Postprint of: Eremeev V., Reccia E., Nonlinear strain gradient and micromorphic one-dimensional elastic continua: Comparison through strong ellipticity conditions, MECHANICS RESEARCH COMMUNICATIONS Vol. 124 (2022), 103909, DOI: 10.1016/j.mechrescom.2022.103909

Nonlinear strain gradient and micromorphic one-dimensional elastic continua: Comparison through strong ellipticity conditions

Victor A. Eremeyev [a,b,c,*], Emanuele Reccia [a]

[a] University of Cagliari, Via Marengo, 2, 09123 Cagliari, Italy
[b] Gdańsk University of Technology, ul. Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland
[c] Institute of Applied Mechanics, Russian Academy of Sciences, Leningradsky prt 7, 125040 Moscow, Russia

Keywords:
Strong ellipticity
Strain gradient elasticity
Micromorphic medium
Nonlinear elasticity
Material instability

A B S T R A C T

We discuss the strong ellipticity (SE) conditions for strain gradient and micromorphic continua considering them as an enhancement of a simple nonlinearly elastic material called in the following primary material. Recently both models are widely used for description of material behavior of beam-lattice metamaterials which may possess various types of material instabilities. We analyze how a possible loss of SE results in the behavior of enhanced models. We shown that SE conditions for a micromorphic medium is more restrictive than for its gradient counterpart. On the other hand we see that a violation of SE for a primary material affects solutions within enhanced models even if the SE conditions are fulfilled for them.

1. Introduction

Nowadays such enhanced models of continuum as the micromorphic medium and the strain gradient elasticity found various applications in description of material behavior of composites and metamaterials with essential difference in mechanical properties [1–5]. In fact, both models could be obtained as a result of homogenization of strongly inhomogeneous materials such as beam-lattices or foams [4,6,7]. The model of micromorphic continuum was proposed in original works by Mindlin [8] and by Eringen and Suhubi [9], see also [10–12]. For a hyperelastic micromorphic medium there exists a strain energy density given as a function of strain and microdeformation tensors. The microdeformations play a role of an additional kinematical descriptor of the model. On the other hand, within the strain gradient elasticity a strain energy density depends on strains and higher-order gradients of placement vector [4,13]. Considering the history of development of these models it is worth to mention [14–16], where further references in the field could be found.

So we can see that the both approaches may successfully model some inhomogeneous materials such as open-cell foams or other beam-lattice materials. Moreover, considering the kinematics within these two models, it is easy to see some similarities between them. Indeed, replacing the microdeformation tensor in the constitutive relations of the micromorphic continuum by the deformation gradient, we immediately come to the constitutive relation of strain gradient elasticity. So the form of the strain energy densities within the both models are similar, in general. On the other hand, the mathematical structure of equilibrium equations is different for these models. Indeed, for the strain gradient elasticity we have a system of three scalar partial differential equations (PDEs) of fourth order, whereas for the micromorphic medium the corresponding system consists of six PDEs of second order. In particular, the strong ellipticity condition for both models are also different. Let us note that the strong ellipticity (SE) condition plays a role of so-called constitutive inequality in nonlinear elasticity, i.e. for simple elastic materials in sense of W. Noll [17,18], which may guarantee some "natural" properties of the static problem under consideration. For example, violation of the SE condition may result in a certain material instabilities in solids [19–21].

Since beam-lattice and some other architected materials undergo large deformations, various kinds of instabilities may occur. These instabilities could be observed at both micro- and macroscales. So an effective medium model has to capture these phenomena, in general. For example, for open-cell foams made of elastomers a local buckling of cell struts results in a plateau in a stress–strain curve similar to plasticity, see [22]. Studying some discrete structures and their strain gradient and micromorphic counterparts, the comparison of these models was provided through description of material instabilities in [23]. A stability analysis of nonlinear boundary-value problems (BVPs) could be rather complex, see e.g. [19,20,24] for simple materials or [25] for micropolar media. In contrast to the complete bifurcation analysis of BVPs, the SE conditions result in an algebraic problem which is more simple, in general, but still could provide some information about material instabilities [26].

The aim of this paper is to discuss the SE conditions for nonlinear strain gradient and micromorphic media in order to formulate corresponding constitutive inequalities and underline the difference between these models. The paper is organized as follows. In Section 2 we briefly recall the basic relations within the strain gradient elasticity and micromorphic continuum. For the both media the SE conditions are formulated. Section 3 is devoted to an one-dimensional (1D) case, i.e. to a stretching of an elastic 1D continuum. In this case one can easily see the difference between models. In particular, we prove that the SE condition for the micromorphic continuum is more restrictive than for a strain gradient bar. Then, in Section 4 we compare both 1D models and discuss the difference in related constitutive inequalities.

In what follows we use the direct (index-free) tensor notations as defined in [19,27,28].

2. Nonlinear continua and strong ellipticity

In the following we briefly recall the basic equations of micromorphic and strain gradient mechanics for solids undergoing finite deformations.

