Since the limit always exists, it is enough to verify it for some subsequence. The notion of distortion comes from geometric group theory and was introduced by Gromov in [7].

The problem of the existence of distorted elements in some groups of homeomorphisms has been intensively studied for many years (see [2, 3–6, 8, 10, 11]). Substantial progress has been achieved for groups of diffeomorphisms of manifolds. In particular, Avila [1] proved that rotations with irrational rotation number are distorted in the group of smooth diffeomorphisms of the circle. In this note we give a constructive proof that all irrational rotations are distorted both in the group of piecewise affine circle homeomorphisms,

$\mathrm{PAff}_+(\mathbb{R}/\mathbb{Z})$, and in the group of smooth circle diffeomorphisms, $\mathrm{Diff}^\infty(\mathbb{R}/\mathbb{Z})$. The result gives an answer to Question 11 in [9] (see also Question 2.5 in [5]). So far it has not even been known whether there exist distorted elements in $\mathrm{PAff}_+(\mathbb{R}/\mathbb{Z})$. Now from [8] it follows that each distorted element is conjugate to a rotation.

From now on let G be either $\mathrm{PAff}_+(\mathbb{R}/\mathbb{Z})$ or $\mathrm{Diff}^\infty(\mathbb{R}/\mathbb{Z})$. We say that $g \in G$ is *trivial on some set* if there exists a non-empty open set $I \subset [0, 1)$ such that $g(x) = x$ for $x \in I$. The set of all homeomorphisms in G which are trivial on some set will be denoted by G_{triv}. By T we denote the set of all rotations, and let T_α be the rotation with rotation number α.

This paper is devoted to the proof of the following theorem.

THEOREM. *All irrational rotations are distorted in G.*

2. *Proofs*

We first formulate two lemmas and deduce the theorem. The proofs of the lemmas will be given at the end of the paper.

LEMMA 1. *For any irrational rotation T_α and $g \in G_{triv} \cup T$ there exist a finite generating set $\mathcal{G}_g \subset G$ and a constant $C > 0$ such that*

$$d_{\mathcal{G}_g}(T_\alpha^n g T_\alpha^{-n}, \mathrm{id}) \le C \log n \quad \textit{for all } n \ge 1.$$

LEMMA 2. *In G there exist $g_1, \ldots, g_l \in G_{triv} \cup T$ and $k, k_1, \ldots, k_l \in \mathbb{Z}$ with $k \ne k_1 + \cdots + k_l$, such that for each sufficiently small $\beta > 0$ the element $x = T_\beta$ satisfies*

$$x^{k_1} g_1 x^{k_2} g_2 \cdots x^{k_l} g_l = x^k. \tag{1}$$

Proof of the theorem. Fix an irrational rotation T_α. From Lemma 2 it follows that in G there exists an equation of the form (1) such that $x = T_\beta$, for all sufficiently small β, is its solution. Let $\mathcal{G} = \mathcal{G}_{g_1} \cup \cdots \cup \mathcal{G}_{g_l}$, where $\mathcal{G}_{g_i}$, $i = 1, \ldots, l$, are finite generating sets derived from Lemma 1 for T_α. We may rewrite equation (1) in the form

$$x^{k_1} g_1 x^{-k_1} x^{k_2+k_1} g_2 x^{-k_2-k_1} \cdots x^{k_1+\cdots+k_l} g_l x^{-k_1-\cdots-k_l} = x^{k-k_1-\cdots-k_l}. \tag{2}$$

Let β_0 be a positive constant such that $x = T_\beta$ for $\beta \in (0, \beta_0)$ satisfies (2). Set $m := k - k_1 - \cdots - k_l$, and let (n_i) be an increasing sequence of integers such that $n_i\alpha \in (0, \beta_0) \pmod 1$. From Lemma 1 it follows that

$$d_{\mathcal{G}}(T_\alpha^{n_i(k_1+\cdots+k_j)} g_j T_\alpha^{-n_i(k_1+\cdots+k_j)}, \mathrm{id}) \le C_j \log n_i \quad \text{for all } i \ge 1 \text{ and } j = 1, \ldots, l.$$

