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A convergence result for mountain pass periodic solutions of perturbed Hamiltonian systems

Marek Izydorek* and Joanna Janczewska†

Institute of Applied Mathematics
Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
Narutowicza 11/12, 80-233 Gdańsk, Poland
**marek.izydorek@pg.edu.pl*
†*joanna.janczewska@pg.edu.pl*

Pedro Soares

Análise e Matemática Financeira
ISEG Lisbon School of Economics & Management
Universidade de Lisboa
Rua do Quelhas 6, 1200-781 Lisbon, Portugal
psoares@iseg.ulisboa.pt

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In this work, we study second-order Hamiltonian systems under small perturbations. We assume that the main term of the system has a mountain pass structure, but do not suppose any condition on the perturbation. We prove the existence of a periodic solution. Moreover, we show that periodic solutions of perturbed systems converge to periodic solutions of the unperturbed systems if the perturbation tends to zero. The assumption on the potential that guarantees the mountain pass geometry of the corresponding action functional is of independent interest as it is more general than those by Rabinowitz [Homoclinic orbits for a class of Hamiltonian systems, *Proc. R. Soc. Edinburgh A* **114** (1990) 33–38] and the authors [M. Izydorek and J. Janczewska, Homoclinic solutions for a class of the second-order Hamiltonian systems, *J. Differ. Equ.* **219** (2005) 375–389].

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†Corresponding author.

1. Introduction

In [7], Kajikiya studied semilinear elliptic equations with a small perturbation

$$\begin{cases} -\Delta u = f(x,u) + \lambda g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{ES_λ}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $n \geq 1$, $f(x,u)$ and $g(x,u)$ are continuous on $\Omega \times [0,\infty)$ and $|\lambda|$ is sufficiently small. He imposed conditions on $f(x,u)$ such that the system (ES_λ) has a mountain pass structure for $\lambda = 0$, and therefore it has a positive solution. He proved the existence of positive solutions of (ES_λ) for sufficiently small $|\lambda|$ without any assumptions on $g(x,u)$.

In this work, we will be concerned with the problem of existence of periodic solutions for a class of perturbed Hamiltonian systems in $\mathbb{R}^n$,

$$\begin{cases} \ddot{u}(t) + V_u(t,u(t)) + \lambda G_u(t,u(t)) = 0, & \text{in } [0,T], \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \end{cases} \tag{HS_λ}$$

where $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $G\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-smooth, T-periodic with respect to the time variable, and λ is a small real parameter. The requirement on V is that (HS_λ) possesses a mountain pass structure for $\lambda = 0$, and hence it has a nontrivial T-periodic solution. The most typical potentials are $V(t,x) = -\frac{1}{2}(L(t)x,x) + W(t,x)$ or $V(t,x) = -K(t,x) + W(t,x)$, where $L\colon \mathbb{R} \to \mathbb{R}^{n^2}$ is a continuous T-periodic positive definite symmetric matrix-valued function and $K, W\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-smooth potentials T-periodic in t such that K satisfies the so-called pinching condition:

- there are $b_1, b_2 > 0$ such that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$b_1|x|^2 \leq K(t,x) \leq b_2|x|^2 \quad \text{and} \quad K(t,x) \leq (K_x(t,x),x) \leq 2K(t,x),$$

$\max\{W(t,x)\colon t \in [0,T], |x| = 1\} < b_1$ and W satisfies the well-known superquadratic growth condition by Ambrosetti and Rabinowitz:

- there is $\mu > 2$ such that for all $t \in \mathbb{R}$ and $x \neq 0$,

$$0 < \mu W(t,x) \leq (W_x(t,x),x).$$

The potential $V(t,x) = -\frac{1}{2}(L(t)x,x) + W(t,x)$ was studied by Rabinowitz [13] and $V(t,x) = -K(t,x) + W(t,x)$ was studied by the authors [6].

The first aim of this paper is to prove the existence of a nontrivial T-periodic solution of (HS_λ) for sufficiently small λ and without any conditions on G. Second, we show that these periodic solutions of the perturbed systems (HS_λ) go to a periodic orbit of the unperturbed system (HS_0) as λ tends to 0.