2.1. Micromorphic continuum

Let $\mathcal{B}$ be an elastic body. Deformation of $\mathcal{B}$ could be described as an invertible mapping from a reference placement κ into a current placement χ. For any material particle z of $\mathcal{B}$ we characterize its positions in κ and χ through vectors $\mathbf{X}$ and $\mathbf{x}$, respectively. So for a static deformation we have

$$\mathbf{x} = \mathbf{x}(\mathbf{X}).$$

For the micromorphic media we introduce a second-order tensor of microdeformations [10] as an additional kinematical descriptor associated to the same material particle z

$$\mathbf{H} = \mathbf{H}(\mathbf{X}).$$

As a result, a strain energy density could be introduced as a function of the deformation gradient $\mathbf{F} = \nabla\mathbf{x}$, $\mathbf{H}$, and $\nabla\mathbf{H}$

$$W = W(\mathbf{F}, \mathbf{H}, \nabla\mathbf{H}),$$

where ∇ is the 3D nabla-operator defined as in [19,27,28]. After application of the material frame indifference principle [18,19] we came to the following form

$$W = W(\mathbf{C}, \mathbf{H}\cdot\mathbf{F}^{-1}, \mathbf{L}), \tag{1}$$

where $\mathbf{C} = \mathbf{F}\cdot\mathbf{F}^T$ is the Cauchy–Green strain tensor, "$\cdot$" stands for the dot product, $\mathbf{L} = \nabla\mathbf{H}\cdot\mathbf{F}^{-1}$ is a third-order tensor, see, e.g., [29,30] for more details. Note that for simplicity we keep the same notation for W.

The Lagrangian equilibrium equations take the form

$$\nabla\cdot\mathbf{P} + \rho\mathbf{f} = \mathbf{0}, \quad \nabla\cdot\mathbf{S} - \frac{\partial W}{\partial \mathbf{H}} + \rho\mathbf{c} = \mathbf{0}, \tag{2}$$

where $\mathbf{P}$ and $\mathbf{S}$ are the first Piola–Kirchhoff stress and hyper-stress tensors, respectively, ρ is a referential mass density, and $\mathbf{f}$ and $\mathbf{c}$ are mass force vector and hyper-force tensor. $\mathbf{P}$ and $\mathbf{S}$ are expressed through W as follows [28]

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{S} = \frac{\partial W}{\partial \nabla\mathbf{H}}. \tag{3}$$

2.2. Strain gradient elastic continuum

Within the strain gradient elasticity approach a strain energy density V depends of $\mathbf{F}$ and its gradient $\mathbf{G} = \nabla\mathbf{F}$ [13,31]:

$$V = V(\mathbf{F}, \mathbf{G}).$$

Applying again the principle of material frame indifference we came to the following form of V [26,32]

$$V = V(\mathbf{C}, \mathbf{K}), \tag{4}$$

where $\mathbf{K} = \nabla\mathbf{F}\cdot\mathbf{F}^T$ is a third-order tensor.

The Lagrangian equations of statics have the form

$$\nabla\cdot\mathbf{T} + \rho\mathbf{f} = \mathbf{0}, \qquad \mathbf{T} = \mathbf{P} - \nabla\cdot\mathbf{M}, \tag{5}$$

where $\mathbf{T}$ is the total stress, $\mathbf{P}$ is the stress, and $\mathbf{M}$ is the hyper-stress tensors, all are of the first Piola–Kirchhoff type. They are defined as follows

$$\mathbf{P} = \frac{\partial V}{\partial \mathbf{F}}, \quad \mathbf{M} = \frac{\partial V}{\partial \mathbf{G}}. \tag{6}$$

2.3. Strong ellipticity conditions

Let us formulate the strong ellipticity (SE) condition in terms of strain energy density for the both models. In the case of the micromorphic continuum the SE condition coincides with the positive definiteness of the following matrix with tensor-valued elements

$$\mathbb{Q} = \begin{pmatrix} \dfrac{\partial^2 W}{\partial \mathbf{F}^2} & \dfrac{\partial^2 W}{\partial \mathbf{F}\partial\nabla\mathbf{H}} \\ \dfrac{\partial^2 W}{\partial\nabla\mathbf{H}\partial\mathbf{F}} & \dfrac{\partial^2 W}{\partial\nabla\mathbf{H}^2} \end{pmatrix}, \tag{7}$$

which could be written as follows [29]

$$\begin{aligned}(\mathbf{k}\otimes\mathbf{a}) : \frac{\partial^2 W}{\partial \mathbf{F}^2} : (\mathbf{k}\otimes\mathbf{a}) + (\mathbf{k}\otimes\mathbf{a}) : \frac{\partial^2 W}{\partial \mathbf{F}\partial\nabla\mathbf{H}} \,\vdots\, (\mathbf{k}\otimes\mathbf{A}) \\ + (\mathbf{k}\otimes\mathbf{A}) \,\vdots\, \frac{\partial^2 W}{\partial\nabla\mathbf{H}\partial\mathbf{F}} : (\mathbf{k}\otimes\mathbf{a}) \\ + (\mathbf{k}\otimes\mathbf{A}) \,\vdots\, \frac{\partial^2 W}{\partial\nabla\mathbf{H}^2} \,\vdots\, (\mathbf{k}\otimes\mathbf{A}) \\ \geq C_1|\mathbf{k}|^2\left(|\mathbf{a}|^2 + |\mathbf{A}|^2\right),\end{aligned}$$

where $\mathbf{k}$ and $\mathbf{a}$ are arbitrary vectors, $\mathbf{A}$ is an arbitrary second-order tensor, "$\otimes$" is the dyadic product, "$:$" and "$\vdots$" are the double dot and triple dot products, respectively, $|\mathbf{a}|$ and $|\mathbf{A}|$ are Euclidean norms for vectors and second-order tensors, and C_1 is a positive constant independent on $\mathbf{k}$, $\mathbf{a}$, and $\mathbf{A}$.

For the strain gradient elasticity the SE condition takes the form [26, 33]

$$(\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a}) \,\vdots\, \frac{\partial^2 V}{\partial \mathbf{G}^2} \,\vdots\, (\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a}) \geq C_2|\mathbf{k}|^4|\mathbf{a}|^2, \tag{8}$$

where again $\mathbf{k}$ and $\mathbf{a}$ are arbitrary vectors, and C_2 is a positive constant independent on $\mathbf{k}$ and $\mathbf{a}$.