Since $x = T_{n_i\alpha}$ satisfies (2), we obtain

$$d_{\mathcal{G}}(T_\alpha^{n_i m}, \mathrm{id}) \le \sum_{j=1}^{l} C_j \log n_i := C \log n_i \quad \text{for all } i \ge 1.$$

Hence

$$\lim_{n\to\infty} \frac{d_{\mathcal{G}}(T_\alpha^n, \mathrm{id})}{n} = \lim_{i\to\infty} \frac{d_{\mathcal{G}}(T_\alpha^{n_i m}, \mathrm{id})}{n_i m} \le \frac{C}{m} \lim_{i\to\infty} \frac{\log n_i}{n_i} = 0$$

and the proof is complete. □

https://doi.org/10.1017/etds.2022.3 Published online by Cambridge University Press

Proof of Lemma 1. The proof relies on the observation that for a given interval $I \subset (0, 1)$ there exists a finite generating set $\mathcal{G} \subset G$ such that for any $n \geq 1$ there exists a homeomorphism h_n with $d_{\mathcal{G}}(h_n, \mathrm{id}) \leq C \log n$ for some constant $C > 0$ independent of n, and $h_n(x) = T_\alpha^n(x)$ for $x \notin I$. Without loss of generality we may assume that $I = (a, 1)$. Let $m \geq 1$ be an integer such that $a + 2/m < 1$. Let $h \in G$ be any homeomorphism such that $h(x) = x/2$ for $x \in [0, a + 2/m)$, and let $r(x) = x + 1/m$.

We shall define h_n by induction. Set $h_0 = \mathrm{id}$. If n is odd we put $h_n = T_\alpha h_{n-1}$. If n is even, we take $s_n := h_{n/2}h$ and observe that $s_n((0, a)) = (n\alpha/2, a/2 + n\alpha/2)$. Let $k \in \{1, \ldots, m\}$ be such that $n\alpha/2 + k/m \in [0, 1/m) \pmod 1$. Then $r^k s_n((0, a)) \subset (0, a/2 + 1/m)$. Therefore

$$h^{-1}r^k h_{n/2}h(x) = 2(x/2 + n\alpha/2 + k/m) = x + n\alpha + 2k/m = T_\alpha^n(x) + 2k/m \tag{3}$$

for $x \in (0, a)$. Put $h_n := r^{-2k}h^{-1}r^k h_{n/2}h$, and let $\mathcal{G} := \{T_\alpha, h, r\}$. Note that

$$d_{\mathcal{G}}(h_n, \mathrm{id}) \leq 3m + 3 + d_{\mathcal{G}}(h_{\lfloor n/2 \rfloor}, \mathrm{id}).$$

Thus we obtain $d_{\mathcal{G}}(h_n, \mathrm{id}) \leq C \log n$. Finally, observe that for any $g \in G_{\mathrm{triv}}$ such that $g(x) = \mathrm{id}$ on I we have

$$T_\alpha^n g T_\alpha^{-n} = h_n g h_n^{-1}. \tag{4}$$

Indeed, from (3) and the definition of h_n and r it follows that $h_n(x) = T_\alpha^n(x)$ for $x \in (0, a)$, and

$$h_n((0, a)) = T_\alpha^n((0, a)) = (n\alpha, a + n\alpha). \tag{5}$$

Therefore, we have

$$h_n^{-1}(x) = T_\alpha^{-n}(x) \in (0, a) \quad \text{for } x \in (n\alpha, a + n\alpha).$$

Since $g(x) = x$ for $x \in (a, 1)$ and g is a homeomorphism, we have $g((0, a)) = (0, a)$.