Set

$$M = \max_{\substack{t \in [0,T] \\ |x| = 1}} V(t,x)$$

and

$$m = \max_{|x|=1} \min_{t\in[0,T]} V(t,x).$$

We will make the following assumptions about V:

$(V1)$ $V(t,0) = 0$ for all $t \in [0,T]$.
$(V2)$ There exist a positive constant α and a negative constant β such that $\alpha+2\beta > 0$, $-(\alpha+2\beta)^{-1} < m \le M < (\alpha+2\beta)^{-1}$ and

$$\alpha V(t,x) + \beta(V_u(t,x),x) \le -|x|^2 \quad \text{for } t\in[0,T],\ x\in\mathbb{R}^n.$$

Our theorem reads as follows.

Theorem 1.1. *Let $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and $G\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be C^1-smooth and T-periodic with respect to the time variable. Under Assumptions $(V1)-(V2)$, the following assertions hold.*

(i) *There exists $\lambda_0 > 0$ such that for all $|\lambda| \le \lambda_0$, (HS_λ) has a T-periodic solution u_λ.*
(ii) *For any sequence $\lambda_m \to 0$ as $m\to\infty$, there exists a subsequence u_{λ_m} converging to u_0 in $W^{1,2}_T(\mathbb{R},\mathbb{R}^n)$, where u_0 is a T-periodic solution of (HS_0).*

The important point to note here is Assumption $(V2)$. This simple condition is essential for the proof as it guarantees that the action functional associated to (HS_0) has a mountain pass geometry and satisfies the compactness condition of Palais and Smale. Let us note that $(V2)$ is more general than the assumptions in [6, 13] provided that $b_2 - b_1$ is sufficiently small. Indeed, for $V = -K + W$ let

$$\alpha = \frac{\mu}{(\mu-2)b_1} \quad \text{and} \quad \beta = -\frac{1}{(\mu-2)b_1}.$$

Then we get

$$\begin{aligned}\alpha V(t,x) + \beta(V_u(t,x),x) &= -\frac{\mu K(t,x)}{(\mu-2)b_1} + \frac{(K_u(t,x),x)}{(\mu-2)b_1} \\ &\quad + \frac{\mu W(t,x)}{(\mu-2)b_1} - \frac{(W_u(t,x),x)}{(\mu-2)b_1} \\ &\le \frac{(2-\mu)K(t,x)}{(\mu-2)b_1} = -\frac{K(t,x)}{b_1} \le -|x|^2.\end{aligned}$$

Moreover, it is seen at once that

$$\alpha + 2\beta = \frac{1}{b_1} \quad \text{and} \quad M < b_1 = \frac{1}{\alpha+2\beta}.$$

If additionally

$$b_2 - b_1 < \max_{|x|=1} \min_{t\in[0,T]} W(t,x),$$

then

$$m + \frac{1}{\alpha + 2\beta} = m + b_1 > -b_1 + b_1 = 0.$$

Similar considerations apply to $V(t,x) = -\frac{1}{2}(L(t)x,x) + W(t,x)$ as the term $\frac{1}{2}(L(t)x,x)$ causes no problem. Indeed, it is enough to take $K(t,x) = -\frac{1}{2}(L(t)x,x)$ and repeat the above argument.

An easy example of V satisfying our assumptions is $V(x) = ax^2 + bx^3$, $x \in \mathbb{R}$, $a < 0$ and $0 < |b| < -2a$. We choose $\beta = 1/a$ and $\alpha = -3\beta$ so that

$$\alpha V(x) + \beta(\nabla V(x), x) = (\alpha a + 2\beta a)x^2 + (\alpha b + 3\beta b)x^3 = -x^2.$$

In the late 1970s, Rabinowitz initiated the use of the minimax approach in the study of periodic solutions of Hamiltonian systems. In the last four decades, a lot of progress has been made in the use of variational methods for investigating periodic orbits for Hamiltonian systems. Both minimization and minimax arguments have been used to obtain them. Some basic material can be found, e.g., in [2, 9, 12, 14]. Periodic solutions for perturbed Hamiltonian systems have been studied in [1, 4, 5, 8, 10].