3. One-dimensional case

In order to illustrate difference between SE conditions let consider one-dimensional (1D) counterparts of considered models. In other words, we restrict ourselves to so-called "1D world", which is similar but not the same as a classic problem of for an elastic bar under tension, as here we have only one dimension. Indeed, now our elastic body $\mathcal{B}$ could be represented as a segment $[0, a]$ in a reference placement. So a position of a material particle z in κ is given by one scalar Lagrangian coordinate $X \in [0, a]$. We assume that $\mathcal{B}$ is clamped at $x = 0$, whereas an external load p is applied at $x = a$. For simplicity we neglect mass forces. The problem under consideration could be considered as an uniaxial strain state, whereas an elastic bar stretching corresponds to uniaxial tension. First, let us consider a hypothetic model with ellipticity loss within nonlinear elasticity.

3.1. Simple material

Let us consider 1D model for a nonlinear elastic material known also as a simple or Cauchy material. In what follows we assume the following 1D strain energy density

$$U = U(\varepsilon), \quad \varepsilon = u_X, \tag{9}$$

where $u \equiv x - X = u(X)$ is a displacement field. For brevity we denote derivatives with respect to X as follows

$$u_X = \frac{du}{dX}, \quad u_{XX} = \frac{d^2u}{dX^2}, \quad \text{etc.}$$

Equilibrium equation takes the form

$$\sigma_X = 0, \quad \sigma = \frac{dU}{d\varepsilon}, \tag{10}$$

with the kinematic and static boundary conditions

$$u(0) = 0, \quad \sigma(a) = p. \tag{11}$$

Here σ is a Piola (nominal or engineering) stress.

For the 1D continuum the SE condition takes simple form

$$\frac{d\sigma}{d\varepsilon} \equiv \frac{d^2U}{d\varepsilon^2} > 0. \tag{12}$$

So within the ellipticity range the tangent elastic modulus is positive, whereas the strain energy density is convex.

In order to demonstrate the loss of strong ellipticity we consider the following strain energy density given in the form of Morse potential [34]

$$U = \frac{1}{2} E \left[1 - \exp(-\varepsilon/\ell)\right]^2, \tag{13}$$

where E is an elastic modulus and ℓ is a characteristic size of the energy well. Typical graphs of U and σ are given in Fig. 1 a) and b), respectively. Here the graph of U has a horizontal asymptote at $\varepsilon \to \infty$, so σ tends to zero at $\varepsilon \to \infty$. We have the non-ellipticity range after the inflexion point at $\varepsilon = \varepsilon^*$ in Fig. 1 a) and for the fading branch of $\sigma - \varepsilon$ curve in Fig. 1 b). In this range the strain energy is non-convex and the tangent elastic modulus is negative. This situation could be treated as a material instability, see [35].

Eqs. (10) and (11) result in an affine deformation

$$u = \varepsilon_0 X,$$

where ε_0 is a solution of $\sigma(\varepsilon) = p$. Obviously, such solution exists if $p \le p_{\max}$ as for $p > p_{\max}$ this equation does not have any solution. Moreover, for $p \in (0, p_{\max})$ we have two solutions $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$. So we see that the loss of the SE condition results in non-uniqueness of solutions as well as in certain material instabilities.

Let us note that 1D problems with non-convex problems strain energy density are studied in nonlinear elasticity in order to model phase transformations, see e.g. [36] and the references therein. Ericksen [37] considered an elastic bar with two-well potential, see also [38,39]. We also underline that the SE condition analysis for uniaxial tension within 3D theory is more complex, in general, see e.g. [40]. In particular, the ellipticity range does not corresponds to the fading branch as observed for some parameters of Ogden's model of material [40].

3.2. Strain gradient 1D continuum

For a strain gradient model of $\mathcal{B}$ a strain energy density and the kinematic boundary conditions take the form

$$V = V(\varepsilon, \varepsilon_X), \tag{14}$$

$$u(0) = 0, \quad u_X(0) \equiv \varepsilon(0) = 0. \tag{15}$$

One-dimensional equilibrium equation and static boundary conditions are given by

$$\sigma_X - \tau_{XX} = 0, \quad \sigma = \frac{\partial V}{\partial \varepsilon}, \quad \tau = \frac{\partial V}{\partial \varepsilon_X}, \tag{16}$$

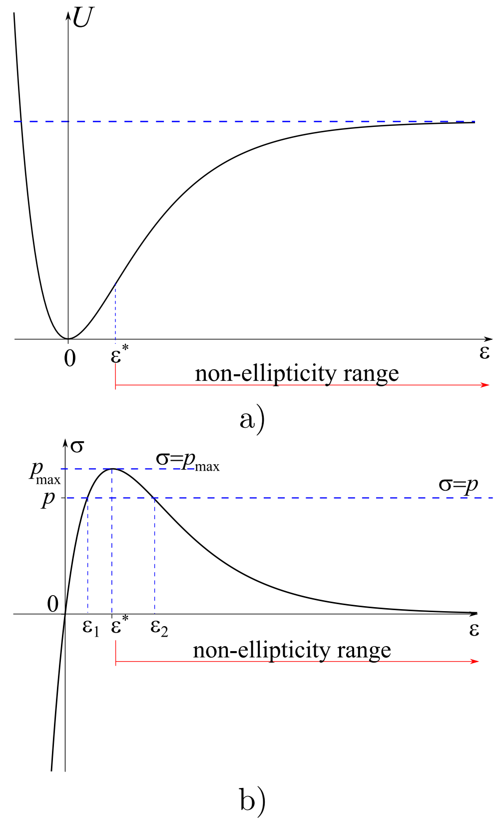

Fig. 1. Simple material: a) strain energy *vs.* strain; b) stress *vs.* strain. For (13) $\varepsilon^* = \ln(2)/\ell$.

$$\sigma(a) - \tau_X(a) = p, \quad \tau(a) = 0, \tag{17}$$

where σ and τ are 1D Piola-type stress and hyper stress (double stress), respectively.