To justify equality (4), first fix $x \in (n\alpha, a + n\alpha)$. Then we have

$$h_n^{-1}(x) = T_\alpha^{-n}(x) \in (0, a)$$

and

$$(gh_n^{-1})(x) = (gT_\alpha^{-n})(x) \in (0, a).$$

Consequently, we obtain

$$h_n g h_n^{-1}(x) = T_\alpha^n g T_\alpha^{-n}(x) \quad \text{for } x \in (n\alpha, a + n\alpha),$$

by the fact that $h_n(x) = T_\alpha^n(x)$ for $x \in (0, a)$. On the other hand, if $x \notin (n\alpha, a + n\alpha)$, from (5) and the fact that T_α^n and h_n are homeomorphisms, we obtain

$$T_\alpha^{-n}(x) \in (a, 1] \quad \text{and} \quad h_n^{-1}(x) \in (a, 1].$$

Since $g(x) = x$ for $x \in (a, 1]$, we have

$$(T_\alpha^n g T_\alpha^{-n})(x) = (T_\alpha^n T_\alpha^{-n})(x) = x$$

https://doi.org/10.1017/etds.2022.3 Published online by Cambridge University Press

and

$$(h_n g h_n^{-1})(x) = (h_n h_n^{-1})(x) = x.$$

Thus equality (4) holds true.

Finally, we obtain

$$d_{\mathcal{G}}(T_\alpha^n g T_\alpha^{-n}, \mathrm{id}) \le C \log n.$$

In the case where g is a rotation the conclusion of the lemma is obvious. □

Proof of Lemma 2. Let $\beta \in (0, 10^{-3})$, and let $f_1 \in G_{\mathrm{triv}}$ be arbitrary such that

$$f_1(x) = 0.4 + 2(x - 0.4) \text{ for } x \in [0.4, 0.6] \quad \text{and} \quad f_1(x) = x \text{ for } x \in [0.9, 1.1].$$

Set

$$H_1 = T_{2\beta}^{-1} f_1 T_{2\beta} f_1^{-1}.$$

It is obvious that

$$H_1(x) = x + 2\beta \text{ for } x \in [0.41, 0.79] \quad \text{and} \quad H_1(x) = x \text{ for } x \in [0.91, 1.09].$$

Define

$$H_2 = T_{1/2} H_1^{-1} T_{1/2} H_1,$$

and observe that

$$H_2(x) = x - 2\beta \quad \text{for } x \in [0.95, 1].$$

Simple computation gives

$$T_{1/2} H_2 T_{1/2} H_2 = \mathrm{id}.$$

Set

$$H_3 = T_{2\beta} H_2.$$

Then we have

$$H_3(x) = x \quad \text{for } x \in [0.95, 1]$$

and

$$T_{2\beta+1/2} H_3 T_{-2\beta-1/2} H_3 = T_{4\beta}. \tag{6}$$

Take an arbitrary $f_2 \in G_{\mathrm{triv}}$ satisfying

$$f_2(x) = 2x \quad \text{for } x \in [0, 0.49],$$

and define

$$H_4 = f_2^{-1} H_3 f_2.$$

https://doi.org/10.1017/etds.2022.3 Published online by Cambridge University Press

It is easy to see that

$$H_4(x) = \begin{cases} H_3(2x)/2 & \text{for } x \in [0, 1/2), \\ x & \text{for } x \in [1/2, 1). \end{cases}$$

Let

$$H_5 = T_{1/2} H_4 T_{1/2} H_4. \tag{7}$$

Observe that the graph of H_5 is built from two scaled copies of H_3, that is,

$$H_5(x) = \begin{cases} H_3(2x)/2 & \text{for } x \in [0, 1/2), \\ H_3(2x-1)/2 + 1/2 & \text{for } x \in [1/2, 1). \end{cases}$$