The variational formulation for Lagrangian systems usually leads to action functionals. To study periodic solutions of the problem (HS_λ), a technical framework will be introduced at the beginning of Sec. 2 to treat the action functional in an appropriate Sobolev space.

The idea of the proof of Theorem 1.1 is vaguely similar to [7] in the sense that we show that for λ sufficiently small the perturbed system (HS_λ) has a mountain pass structure. However, as mentioned before, Kajikiya studied semilinear elliptic equations. We study Hamiltonian systems and this requires serious modification of every single argument of Kajikiya's work [7].

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. The proof will be divided into a sequence of lemmas. From now on we assume $(V1)-(V2)$. We emphasize that our theorem does not need any assumptions on G.

Let $W_T^{1,2}$ be the Sobolev space of functions $u \in L^2(0,T;\mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^2(0,T;\mathbb{R}^n)$. Let us recall that, if $u \in W_T^{1,2}$,

$$u(t) = \int_0^t \dot{u}(s)ds + c$$

and $u(0) = u(T)$. The norm over $W_T^{1,2}$ is defined by

$$\|u\| = \left(\int_0^T (|u(t)|^2 + |\dot{u}(t)|^2)dt\right)^{\frac{1}{2}}.$$

Let us recall that

$$\|u\|_{L^2} = \left(\int_0^T |u(t)|^2 dt\right)^{\frac{1}{2}}$$

and

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Proposition 2.1 (see [9, Proposition 1.1]). *There exists* $C_0 > 0$ *such that, if* $u \in W_T^{1,2}$, *then*

$$\|u\|_\infty \leq C_0\|u\|. \tag{1}$$

For (HS_λ) with $\lambda = 0$ we define the Lagrangian functional $I_0\colon W_T^{1,2} \to \mathbb{R}$ by

$$I_0(u) = \int_0^T \left(\frac{1}{2}|\dot{u}(t)|^2 - V(t, u(t))\right) dt.$$

Lemma 2.2. I_0 *satisfies the Palais–Smale condition.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset W_T^{1,2}$ be a sequence such that $\{I_0(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I_0'(u_k) \to 0$ in $W^* := \mathcal{L}(W_T^{1,2}, \mathbb{R})$ as $k \to \infty$. Hence there is $C > 0$ such that

$$|I_0(u_k)| \leq C \quad \text{and} \quad \|I_0'(u_k)\|_{W^*} \leq C,$$

for each $k \in \mathbb{N}$. An easy computation shows that

$$I_0'(u)v = \int_0^T ((\dot{u}(t), \dot{v}(t)) - (V_u(t, u(t)), v(t)))dt,$$

which gives

$$\begin{aligned} &\alpha I_0(u_k) + \beta I_0'(u_k)u_k \\ &\quad = \left(\frac{\alpha}{2} + \beta\right)\int_0^T |\dot{u}_k(t)|^2 dt - \int_0^T (\alpha V(t, u_k(t)) + \beta(V_u(t, u_k(t)), u_k(t)))\, dt \\ &\quad \geq \min\left\{\left(\frac{\alpha}{2} + \beta\right), 1\right\}\|u_k\|^2, \end{aligned}$$

by $(V2)$. Consequently, for each $k \in \mathbb{N}$, we have

$$\min\left\{\left(\frac{\alpha}{2} + \beta\right), 1\right\}\|u_k\|^2 \leq \alpha C + |\beta|C\|u_k\|,$$

which implies $\{u_k\}_{k\in\mathbb{N}}$ is bounded in the Sobolev space $W_T^{1,2}$. Then there is a weakly convergent subsequence of $\{u_k\}_{k\in\mathbb{N}}$. Using the standard method (see the proof of [6, Lemma 2.4, p. 7], we can prove that this convergence is strong. □

Lemma 2.3. *I_0 has a mountain pass geometry, i.e. there exist constants $a, \varrho > 0$ and $u_1 \in W_T^{1,2}$ such that $I_0(u_1) < 0$, $\|u_1\| > \varrho$ and*

$$I_0(u) \geq a \quad \text{when } \|u\| = \varrho.$$

Proof. Set $\gamma = \alpha + 2\beta$. By $(V2)$, $\gamma > 0$. First we prove that for all $\varepsilon \in (0,1)$ and for all $0 < |x| \leq \varepsilon$,

$$V(t,x) \leq V\left(t, \frac{x}{|x|}\right)|x|^{-\frac{\alpha}{\beta}} - \frac{c(\varepsilon)}{\gamma}|x|^2,$$

where $c(\varepsilon)$ is a decreasing function, $c(\varepsilon) > 0$ and $c(\varepsilon) \to 1$ as $\varepsilon \to 0$.