The 1D SE condition takes the form

$$\frac{d\tau}{d\varepsilon_X} \equiv \frac{d^2V}{d\varepsilon_X^2} > 0. \tag{18}$$

Obviously, Eq. (18) is different from (12) as it does not apply any constraint on the dependence on ε.

Let us consider a particular form of (14) given by

$$V = U(\varepsilon) + \frac{1}{2}\alpha\varepsilon_X^2, \tag{19}$$

where U is defined as for simple material, and α is an additional elastic modulus. In fact, (19) could be treated as a regularization of (9). In this case (18) transforms into the inequality

$$\alpha > 0. \tag{20}$$

Note, that inequality (20) results in the positive definiteness of V. Now the 1D boundary-value problem has the form

$$\sigma_X - \alpha\varepsilon_{XXX} = 0, \quad \sigma = \frac{dU}{d\varepsilon}, \tag{21}$$

$$u(0) = 0, \quad \varepsilon(0) = 0, \quad \sigma(a) - \varepsilon_{XX}(a) = p, \quad \varepsilon_X(a) = 0, \tag{22}$$

Obviously, (21) and $(22)_{2,3,4}$ constitute a BVP with respect to ε, which could be solved as follows. Integrating (21) and taking into account $(22)_3$, we get a new BVP

$$\sigma(\varepsilon) - \alpha\varepsilon_{XX} = p, \tag{23}$$

$$\varepsilon(0) = 0, \quad \varepsilon_X(a) = 0. \tag{24}$$

Using the standard technique for ordinary differential equations (ODEs) of second-order [41] we come to the first integral

$$\frac{\alpha}{2}\varepsilon_X^2 = U(\varepsilon) - p\varepsilon + C, \tag{25}$$

where C is an integration constant. It should be found from (22)$_4$. Then the solution is given by the implicit dependence

$$\int_0^{\varepsilon} \frac{d\varepsilon}{\sqrt{2\,[U(\varepsilon) - p\varepsilon + C]}} = \pm X. \tag{26}$$

Finally, u has the form $u(X) = \int_0^X \varepsilon(X)\,dX$.

3.3. Micromorphic approach

Let us now consider an extension of the 1D simple material using micromorphic approach. For the 1D micromorphic continuum a strain energy density is given by

$$W = W(\varepsilon, \eta, \eta_X), \tag{27}$$

where $\eta = \eta(X)$ is a scalar microdeformation field. In fact, for 1D case we have a model with a scalar microstructure as defined by Capriz [42], such as for example Nunziato–Cowin poroelasticity [43].

For 1D micromorphic body $\mathcal{B}$ the BVP consists of equilibrium equations

$$\sigma_X = 0, \quad \mu_X - \frac{\partial W}{\partial \eta} = 0; \quad \sigma = \frac{\partial W}{\partial \varepsilon}, \quad \mu = \frac{\partial W}{\partial \eta_X}, \tag{28}$$

and the following boundary conditions

$$u(0) = 0, \quad \eta(0) = 0; \quad \sigma(a) = p, \quad \mu(a) = 0. \tag{29}$$

The SE conditions coincide with the positive definiteness of the matrix

$$\mathbb{Q} = \begin{pmatrix} \dfrac{\partial^2 W}{\partial \varepsilon^2} & \dfrac{\partial^2 W}{\partial \varepsilon \partial \eta_X} \\ \dfrac{\partial^2 W}{\partial \eta_X \partial \varepsilon} & \dfrac{\partial^2 W}{\partial \eta_X^2} \end{pmatrix}. \tag{30}$$

In what follows similar to (19) we restrict ourselves to the micromorphic extension of (9)

$$W = U(\varepsilon) + \frac{\gamma}{2}\eta_X^2 + \frac{\beta}{2}(\varepsilon - \eta)^2, \tag{31}$$

where γ and β are new elastic moduli. In this case the positive definiteness of $\mathbb{Q}$ is equivalent to the inequalities

$$\gamma > 0, \quad \frac{d^2U}{d\varepsilon^2} + \beta > 0. \tag{32}$$

So the SE conditions are determined by both parts of the strain energy density, i.e. by nonlinearly elastic and micromorphic parts. Note, that positive definiteness of W requires more strong inequalities: $\gamma > 0$, $\beta \geq 0$. The case $\beta = 0$ corresponds to a decoupled problem, so we assume that $\beta > 0$. For U given by (13) we have that $d^2U/d\varepsilon^2 \geq -E/8\ell^2$. So Eq. (32)$_2$ results in the inequality $\beta > \beta^*$, where $\beta^* = E/8\ell^2$.

The corresponding 1D BVP has the form

$$\left[\sigma_0(\varepsilon) + \beta(\varepsilon - \eta)\right]_X = 0, \quad \sigma_0 = \frac{\partial U}{\partial \varepsilon}, \tag{33}$$

$$\gamma \eta_{XX} + \beta(\varepsilon - \eta) = 0; \tag{34}$$

$$u(0) = 0, \quad \eta(0) = 0; \tag{35}$$

$$\sigma_0(\varepsilon(a)) + \beta(\varepsilon(a) - \eta(a)) = p, \quad \eta_X(a) = 0. \tag{36}$$

