



OPEN ACCESS  
Communications in Contemporary Mathematics  
(2022) 2250011 (10 pages)  
© The Author(s)  
DOI: 10.1142/S0219199722500110



## A convergence result for mountain pass periodic solutions of perturbed Hamiltonian systems

Marek Izydorek\* and Joanna Janczewska†

*Institute of Applied Mathematics  
Faculty of Applied Physics and Mathematics  
Gdańsk University of Technology  
Narutowicza 11/12, 80-233 Gdańsk, Poland  
\*marek.izydorek@pg.edu.pl  
†joanna.janczewska@pg.edu.pl*

Pedro Soares

*Análise e Matemática Financeira  
ISEG Lisbon School of Economics & Management  
Universidade de Lisboa  
Rua do Quelhas 6, 1200-781 Lisbon, Portugal  
psoares@iseg.ulisboa.pt*

Received 1 March 2021

Revised 18 January 2022

Accepted 12 February 2022

Published 28 May 2022

In this work, we study second-order Hamiltonian systems under small perturbations. We assume that the main term of the system has a mountain pass structure, but do not suppose any condition on the perturbation. We prove the existence of a periodic solution. Moreover, we show that periodic solutions of perturbed systems converge to periodic solutions of the unperturbed systems if the perturbation tends to zero. The assumption on the potential that guarantees the mountain pass geometry of the corresponding action functional is of independent interest as it is more general than those by Rabinowitz [Homoclinic orbits for a class of Hamiltonian systems, *Proc. R. Soc. Edinburgh A* **114** (1990) 33–38] and the authors [M. Izydorek and J. Janczewska, Homoclinic solutions for a class of the second-order Hamiltonian systems, *J. Differ. Equ.* **219** (2005) 375–389].

*Keywords:* Mountain pass lemma; periodic solution; perturbation problem; Hamiltonian system; variational method.

Mathematics Subject Classification 2020: 34C25, 37J46, 49J35

†Corresponding author.

This is an Open Access article published by World Scientific Publishing Company. It is distributed under the terms of the Creative Commons Attribution 4.0 (CC BY) License which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

### 1. Introduction

In [7], Kajikiya studied semilinear elliptic equations with a small perturbation

$$\begin{cases} -\Delta u = f(x, u) + \lambda g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{ES_\lambda}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ ,  $n \geq 1$ ,  $f(x, u)$  and  $g(x, u)$  are continuous on  $\Omega \times [0, \infty)$  and  $|\lambda|$  is sufficiently small. He imposed conditions on  $f(x, u)$  such that the system  $(ES_\lambda)$  has a mountain pass structure for  $\lambda = 0$ , and therefore it has a positive solution. He proved the existence of positive solutions of  $(ES_\lambda)$  for sufficiently small  $|\lambda|$  without any assumptions on  $g(x, u)$ .

In this work, we will be concerned with the problem of existence of periodic solutions for a class of perturbed Hamiltonian systems in  $\mathbb{R}^n$ ,

$$\begin{cases} \ddot{u}(t) + V_u(t, u(t)) + \lambda G_u(t, u(t)) = 0, & \text{in } [0, T], \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \end{cases} \tag{HS_\lambda}$$

where  $V: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $G: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $C^1$ -smooth,  $T$ -periodic with respect to the time variable, and  $\lambda$  is a small real parameter. The requirement on  $V$  is that  $(HS_\lambda)$  possesses a mountain pass structure for  $\lambda = 0$ , and hence it has a nontrivial  $T$ -periodic solution. The most typical potentials are  $V(t, x) = -\frac{1}{2}(L(t)x, x) + W(t, x)$  or  $V(t, x) = -K(t, x) + W(t, x)$ , where  $L: \mathbb{R} \rightarrow \mathbb{R}^{n^2}$  is a continuous  $T$ -periodic positive definite symmetric matrix-valued function and  $K, W: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $C^1$ -smooth potentials  $T$ -periodic in  $t$  such that  $K$  satisfies the so-called pinching condition:

- there are  $b_1, b_2 > 0$  such that for all  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ ,

$$b_1|x|^2 \leq K(t, x) \leq b_2|x|^2 \quad \text{and} \quad K(t, x) \leq (K_x(t, x), x) \leq 2K(t, x),$$

$\max\{W(t, x) : t \in [0, T], |x| = 1\} < b_1$  and  $W$  satisfies the well-known superquadratic growth condition by Ambrosetti and Rabinowitz:

- there is  $\mu > 2$  such that for all  $t \in \mathbb{R}$  and  $x \neq 0$ ,

$$0 < \mu W(t, x) \leq (W_x(t, x), x).$$

The potential  $V(t, x) = -\frac{1}{2}(L(t)x, x) + W(t, x)$  was studied by Rabinowitz [13] and  $V(t, x) = -K(t, x) + W(t, x)$  was studied by the authors [6].