Fix $x \in \mathbb{R}^n$. Let $g\colon (0,\infty) \to \mathbb{R}$ be given by

$$g(r) = (\gamma V(t, r^\beta x) + |r^\beta x|^2)r^\alpha.$$

Differentiating g we obtain

$$g'(r) = r^{\alpha-1}\gamma(\alpha V(t, r^\beta x) + \beta(V_u(t, r^\beta x), r^\beta x) + |r^\beta x|^2).$$

By $(V2)$ it follows that g is nonincreasing. For $|x| < 1$ we have $g(1) \leq g(|x|^{-\frac{1}{\beta}})$, and hence

$$V(t,x) \leq V\left(t, \frac{x}{|x|}\right)|x|^{-\frac{\alpha}{\beta}} - \frac{1}{\gamma}(1 - |x|^{-\frac{\alpha}{\beta}-2})|x|^2.$$

Define

$$c(\varepsilon) = (1 - \varepsilon^{-\frac{\alpha}{\beta}-2}),$$

for $\varepsilon \in (0,1)$. It is obvious that $c(\varepsilon)$ is a desired function.

Fix $\varepsilon \in (0,1)$. Then for $\|u\|_\infty \leq \varepsilon$ we get

$$\begin{aligned} I_0(u) &\geq \frac{1}{2}\|\dot{u}\|_2^2 + \frac{c(\varepsilon)}{\gamma}\|u\|_2^2 - \int_0^T V\left(t, \frac{u(t)}{|u(t)|}\right)|u(t)|^{-\frac{\alpha}{\beta}}dt \\ &\geq \frac{1}{2}\|\dot{u}\|_2^2 + \frac{c(\varepsilon)}{\gamma}\|u\|_2^2 - M\|u\|_2^2 \\ &\geq \min\left(\frac{1}{2}, \frac{c(\varepsilon)}{\gamma} - M\right)\|u\|^2. \end{aligned}$$

Choose $\varrho > 0$ such that $\frac{c(\varrho)}{\gamma} - M > 0$ and $\varrho \leq \frac{\varepsilon}{C_0}$, where C_0 is given by (1). We thus get

$$I_0(u) \geq \min\left(\frac{1}{2}, \frac{c(\varrho)}{\gamma} - M\right)\|u\|^2$$

for all $\|u\| \leq \varrho$. In this way, we obtain

$$a := \min\left(\frac{1}{2}, \frac{c(\varrho)}{\gamma} - M\right)\varrho^2.$$

The task is now to find u_1. For $|x| > 1$ we have $g(|x|^{-\frac{1}{\beta}}) \leq g(1)$, which implies

$$V\left(t, \frac{x}{|x|}\right)|x|^{-\frac{\alpha}{\beta}} - \frac{1}{\gamma}(1 - |x|^{-\frac{\alpha}{\beta}-2})|x|^2 \leq V(t,x).$$

Consider $u_0(t) = x_0 \sin^2 \frac{\pi t}{T}$, where $x_0 \in S^{n-1}$ is a point such that $m = \min\{V(t,x_0)\colon t \in [0,T]\}$. Fix $\xi > 0$. Set

$$A = \{t \in [0,T] : |\xi u_0(t)| \leq 1\}$$

and

$$B = \{t \in [0,T] : |\xi u_0(t)| > 1\}.$$

By assumption, there is $C > 0$ such that $|V(t,x)| \leq C$ for all $|x| \leq 1$ and $t \in \mathbb{R}$. It follows that

$$\begin{aligned} I_0(\xi u_0) &\leq \frac{1}{2}\|\xi \dot{u}_0\|_2^2 + \frac{1}{\gamma}\|\xi u_0\|_2^2 - \int_A V(t, \xi u_0(t))dt + \\ &\quad - \int_B \left(V(t,x_0) + \frac{1}{\gamma}\right)|\xi u_0(t)|^{-\frac{\alpha}{\beta}} dt \\ &\leq \max\left(\frac{1}{2}, \frac{1}{\gamma}\right)\xi^2\|u_0\|^2 + TC + \\ &\quad - \left(m + \frac{1}{\gamma}\right)\xi^{-\frac{\alpha}{\beta}}\int_0^T |u_0(t)|^{-\frac{\alpha}{\beta}} dt + \left(m + \frac{1}{\gamma}\right)T. \end{aligned}$$