From (33) and (36)$_1$ we get that

$$\sigma_0(\varepsilon) + \beta(\varepsilon - \eta) = p \tag{37}$$

for all $X \in (0, a]$. Extracting η from (37) and substituting the result into (34), (35)$_2$, and (36)$_2$ we get again the nonlinear BVP with respect to ε

$$\frac{\gamma}{\beta}\left[\frac{d\sigma_0}{d\varepsilon} + \beta\right]\varepsilon_{XX} + \frac{\gamma}{\beta}\frac{d^2\sigma_0}{d\varepsilon^2}\varepsilon_X^2 - \sigma_0(\varepsilon) + p = 0, \tag{38}$$

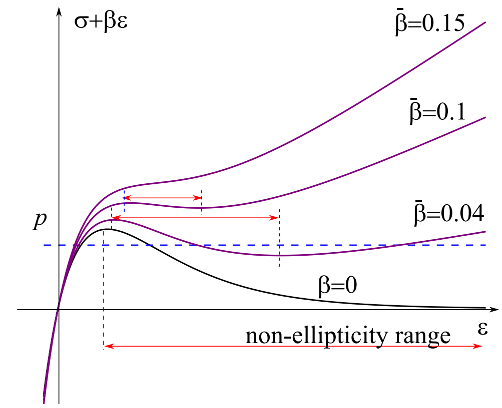

Fig. 2. To violation of the SE condition (32)$_2$: $\sigma_0(\varepsilon)+\beta\varepsilon$ *vs.* strain, non-ellipticity range is shown in red. Here $\bar{\beta} = \beta/E$.

$$\left[\left(\frac{d\sigma_0}{d\varepsilon} + \beta\right)\varepsilon_X\right]\Big|_{X=a} = 0, \tag{39}$$

$$\left[\sigma_0(\varepsilon) + \beta\varepsilon\right]\Big|_{X=0} = p. \tag{40}$$

Obviously, the latter BVP differs essentially from (21) and (22). So we have not a solution in a form similar to (26). The main difference consists of a possible singularity in multiplier $\frac{d\sigma_0}{d\varepsilon} + \beta$ before ε_{XX}, i.e when $\frac{d\sigma_0}{d\varepsilon} + \beta = 0$ or, in other words, when (32)$_2$ is violated. In addition, in this case (39) becomes degenerated and does not constitute a boundary condition. Moreover, violation of (32)$_2$ result in multiple solutions of (40) for ε. The dependencies of $\sigma_0(\varepsilon) + \beta\varepsilon$ vs. ε is shown in Fig. 2 for some values of β. Here we can see how the non-ellipticity of the simple material was inherited by the micromorphic model.

4. Comparison of the models through SE conditions

Let us consider these two 1D models in more details. First, we shall underline some obvious similarities between strain gradient and micromorphic approaches. In fact, equating η to ε, from (27) we immediately get (14). So the micromorphic model with the constraint $\eta = \varepsilon$ (or $\mathbf{H} = \mathbf{F}$) could be treated as s strain gradient medium, see [44] for application of Lagrange multipliers technique. Moreover, let us note that a static solutions could be obtained minimizing the total energy functionals, i.e. from variational equations

$$\delta\mathcal{E}_G = 0, \quad \delta\mathcal{E}_M = 0,$$

$$\mathcal{E}_G = \int_0^a V\,dX - pu(a), \quad \mathcal{E}_M = \int_0^a W\,dX - pu(a) = 0,$$

where in $\mathcal{E}_G$ and $\mathcal{E}_M$ the energy densities given by (19) and (31), respectively.

Using the penalty technique we can treat the term $\beta(\varepsilon - \eta)^2$ with large enough value of β as a penalty function. So a minimizer of $\mathcal{E}_M$ should be close to a minimizer of $\mathcal{E}_G$ when $\beta \to \infty$.

On the other hand, we also see obvious differences between the models. Both models are reduced to 1D BVPs for second-order ODEs, which differ from each other in the form of ODE and in the boundary conditions. In order to compare possible solutions of (23), (24), and (38)–(40) we consider their phase portraits. Any solution of (23), (24), or (38)–(40) could be represented as an integral curve on $\varepsilon - \varepsilon_X$-plane. In what follows we call a simple material model with strain energy U the *primary* material.

Typical phase portraits for (23) and (24) are given in Fig. 3(a) and (b). Fig. 3(a) corresponds to $p \in (0, p_{\max})$, i.e to the case when for the primary material there are two solutions ε_1 and ε_2, whereas Fig. 3(b) relates to $p > p_{\max}$ (for the primary material a static solution does not