The first aim of this paper is to prove the existence of a nontrivial  $T$ -periodic solution of  $(HS_\lambda)$  for sufficiently small  $\lambda$  and without any conditions on  $G$ . Second, we show that these periodic solutions of the perturbed systems  $(HS_\lambda)$  go to a periodic orbit of the unperturbed system  $(HS_0)$  as  $\lambda$  tends to 0.

Set

$$M = \max_{\substack{t \in [0, T] \\ |x|=1}} V(t, x)$$

Commun. Contemp. Math. Downloaded from www.worldscientific.com  
 MATHS DEPT, ZHANG YAN PAN on 05/29/22. Re-use and distribution is strictly not permitted, except for Open Access articles.  
 MOST WIEDZY

and

$$m = \max_{|x|=1} \min_{t \in [0, T]} V(t, x).$$

We will make the following assumptions about  $V$ :

(V1)  $V(t, 0) = 0$  for all  $t \in [0, T]$ .

(V2) There exist a positive constant  $\alpha$  and a negative constant  $\beta$  such that  $\alpha + 2\beta > 0$ ,  $-(\alpha + 2\beta)^{-1} < m \leq M < (\alpha + 2\beta)^{-1}$  and

$$\alpha V(t, x) + \beta(V_u(t, x), x) \leq -|x|^2 \quad \text{for } t \in [0, T], \quad x \in \mathbb{R}^n.$$

Our theorem reads as follows.

**Theorem 1.1.** *Let  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $G : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^1$ -smooth and  $T$ -periodic with respect to the time variable. Under Assumptions (V1) – (V2), the following assertions hold.*

- (i) *There exists  $\lambda_0 > 0$  such that for all  $|\lambda| \leq \lambda_0$ ,  $(HS_\lambda)$  has a  $T$ -periodic solution  $u_\lambda$ .*
- (ii) *For any sequence  $\lambda_m \rightarrow 0$  as  $m \rightarrow \infty$ , there exists a subsequence  $u_{\lambda_m}$  converging to  $u_0$  in  $W_T^{1,2}(\mathbb{R}, \mathbb{R}^n)$ , where  $u_0$  is a  $T$ -periodic solution of  $(HS_0)$ .*

The important point to note here is Assumption (V2). This simple condition is essential for the proof as it guarantees that the action functional associated to  $(HS_0)$  has a mountain pass geometry and satisfies the compactness condition of Palais and Smale. Let us note that (V2) is more general than the assumptions in [6, 13] provided that  $b_2 - b_1$  is sufficiently small. Indeed, for  $V = -K + W$  let

$$\alpha = \frac{\mu}{(\mu - 2)b_1} \quad \text{and} \quad \beta = -\frac{1}{(\mu - 2)b_1}.$$

Then we get

$$\begin{aligned} \alpha V(t, x) + \beta(V_u(t, x), x) &= -\frac{\mu K(t, x)}{(\mu - 2)b_1} + \frac{(K_u(t, x), x)}{(\mu - 2)b_1} \\ &\quad + \frac{\mu W(t, x)}{(\mu - 2)b_1} - \frac{(W_u(t, x), x)}{(\mu - 2)b_1} \\ &\leq \frac{(2 - \mu)K(t, x)}{(\mu - 2)b_1} = -\frac{K(t, x)}{b_1} \leq -|x|^2. \end{aligned}$$

Moreover, it is seen at once that

$$\alpha + 2\beta = \frac{1}{b_1} \quad \text{and} \quad M < b_1 = \frac{1}{\alpha + 2\beta}.$$

If additionally

$$b_2 - b_1 < \max_{|x|=1} \min_{t \in [0, T]} W(t, x),$$

then

$$m + \frac{1}{\alpha + 2\beta} = m + b_1 > -b_1 + b_1 = 0.$$

Similar considerations apply to  $V(t, x) = -\frac{1}{2}(L(t)x, x) + W(t, x)$  as the term  $\frac{1}{2}(L(t)x, x)$  causes no problem. Indeed, it is enough to take  $K(t, x) = -\frac{1}{2}(L(t)x, x)$  and repeat the above argument.