From this we see that there is $\hat{\xi} > 0$ large enough such that $\|\hat{\xi} u_0\| > \varrho$ and $I_0(\hat{\xi} u_0) < 0$. We set $u_1 \equiv \hat{\xi} u_0$, which completes the proof. □

Define

$$\Gamma = \{g \in C([0,1], W_T^{1,2})\colon g(0) = 0, g(1) = u_1\},$$

where $u_1 \in W_T^{1,2}$ is given in Lemma 2.3, and let

$$c_0 = \inf_{g\in\Gamma} \max_{t\in[0,1]} I_0(g(t)).$$

Lemma 2.4. *c_0 is a critical value of I_0.*

This follows by Mountain Pass lemma (see [3, 9, 11]). This lemma has been widely applied and generalized (see [11] for a clear survey and references).

An immediate consequence of Lemma 2.2 is the following.

Lemma 2.5. *There exists a constant $K > 0$ such that if $u \in W_T^{1,2}$ is a critical point of I_0 corresponding to the critical value c_0 then $\|u\| \leq K$.*

By Lemma 2.5, we have $K > 0$ such that

$$\|u\| \leq K,$$

for any critical point of I_0 corresponding to the critical value c_0. Let us choose a smooth function $h\colon \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions: $h(x) = 1$ for

$|x| \le 2KC_0$, $h(x) = 0$ for $|x| \ge 4KC_0$ and $0 \le h(x) \le 1$ for $x \in \mathbb{R}^n$. We will consider the behavior of the family of functionals I_λ defined as follows. For $u \in W_T^{1,2}$,

$$I_\lambda(u) = \int_0^T \left(\frac{1}{2}|\dot{u}(t)|^2 - V(t,u(t)) - \lambda h(u(t))G(t,u(t))\right) dt.$$

Critical points of a functional I_λ are solutions of

$$\begin{cases} \ddot{u}(t) + V_u(t,u(t)) + \lambda h(u(t))G_u(t,u(t)) \\ \qquad + \lambda \nabla h(u(t))G(t,u(t)) = 0, \quad \text{in } [0,T] \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T). \end{cases} \tag{AHS_λ}$$

Let us outline the proof of Theorem 1.1. We first observe that for λ small enough, I_λ has a mountain pass geometry. Next we prove the existence of a mountain pass type solution u_λ of an auxiliary system (AHS_λ). Finally, we show that $\|u_\lambda\|_\infty \le 2KC_0$, hence that $h(u_\lambda) = 1$ and $\nabla h(u_\lambda) = 0$, and consequently, u_λ satisfies (HS_λ).

Applying the same arguments as in the proof of Lemma 2.2 with the fact that $h(x)G(t,x)$ and its derivative are bounded, we get the following lemma.

Lemma 2.6. *For each $\lambda \in \mathbb{R}$, I_λ satisfies the Palais–Smale condition.*

Lemma 2.7. *There is $\lambda_0 > 0$ such that for all $|\lambda| \le \lambda_0$, I_λ has a mountain pass geometry.*

Proof. Let a, ϱ and u_1 be the same as in Lemma 2.3. Fix $R > \varrho$ such that $\|u_1\| < R$. Since $G(t,x)$ is bounded on compact subsets of $\mathbb{R} \times \mathbb{R}^n$, there is $L > 0$ such that $|G(t,x)| \le L$ for all $t \in [0,T]$ and $|x| \le C_0 R$. We obtain

$$I_0(u) - |\lambda|TL \le I_\lambda(u) \le I_0(u) + |\lambda|TL, \tag{2}$$

for $\|u\| \le R$. From this it follows that for λ small enough,

$$I_\lambda(u) \ge a - |\lambda|TL > \frac{a}{2} \quad \text{when } \|u\| = \varrho$$

and

$$I_\lambda(u_1) \le I_0(u_1) + |\lambda|TL < 0,$$

which completes the proof. □

Define

$$c_\lambda = \inf_{g \in \Gamma} \max_{t \in [0,1]} I_\lambda(g(t)).$$

By (2), $c_\lambda \to c_0$ as $\lambda \to 0$.