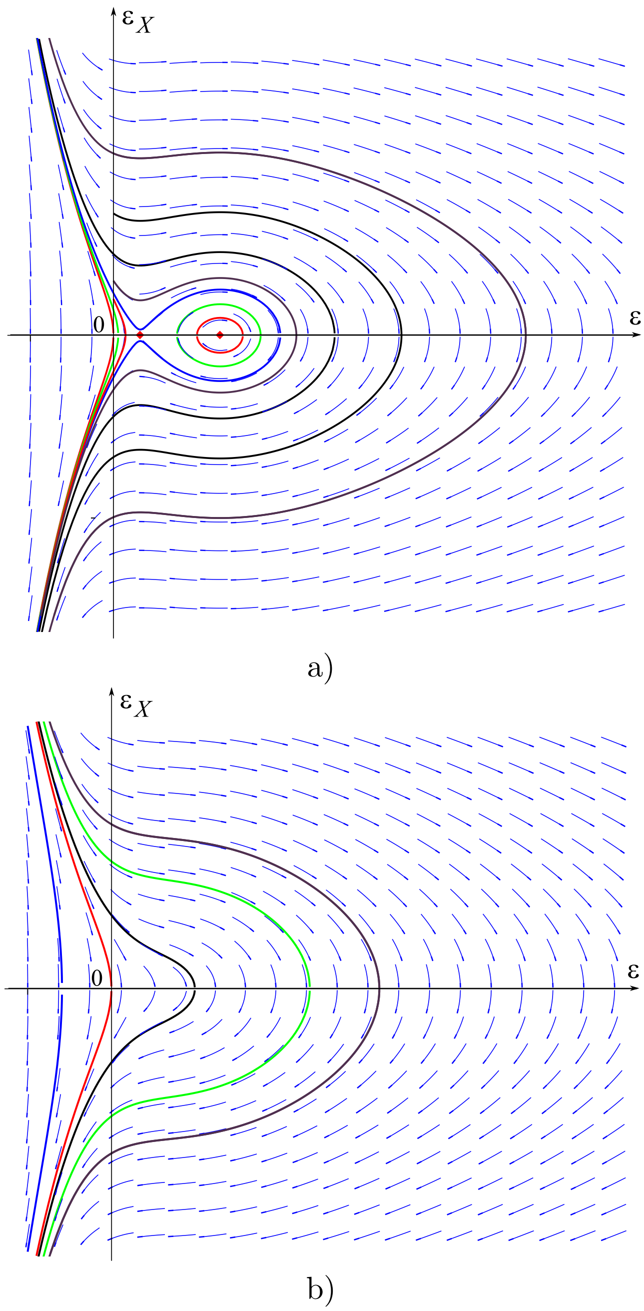

Fig. 3. Phase portraits for the strain gradient BVP: a) $p \in (0, p_{\max})$, two stationary points at $(\varepsilon_1, 0)$ and $(\varepsilon_2, 0)$ are pointed red diamonds; b) $p > p_{\max}$, no stationary points. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

exist). Two red diamonds in Fig. 3(a) describe stationary points of (23) and (24), that are solutions of $\sigma(\varepsilon) = p$, see Fig. 1(b). The latter are a saddle point and a center, respectively. For Fig. 3(b) BVP (23) and (24) does not have a stationary point. A solution of (23) and (24) could be represented as a curve which begins at the vertical line $\varepsilon = 0$ and ends at the horizontal line $\varepsilon_X = 0$. Some integral curves are shown. Note that here we restrict ourselves to the elliptic case with $\alpha > 0$. As (23) and (24) has a solution for any p, we can say that (19) is a certain *gradient regularization* of a primary material independently on its ellipticity loss.

Under SE conditions phase portraits for the micromorphic model is given in Figs. 4 and 5. Now the behavior of integral curves is more complex and depends not only on p but also on β. For $p \in (0, p_{\max})$ we again have saddle and center points (Figs. 4 and 5 a), whereas for $p > p_{\max}$ stationary points do not exist. A solution of (38)–(40) could be represented as an integral curve which starts on the vertical line $\varepsilon = \varepsilon_\beta$ and ends on the line $\varepsilon_X = 0$. Here ε_β is a solution of (40). Obviously, $\varepsilon_\beta \to 0$ at $\beta \to \infty$. So for relatively small values of β, i.e. $\beta \sim \beta^*$, integral curves are similar only qualitatively as shown in Fig. 3, whereas for relatively large values of β, i.e for $\beta \gg \beta^*$, the shape of integral curves are quite similar, see Figs. 3 and 5. Here the values $\bar{\beta} \equiv \beta/E = 0.15$ and $\bar{\beta} = 10$ are used for Figs. 4 and 5, respectively, whereas $\bar{\beta}^* = 0.125$. So we can confirm a convergence of solutions of (38)–(40) to their gradient counterparts followed from (23) and (24) at $\beta \to \infty$ and under assumption $\alpha = \gamma$.

Note that a solution of (38)–(40) exists for any p, whereas for the primary material there a solution does not exist for $p > p_{\max}$. So we can also call Eq. (31) a *micromorphic regularization* of the primary material.

5. Conclusions

Considering gradient and micromorphic "regularizations" of a primary nonlinear elastic simple material, we have discussed the strong ellipticity conditions for these media, that could be related to a certain material instability. The considered primary material may loss ellipticity which results in non-existence of solutions under some loads. We can conclude that for a strain gradient material the SE conditions are more simple and entirely independent on the SE conditions for the primary material. Under the SE conditions the strain gradient approach could be considered as a regularization of constitutive equations of a simple material. Indeed, in this case one avoid a non-existence issue as existence depends on the higher-order terms. Nevertheless, one can see that a solution of 1D BVP within gradient approach reflects some properties of a primary material including its ellipticity.

Instead, for a micromorphic material the SE conditions inherit SE conditions of the primary material. In other words, a violation of the SE conditions for the primary material may result in the consequent violation of the SE conditions for micromorphic materials. But under the SE conditions we again can solve the non-existence issue.

So considering strain gradient and micromorphic continua as models of some microstructured materials such as beam-lattice metamaterials, we see that the SE conditions for micromorphic materials are more restrictive and may correspond to material instabilities at different scales. On the other hand, as we can see above, the both regularizations could produce similar results, at least for some cases.

Finally, we can conclude that the strong ellipticity plays an important role as constitutive inequality with enhanced models of continua. In particular, violation of SE conditions may signal a certain material instability. On the other hand, one should be aware of transmission of results from one model to another one without detailed analysis. For example, in order to avoid material instability within the strain gradient elasticity the SE condition should be complemented by additional inequality as in [26], whereas SE conditions for a micromorphic medium does not require such a complement.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The first author acknowledges the support by the Russian Science Foundation under grant 20-41-04404 issued to the Institute of Applied Mechanics of Russian Academy of Sciences.