An easy example of  $V$  satisfying our assumptions is  $V(x) = ax^2 + bx^3$ ,  $x \in \mathbb{R}$ ,  $a < 0$  and  $0 < |b| < -2a$ . We choose  $\beta = 1/a$  and  $\alpha = -3\beta$  so that

$$\alpha V(x) + \beta(\nabla V(x), x) = (\alpha a + 2\beta a)x^2 + (\alpha b + 3\beta b)x^3 = -x^2.$$

In the late 1970s, Rabinowitz initiated the use of the minimax approach in the study of periodic solutions of Hamiltonian systems. In the last four decades, a lot of progress has been made in the use of variational methods for investigating periodic orbits for Hamiltonian systems. Both minimization and minimax arguments have been used to obtain them. Some basic material can be found, e.g., in [2, 9, 12, 14]. Periodic solutions for perturbed Hamiltonian systems have been studied in [1, 4, 5, 8, 10].

The variational formulation for Lagrangian systems usually leads to action functionals. To study periodic solutions of the problem  $(HS_\lambda)$ , a technical framework will be introduced at the beginning of Sec. 2 to treat the action functional in an appropriate Sobolev space.

The idea of the proof of Theorem 1.1 is vaguely similar to [7] in the sense that we show that for  $\lambda$  sufficiently small the perturbed system  $(HS_\lambda)$  has a mountain pass structure. However, as mentioned before, Kajikiya studied semilinear elliptic equations. We study Hamiltonian systems and this requires serious modification of every single argument of Kajikiya's work [7].

## 2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. The proof will be divided into a sequence of lemmas. From now on we assume (V1) – (V2). We emphasize that our theorem does not need any assumptions on  $G$ .

Let  $W_T^{1,2}$  be the Sobolev space of functions  $u \in L^2(0, T; \mathbb{R}^n)$  having a weak derivative  $\dot{u} \in L^2(0, T; \mathbb{R}^n)$ . Let us recall that, if  $u \in W_T^{1,2}$ ,

$$u(t) = \int_0^t \dot{u}(s) ds + c$$

and  $u(0) = u(T)$ . The norm over  $W_T^{1,2}$  is defined by

$$\|u\| = \left( \int_0^T (|u(t)|^2 + |\dot{u}(t)|^2) dt \right)^{\frac{1}{2}}.$$

Let us recall that

$$\|u\|_{L^2} = \left( \int_0^T |u(t)|^2 dt \right)^{\frac{1}{2}}$$

and

$$\|u\|_{\infty} = \max_{t \in [0, T]} |u(t)|.$$

**Proposition 2.1** (see [9, Proposition 1.1]). *There exists  $C_0 > 0$  such that, if  $u \in W_T^{1,2}$ , then*

$$\|u\|_{\infty} \leq C_0 \|u\|. \tag{1}$$

For  $(HS_{\lambda})$  with  $\lambda = 0$  we define the Lagrangian functional  $I_0: W_T^{1,2} \rightarrow \mathbb{R}$  by

$$I_0(u) = \int_0^T \left( \frac{1}{2} |\dot{u}(t)|^2 - V(t, u(t)) \right) dt.$$

**Lemma 2.2.**  *$I_0$  satisfies the Palais–Smale condition.*

**Proof.** Let  $\{u_k\}_{k \in \mathbb{N}} \subset W_T^{1,2}$  be a sequence such that  $\{I_0(u_k)\}_{k \in \mathbb{N}}$  is bounded and  $I_0'(u_k) \rightarrow 0$  in  $W^* := \mathcal{L}(W_T^{1,2}, \mathbb{R})$  as  $k \rightarrow \infty$ . Hence there is  $C > 0$  such that