Therefore, by (6) and (7), we finally obtain

$$T_{\beta+1/4} H_5 T_{-\beta-1/4} H_5 = T_{2\beta}. \tag{8}$$

Indeed, this is easy to see if we realize that (8) is simply equation (6) rewritten in the new coordinates $(x/2, y/2)$. Subsequently plugging H_5, H_4, H_3, H_2 and H_1 into formula (8), we have

$$\begin{aligned} &T_\beta T_{1/4} T_{1/2} f_2^{-1} T_\beta^2 T_{1/2} f_1 T_\beta^{-2} f_1^{-1} T_\beta^2 T_{1/2} T_\beta^{-2} f_1 T_\beta^2 f_1^{-1} f_2 T_{1/2} f_2^{-1} T_\beta^2 T_{1/2} f_1 T_\beta^{-2} \\ &\cdot f_1^{-1} T_\beta^2 T_{1/2} T_\beta^{-2} f_1 T_\beta^2 f_1^{-1} f_2 T_\beta^{-1} T_{-1/4} T_{1/2} f_2^{-1} T_\beta^2 T_{1/2} f_1 T_\beta^{-2} f_1^{-1} T_\beta^2 T_{1/2} T_\beta^{-2} f_1 T_\beta^2 \\ &\cdot f_1^{-1} f_2 T_{1/2} f_2^{-1} T_\beta^2 T_{1/2} f_1 T_\beta^{-2} f_1^{-1} T_\beta^2 T_{1/2} T_\beta^{-2} f_1 T_\beta^2 f_1^{-1} f_2 = T_\beta^2. \end{aligned}$$

Since $\beta \in (0, 10^{-3})$ was arbitrary, we obtain that each T_β sufficiently small satisfies equation (1) with the functions $g_1, \ldots, g_l \in \{f_1, f_2, f_1^{-1}, f_2^{-1}, T_{1/2}, T_{-1/2}, T_{1/4}, T_{-1/4}\} \subset G_{\text{triv}} \cup \mathrm{T}$ and $k_1, \ldots, k_l \in \mathbb{Z}$. Obviously, some of the k_i are equal to 0 (k_2, for instance) but $k_1 + \cdots + k_l = 8$. Since $k = 2$, the proof of the lemma is complete. $\square$

REFERENCES

[1] A. Avila. Distortion elements in $Diff^\infty(\mathbb{R}/\mathbb{Z})$. *Preprint*, 2008, arXiv:0808.2334.

[2] D. Calegari and M. Freedman. Distortion in transformation groups. *Geom. Topol.* **10** (2006), 267–293, with an appendix by Y. de Cornulier.

[3] L. Dinamarca and M. Escayola. Some examples of distorted interval diffeomorphisms of intermediate regularity. *Ergod. Th. & Dynam. Sys.*, to appear.

[4] H. Eynard-Bontemps and A. Navas. *Mather invariant, distortion, and conjugates for diffeomorphisms of the interval*. *J. Funct. Anal.* **281**(9) (2021), 109149.

[5] J. Franks. Distortion in groups of circle and surface diffeomorphisms. *Dynamique des Difféomorphismes Conservatifs des Surfaces: Un Point de vue Topologique (Panoramas et Synthèses, 21)*. Eds. S. Crovisier, J. Franks, J.-M. Gambaudo and P. Le Calvez. Société Mathématique de France, Paris, 2006, pp. 35–52.

[6] J. Franks and M. Handel. Distortion elements in group actions on surface. *Duke Math. J.* **131**(3) (2006), 441–468.

[7] M. Gromov. Asymptotic invariants of infinite groups. *Geometric Group Theory (Sussex, 1991) (London Mathematical Society Lecture Notes Series, 182)*. Vol. 2. Cambridge University Press, Cambridge, 1993, pp. 1–295.

[8] N. Guelmann and I. Liousse. Distortion in group of interval exchange transformations. *Groups Geom. Dyn.* **13** (2019), 795–819.

[9] A. Navas. Group actions on 1-manifolds: a list of very concrete open questions. *Proc. Int. Congress of Mathematicians (Rio de Janeiro 2018). Vol. III. Invited Lectures*. World Scientific, Hackensack, NJ, 2018, pp. 2035–2062.

[10] A. Navas. (Un)distorted diffeomorphisms in different regularities. *Israel J. Math*. **244**(2) (2021), 727–741.

[11] L. Polterovich. Growth of maps, distortion in groups and symplectic geometry. *Invent. Math.* **150** (2002), 655–686.