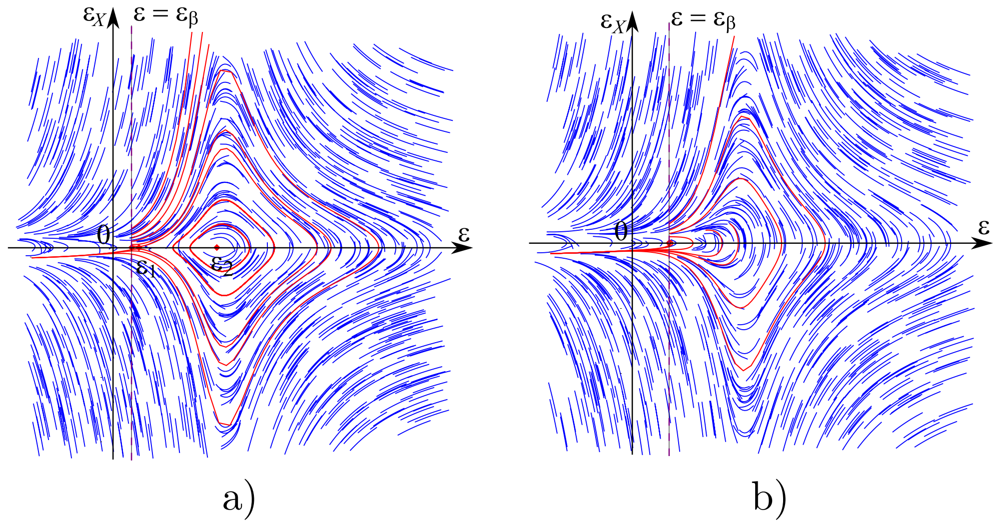

Fig. 4. Phase portraits for the micromorphic BVP: "small" β. (a) $p \in (0, p_{\max})$, two stationary points at $(\varepsilon_1, 0)$ and $(\varepsilon_2, 0)$ are pointed as red diamonds; b) $p > p_{\max}$, no stationary points. Integral curves begins on the vertical dashed line $\varepsilon = \varepsilon_\beta$, $\bar{\beta} = 0.15$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

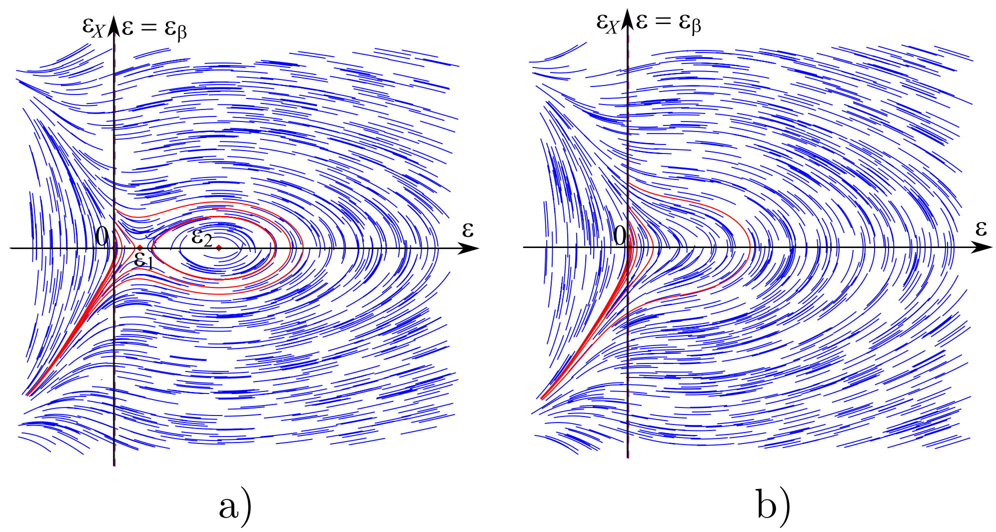

Fig. 5. Phase portraits for the micromorphic BVP: "large" β. (a) $p \in (0, p_{\max})$, two stationary points at $(\varepsilon_1, 0)$ and $(\varepsilon_2, 0)$ are pointed as red diamonds; b) $p > p_{\max}$, no stationary points. Integral curves begins on the vertical line $\varepsilon = \varepsilon_\beta$, which is close to the line $\varepsilon = 0$, $\bar{\beta} = 10$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

References

[1] G. Hütter, Homogenization of a Cauchy continuum towards a micromorphic continuum, J. Mech. Phys. Solids 99 (2017) 394–408.

[2] R. Biswas, L.H. Poh, A micromorphic computational homogenization framework for heterogeneous materials, J. Mech. Phys. Solids 102 (2017) 187–208.

[3] S.E. Alavi, J.F. Ganghoffer, H. Reda, M. Sadighi, Construction of micromorphic continua by homogenization based on variational principles, J. Mech. Phys. Solids 153 (2021) 104278.

[4] F. dell'Isola, D.J. Steigmann, Discrete and Continuum Models for Complex Metamaterials, Cambridge University Press, Cambridge, 2020.

[5] A. Misra, L. Placidi, F. dell'Isola, E. Barchiesi, Identification of a geometrically nonlinear micromorphic continuum via granular micromechanics, ZAMP 72 (4) (2021) 1–21.

[6] H. Abdoul-Anziz, P. Seppecher, Strain gradient and generalized continua obtained by homogenizing frame lattices, Math. Mech. Complex Syst. 6 (3) (2018) 213–250.

[7] F. dell'Isola, D. Steigmann, A two-dimensional gradient-elasticity theory for woven fabrics, J. Elasticity 118 (1) (2015) 113–125.

[8] R.D. Mindlin, Micro-structure in linear elasticity, Arch. Ration. Mech. Anal. 16 (1) (1964) 51–78.

[9] A.C. Eringen, E.S. Suhubi, Nonlinear theory of simple micro-elastic solids–I, Internat. J. Engrg. Sci. 2 (2) (1964) 189–203.