$$|I_0(u_k)| \leq C \quad \text{and} \quad \|I_0'(u_k)\|_{W^*} \leq C,$$

for each  $k \in \mathbb{N}$ . An easy computation shows that

$$I_0'(u)v = \int_0^T ((\dot{u}(t), \dot{v}(t)) - (V_u(t, u(t)), v(t))) dt,$$

which gives

$$\begin{aligned} & \alpha I_0(u_k) + \beta I_0'(u_k)u_k \\ &= \left( \frac{\alpha}{2} + \beta \right) \int_0^T |\dot{u}_k(t)|^2 dt - \int_0^T (\alpha V(t, u_k(t)) + \beta (V_u(t, u_k(t)), u_k(t))) dt \\ &\geq \min \left\{ \left( \frac{\alpha}{2} + \beta \right), 1 \right\} \|u_k\|^2, \end{aligned}$$

by (V2). Consequently, for each  $k \in \mathbb{N}$ , we have

$$\min \left\{ \left( \frac{\alpha}{2} + \beta \right), 1 \right\} \|u_k\|^2 \leq \alpha C + |\beta|C \|u_k\|,$$

which implies  $\{u_k\}_{k \in \mathbb{N}}$  is bounded in the Sobolev space  $W_T^{1,2}$ . Then there is a weakly convergent subsequence of  $\{u_k\}_{k \in \mathbb{N}}$ . Using the standard method (see the proof of [6, Lemma 2.4, p. 7], we can prove that this convergence is strong.  $\square$

**Lemma 2.3.**  $I_0$  has a mountain pass geometry, i.e. there exist constants  $a, \varrho > 0$  and  $u_1 \in W_T^{1,2}$  such that  $I_0(u_1) < 0$ ,  $\|u_1\| > \varrho$  and

$$I_0(u) \geq a \quad \text{when } \|u\| = \varrho.$$

**Proof.** Set  $\gamma = \alpha + 2\beta$ . By (V2),  $\gamma > 0$ . First we prove that for all  $\varepsilon \in (0, 1)$  and for all  $0 < |x| \leq \varepsilon$ ,

$$V(t, x) \leq V\left(t, \frac{x}{|x|}\right) |x|^{-\frac{\alpha}{\beta}} - \frac{c(\varepsilon)}{\gamma} |x|^2,$$

where  $c(\varepsilon)$  is a decreasing function,  $c(\varepsilon) > 0$  and  $c(\varepsilon) \rightarrow 1$  as  $\varepsilon \rightarrow 0$ .

Fix  $x \in \mathbb{R}^n$ . Let  $g: (0, \infty) \rightarrow \mathbb{R}$  be given by

$$g(r) = (\gamma V(t, r^\beta x) + |r^\beta x|^2) r^\alpha.$$

Differentiating  $g$  we obtain

$$g'(r) = r^{\alpha-1} \gamma (\alpha V(t, r^\beta x) + \beta (V_u(t, r^\beta x), r^\beta x) + |r^\beta x|^2).$$

By (V2) it follows that  $g$  is nonincreasing. For  $|x| < 1$  we have  $g(1) \leq g(|x|^{-\frac{1}{\beta}})$ , and hence

$$V(t, x) \leq V\left(t, \frac{x}{|x|}\right) |x|^{-\frac{\alpha}{\beta}} - \frac{1}{\gamma} (1 - |x|^{-\frac{\alpha}{\beta}-2}) |x|^2.$$

Define

$$c(\varepsilon) = (1 - \varepsilon^{-\frac{\alpha}{\beta}-2}),$$

for  $\varepsilon \in (0, 1)$ . It is obvious that  $c(\varepsilon)$  is a desired function.

Fix  $\varepsilon \in (0, 1)$ . Then for  $\|u\|_\infty \leq \varepsilon$  we get

$$\begin{aligned} I_0(u) &\geq \frac{1}{2} \|\dot{u}\|_2^2 + \frac{c(\varepsilon)}{\gamma} \|u\|_2^2 - \int_0^T V\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{-\frac{\alpha}{\beta}} dt \\ &\geq \frac{1}{2} \|\dot{u}\|_2^2 + \frac{c(\varepsilon)}{\gamma} \|u\|_2^2 - M \|u\|_2^2 \\ &\geq \min\left(\frac{1}{2}, \frac{c(\varepsilon)}{\gamma} - M\right) \|u\|^2. \end{aligned}$$