Lemma 2.8. *Let $\lambda_m \in \mathbb{R}$, $m \in \mathbb{N}$, be a sequence converging to 0. Assume that v_m is a critical point of I_{λ_m} corresponding to the critical value c_{λ_m}.*

Then, up to a subsequence,

$$v_m \to v_0 \quad in\ W_T^{1,2} \quad as\ m \to \infty,$$

where v_0 is a critical point of I_0 corresponding to the critical value c_0.

Proof. By assumption, $I_{\lambda_m}(v_m) = c_{\lambda_m}$ and $I'_{\lambda_m}(v_m) = 0$. We will prove that $\{v_m\}_{m\in\mathbb{N}}$ is bounded in $W_T^{1,2}$. There is $C > 0$ such that $|G(t,x)\nabla h(x)| \le C$ and $|h(x)G_u(t,x)| \le C$ for all $t \in [0,T]$ and $|x| \le 4KC_0$. There is also $\widetilde{C} > 0$ such that $c_{\lambda_m} \le \widetilde{C}$ for all $m \in \mathbb{N}$. We have

$$\begin{aligned}
\alpha\widetilde{C} \ge \alpha c_{\lambda_m} &= \alpha I_{\lambda_m}(v_m) + \beta I'_{\lambda_m}(v_m)v_m \\
&= \alpha I_0(v_m) + \beta I'_0(v_m)v_m - \alpha\lambda_m \int_0^T h(v_m(t))G(t,v_m(t))dt + \\
&\quad - \beta\lambda_m \int_0^T h(v_m(t))(G_u(t,v_m(t)),v_m(t))dt + \\
&\quad - \beta\lambda_m \int_0^T (\nabla h(v_m(t)),v_m(t))G(t,v_m(t))dt \\
&\ge \min\left\{\left(\frac{\alpha}{2}+\beta\right),1\right\}\|v_m\|^2 - T\alpha\lambda_0 C + 2\sqrt{T}\beta\lambda_0 C\|v_m\|.
\end{aligned}$$

From this it follows that $\{v_m\}_{m=1}^{\infty}$ is bounded, and consequently a subsequence of $\{v_m\}_{m=1}^{\infty}$ converges weakly to a limit $v_0 \in W_T^{1,2}$. By the compact embedding of $W_T^{1,2}$ into $C([0,T],\mathbb{R}^n)$, we obtain $v_m \to v_0$, up to a subsequence, in $C([0,T],\mathbb{R}^n)$. Standard arguments (compare the proof of [6, Lemma 2.4, p. 7]) show that up to a subsequence v_m converges strongly to v_0 in $W_T^{1,2}$. Hence $I_0(v_0) = c_0$ and $I'_0(v_0) = 0$. □

Lemma 2.9. *Any mountain pass solution u_λ of I_λ with λ sufficiently small satisfies (HS_λ).*

Proof. As mentioned before, it suffices to show that $\|u_\lambda\|_\infty \le 2KC_0$ for all $|\lambda| \le \lambda_0$.

On the contrary, suppose that there are sequences $\{\lambda_m\}_{m=1}^{\infty}$ and $\{u_m\}_{m=1}^{\infty}$ such that λ_m goes to 0, u_m is a mountain pass solution of I_{λ_m} and $\|u_m\|_\infty > 2KC_0$. By Lemma 2.8, along a subsequence u_m converges uniformly to a critical point u_0 of I_0 corresponding to the critical value c_0. Since $\|u_0\| \le K$ by Lemma 2.5, we get $\|u_0\|_\infty \le C_0K$, and consequently $\|u_m\|_\infty \le 2C_0K$ for m large enough. A contradiction occurs. □

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