[10] A.C. Eringen, Microcontinuum Field Theory. I. Foundations and Solids, Springer, New York, 1999.

[11] S. Forest, Micromorphic approach for gradient elasticity, viscoplasticity, and damage, J. Engng. Mech. 135 (3) (2009) 117–131.

[12] S. Forest, Micromorphic media, in: H. Altenbach, V.A. Eremeyev (Eds.), Generalized Continua from the Theory to Engineering Applications, in: 541 of CISM International Centre for Mechanical Sciences, Springer, Vienna, 2013, pp. 249–300.

[13] A. Bertram, S. Forest (Eds.), Mechanics of Strain Gradient Materials, Springer, Cham, 2020.

[14] G.A. Maugin, Generalized continuum mechanics: Various paths, in: Continuum Mechanics Through the Twentieth Century: A Concise Historical Perspective, Springer, Dordrecht, 2013, pp. 223–241.

[15] G.A. Maugin, Non-Classical Continuum Mechanics: A Dictionary, Springer Singapore, Singapore, 2017.

[16] F. dell'Isola, A. Della Corte, I. Giorgio, Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives, Math. Mech. Solids 22 (4) (2017) 852–872.

[17] W. Noll, A new mathematical theory of simple materials, Arch. Ration. Mech. Anal. 48 (1972) 243–292.

[18] C. Truesdell, W. Noll, The Non-Linear Field Theories of Mechanics, third ed., Springer, Berlin, 2004.

[19] A.I. Lurie, Non-Linear Theory of Elasticity, North-Holland, Amsterdam, 1990.

[20] R.W. Ogden, Non-Linear Elastic Deformations, Dover, Mineola, 1997.
[21] C. Truesdell, The Elements of Continuum Mechanics, Springer, New York, 1966.
[22] L.J. Gibson, M.F. Ashby, Cellular Solids: Structure and Properties, second ed., in: Cambridge Solid State Science Series, Cambridge University Press, Cambridge, 1997.
[23] C. Combescure, Generalized continuum models confronted to cell-commensurate instabilities in structured media, in: IUTAM Symposium on Generalized Continua Emerging from Microstructures, Paris, France, 2021, pp. 1–39, https://hal-upec-upem.archives-ouvertes.fr/hal-03403455.
[24] Y.B. Fu, R.W. Ogden, Nonlinear stability analysis of pre-stressed elastic bodies, Contin. Mech. Thermodyn. 11 (1999) 141–172.
[25] D.N. Sheydakov, Stability of circular micropolar rod with prestressed two-layer coating, Contin. Mech. Thermodyn. 33 (4) (2021) 1313–1329.
[26] V.A. Eremeyev, Strong ellipticity conditions and infinitesimal stability within nonlinear strain gradient elasticity, Mech. Res. Commun. 117 (2021) 103782.
[27] J.G. Simmonds, A Brief on Tensor Analysis, second ed., Springer, New Yourk, 1994.
[28] V.A. Eremeyev, M.J. Cloud, L.P. Lebedev, Applications of Tensor Analysis in Continuum Mechanics, World Scientific, New Jersey, 2018.
[29] V.A. Eremeyev, L.P. Lebedev, M.J. Cloud, Acceleration waves in the nonlinear micromorphic continuum, Mech. Res. Commun. 93 (2018) 70–74.
[30] V.A. Eremeyev, On the material symmetry group for micromorphic media with applications to granular materials, Mech. Res. Commun. 94 (2018) 8–12.
[31] J.C. Reiher, A. Bertram, Finite third-order gradient elasticity and thermoelasticity, J. Elasticity 133 (2) (2018) 223–252.
[32] V.A. Eremeyev, Local material symmetry group for first- and second-order strain gradient fluids, Math. Mech. Solids 26 (8) (2021) 1173–1190.
[33] A. Mareno, T.J. Healey, Global continuation in second-gradient nonlinear elasticity, SIAM J. Math. Anal. 38 (1) (2006) 103–115.
[34] R.C. Lincoln, K.M. Koliwad, P.B. Ghate, Morse-potential evaluation of second- and third-order elastic constants of some cubic metals, Phys. Rev. 157 (1967) 463–466.
[35] M. Born, K. Huang, Dynamical Theory of Crystal Lattices, The Clarendon Press, Oxford University Press, Oxford, 1985.
[36] R. Abeyaratne, J.K. Knowles, Evolution of Phase Transitions. A Continuum Theory, Cambridge University Press, Cambridge, 2006.
[37] J.L. Ericksen, Equilibrium of bars, J. Elasticity 5 (3) (1975) 191–201.
[38] L. Truskinovsky, G. Zanzotto, Finite-scale microstructures and metastability in one-dimensional elasticity, Meccanica 30 (5) (1995) 577–589.
[39] L. Truskinovsky, G. Zanzotto, Ericksen's bar revisited: Energy wiggles, J. Mech. Phys. Solids 44 (8) (1996) 1371–1408.
[40] L. Zee, E. Sternberg, Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids, Arch. Ration. Mech. Anal. 83 (1) (1983) 53–90.
[41] V.I. Arnold, Mathematical Methods of Classical Mechanics, second ed., Springer, New York, 1989.
[42] G. Capriz, Continua with Microstructure, Springer, New York, 1989.
[43] J.W. Nunziato, S.C. Cowin, A nonlinear theory of elastic materials with voids, Arch. Ration. Mech. Anal. 72 (2) (1979) 175–201.
[44] F. dell'Isola, F. Di Cosmo, Lagrange multipliers in infinite-dimensional systems, methods of, in: H. Altenbach, A. Öchsner (Eds.), Encyclopedia of Continuum Mechanics, Springer, Berlin, 2020, pp. 1–9.