Choose  $\varrho > 0$  such that  $\frac{c(\varrho)}{\gamma} - M > 0$  and  $\varrho \leq \frac{\varepsilon}{C_0}$ , where  $C_0$  is given by (1). We thus get

$$I_0(u) \geq \min\left(\frac{1}{2}, \frac{c(\varrho)}{\gamma} - M\right) \|u\|^2$$

for all  $\|u\| \leq \varrho$ . In this way, we obtain

$$a := \min\left(\frac{1}{2}, \frac{c(\varrho)}{\gamma} - M\right) \varrho^2.$$

The task is now to find  $u_1$ . For  $|x| > 1$  we have  $g(|x|^{-\frac{1}{\beta}}) \leq g(1)$ , which implies

$$V\left(t, \frac{x}{|x|}\right) |x|^{-\frac{\alpha}{\beta}} - \frac{1}{\gamma} (1 - |x|^{-\frac{\alpha}{\beta}-2}) |x|^2 \leq V(t, x).$$

Consider  $u_0(t) = x_0 \sin^2 \frac{\pi t}{T}$ , where  $x_0 \in S^{n-1}$  is a point such that  $m = \min\{V(t, x_0) : t \in [0, T]\}$ . Fix  $\xi > 0$ . Set

$$A = \{t \in [0, T] : |\xi u_0(t)| \leq 1\}$$

and

$$B = \{t \in [0, T] : |\xi u_0(t)| > 1\}.$$

By assumption, there is  $C > 0$  such that  $|V(t, x)| \leq C$  for all  $|x| \leq 1$  and  $t \in \mathbb{R}$ . It follows that

$$\begin{aligned} I_0(\xi u_0) &\leq \frac{1}{2} \|\xi \dot{u}_0\|_2^2 + \frac{1}{\gamma} \|\xi u_0\|_2^2 - \int_A V(t, \xi u_0(t)) dt + \\ &\quad - \int_B \left( V(t, x_0) + \frac{1}{\gamma} \right) |\xi u_0(t)|^{-\frac{\alpha}{\beta}} dt \\ &\leq \max\left(\frac{1}{2}, \frac{1}{\gamma}\right) \xi^2 \|u_0\|^2 + TC + \\ &\quad - \left(m + \frac{1}{\gamma}\right) \xi^{-\frac{\alpha}{\beta}} \int_0^T |u_0(t)|^{-\frac{\alpha}{\beta}} dt + \left(m + \frac{1}{\gamma}\right) T. \end{aligned}$$

From this we see that there is  $\hat{\xi} > 0$  large enough such that  $\|\hat{\xi} u_0\| > \varrho$  and  $I_0(\hat{\xi} u_0) < 0$ . We set  $u_1 \equiv \hat{\xi} u_0$ , which completes the proof.  $\square$

Define

$$\Gamma = \{g \in C([0, 1], W_T^{1,2}) : g(0) = 0, g(1) = u_1\},$$

where  $u_1 \in W_T^{1,2}$  is given in Lemma 2.3, and let

$$c_0 = \inf_{g \in \Gamma} \max_{t \in [0, 1]} I_0(g(t)).$$

**Lemma 2.4.**  $c_0$  is a critical value of  $I_0$ .

This follows by Mountain Pass lemma (see [3, 9, 11]). This lemma has been widely applied and generalized (see [11] for a clear survey and references).

An immediate consequence of Lemma 2.2 is the following.

**Lemma 2.5.** *There exists a constant  $K > 0$  such that if  $u \in W_T^{1,2}$  is a critical point of  $I_0$  corresponding to the critical value  $c_0$  then  $\|u\| \leq K$ .*

By Lemma 2.5, we have  $K > 0$  such that

$$\|u\| \leq K,$$

for any critical point of  $I_0$  corresponding to the critical value  $c_0$ . Let us choose a smooth function  $h: \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the following conditions:  $h(x) = 1$  for

$|x| \leq 2KC_0$ ,  $h(x) = 0$  for  $|x| \geq 4KC_0$  and  $0 \leq h(x) \leq 1$  for  $x \in \mathbb{R}^n$ . We will consider the behavior of the family of functionals  $I_\lambda$  defined as follows. For  $u \in W_T^{1,2}$ ,

$$I_\lambda(u) = \int_0^T \left( \frac{1}{2} |\dot{u}(t)|^2 - V(t, u(t)) - \lambda h(u(t))G(t, u(t)) \right) dt.$$

Critical points of a functional  $I_\lambda$  are solutions of

$$\begin{cases} \ddot{u}(t) + V_u(t, u(t)) + \lambda h(u(t))G_u(t, u(t)) \\ \quad + \lambda \nabla h(u(t))G(t, u(t)) = 0, & \text{in } [0, T] \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T). \end{cases} \quad (AHS_\lambda)$$

Let us outline the proof of Theorem 1.1. We first observe that for  $\lambda$  small enough,  $I_\lambda$  has a mountain pass geometry. Next we prove the existence of a mountain pass type solution  $u_\lambda$  of an auxiliary system  $(AHS_\lambda)$ . Finally, we show that  $\|u_\lambda\|_\infty \leq 2KC_0$ , hence that  $h(u_\lambda) = 1$  and  $\nabla h(u_\lambda) = 0$ , and consequently,  $u_\lambda$  satisfies  $(HS_\lambda)$ .

Applying the same arguments as in the proof of Lemma 2.2 with the fact that  $h(x)G(t, x)$  and its derivative are bounded, we get the following lemma.

**Lemma 2.6.** *For each  $\lambda \in \mathbb{R}$ ,  $I_\lambda$  satisfies the Palais–Smale condition.*

**Lemma 2.7.** *There is  $\lambda_0 > 0$  such that for all  $|\lambda| \leq \lambda_0$ ,  $I_\lambda$  has a mountain pass geometry.*

**Proof.** Let  $a, \varrho$  and  $u_1$  be the same as in Lemma 2.3. Fix  $R > \varrho$  such that  $\|u_1\| < R$ . Since  $G(t, x)$  is bounded on compact subsets of  $\mathbb{R} \times \mathbb{R}^n$ , there is  $L > 0$  such that  $|G(t, x)| \leq L$  for all  $t \in [0, T]$  and  $|x| \leq C_0R$ . We obtain

$$I_0(u) - |\lambda|TL \leq I_\lambda(u) \leq I_0(u) + |\lambda|TL, \tag{2}$$

for  $\|u\| \leq R$ . From this it follows that for  $\lambda$  small enough,

$$I_\lambda(u) \geq a - |\lambda|TL > \frac{a}{2} \quad \text{when } \|u\| = \varrho$$

and

$$I_\lambda(u_1) \leq I_0(u_1) + |\lambda|TL < 0,$$

which completes the proof. □

Define

$$c_\lambda = \inf_{g \in \Gamma} \max_{t \in [0, 1]} I_\lambda(g(t)).$$

By (2),  $c_\lambda \rightarrow c_0$  as  $\lambda \rightarrow 0$ .

**Lemma 2.8.** *Let  $\lambda_m \in \mathbb{R}$ ,  $m \in \mathbb{N}$ , be a sequence converging to 0. Assume that  $v_m$  is a critical point of  $I_{\lambda_m}$  corresponding to the critical value  $c_{\lambda_m}$ .*



Then, up to a subsequence,

$$v_m \rightarrow v_0 \quad \text{in } W_T^{1,2} \quad \text{as } m \rightarrow \infty,$$

where  $v_0$  is a critical point of  $I_0$  corresponding to the critical value  $c_0$ .

**Proof.** By assumption,  $I_{\lambda_m}(v_m) = c_{\lambda_m}$  and  $I'_{\lambda_m}(v_m) = 0$ . We will prove that  $\{v_m\}_{m \in \mathbb{N}}$  is bounded in  $W_T^{1,2}$ . There is  $C > 0$  such that  $|G(t, x)\nabla h(x)| \leq C$  and  $|h(x)G_u(t, x)| \leq C$  for all  $t \in [0, T]$  and  $|x| \leq 4KC_0$ . There is also  $\tilde{C} > 0$  such that  $c_{\lambda_m} \leq \tilde{C}$  for all  $m \in \mathbb{N}$ . We have

$$\begin{aligned} \alpha\tilde{C} &\geq \alpha c_{\lambda_m} = \alpha I_{\lambda_m}(v_m) + \beta I'_{\lambda_m}(v_m)v_m \\ &= \alpha I_0(v_m) + \beta I'_0(v_m)v_m - \alpha\lambda_m \int_0^T h(v_m(t))G(t, v_m(t))dt + \\ &\quad - \beta\lambda_m \int_0^T h(v_m(t))(G_u(t, v_m(t)), v_m(t))dt + \\ &\quad - \beta\lambda_m \int_0^T (\nabla h(v_m(t)), v_m(t))G(t, v_m(t))dt \\ &\geq \min\left\{\left(\frac{\alpha}{2} + \beta\right), 1\right\} \|v_m\|^2 - T\alpha\lambda_0 C + 2\sqrt{T}\beta\lambda_0 C \|v_m\|. \end{aligned}$$

From this it follows that  $\{v_m\}_{m=1}^\infty$  is bounded, and consequently a subsequence of  $\{v_m\}_{m=1}^\infty$  converges weakly to a limit  $v_0 \in W_T^{1,2}$ . By the compact embedding of  $W_T^{1,2}$  into  $C([0, T], \mathbb{R}^n)$ , we obtain  $v_m \rightarrow v_0$ , up to a subsequence, in  $C([0, T], \mathbb{R}^n)$ . Standard arguments (compare the proof of [6, Lemma 2.4, p. 7]) show that up to a subsequence  $v_m$  converges strongly to  $v_0$  in  $W_T^{1,2}$ . Hence  $I_0(v_0) = c_0$  and  $I'_0(v_0) = 0$ .  $\square$

**Lemma 2.9.** Any mountain pass solution  $u_\lambda$  of  $I_\lambda$  with  $\lambda$  sufficiently small satisfies  $(HS_\lambda)$ .

**Proof.** As mentioned before, it suffices to show that  $\|u_\lambda\|_\infty \leq 2KC_0$  for all  $|\lambda| \leq \lambda_0$ .

On the contrary, suppose that there are sequences  $\{\lambda_m\}_{m=1}^\infty$  and  $\{u_m\}_{m=1}^\infty$  such that  $\lambda_m$  goes to 0,  $u_m$  is a mountain pass solution of  $I_{\lambda_m}$  and  $\|u_m\|_\infty > 2KC_0$ . By Lemma 2.8, along a subsequence  $u_m$  converges uniformly to a critical point  $u_0$  of  $I_0$  corresponding to the critical value  $c_0$ . Since  $\|u_0\| \leq K$  by Lemma 2.5, we get  $\|u_0\|_\infty \leq C_0K$ , and consequently  $\|u_m\|_\infty \leq 2C_0K$  for  $m$  large enough. A contradiction occurs.  $\square$

### Acknowledgments

Our research is supported by Grant BEETHOVEN 2 of the National Science Centre, Poland, no. 2016/23/G/ST1/04081.

## References

- [1] A. Abbondandolo, Sub-harmonics for two-dimensional Hamiltonian systems, *NoDEA Nonlinear Differ. Equ. Appl.* **6**(4) (1999) 341–355.
- [2] A. Abbondandolo, *Morse Theory for Hamiltonian Systems*, Chapman & Hall/CRC Research Notes in Mathematics, Vol. 425 (Chapman & Hall/CRC, Boca Raton, FL, 2001).
- [3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973) 349–381.
- [4] A. Bahri and H. Berestycki, Forced vibrations of superquadratic Hamiltonian systems, *Acta Math.* **152**(3–4) (1984) 143–197.
- [5] A. Bahrouni, M. Izydorek and J. Janczewska, Subharmonic solutions for a class of Lagrangian systems, *Discrete Contin. Dyn. Syst. Ser. S* **12**(7) (2019) 1841–1850.
- [6] M. Izydorek and J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems, *J. Differ. Equ.* **219** (2005) 375–389.
- [7] R. Kajikiya, Positive solutions of semilinear elliptic equations with small perturbations, *Proc. Amer. Math. Soc.* **141**(4) (2013) 1335–1342.
- [8] E. Kirr, Periodic solutions for perturbed Hamiltonian systems with superlinear growth and impulsive effects, *Studia Univ. Babeş-Bolyai Math.* **41**(4) (1996) 55–65.
- [9] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, Vol. 74 (Springer-Verlag, New York, 1989).
- [10] M. Matzeu and I. Scarascia, Convergence results for mountain pass periodic solutions of autonomous Hamiltonian systems, *Boll. Un. Mat. Ital. A* **10**(2) (1996) 445–459.
- [11] P. H. Rabinowitz, The mountain pass theorem: Theme and variations, in *Differential Equations*, ed. de Figueiredo, Hoing, Lecture Notes Mathematics, Vol. 957 (Springer, Berlin, 1982), pp. 237–271.
- [12] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, Vol. 65 (American Mathematical Society, 1986).
- [13] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. R. Soc. Edinburgh A* **114** (1990) 33–38.
- [14] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **31** (1978) 157–184.

