Bell-Type Inequalities from the Perspective of Non-Newtonian Calculus

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Abstract

A class of quantum probabilities is reformulated in terms of non-Newtonian calculus and projective arithmetic. The model generalizes spin-1/2 singlet state probabilities discussed in Czachor (Acta Physica Polonica:139 70–83, 2021) to arbitrary spins s. For $s \to \infty$ the formalism reduces to ordinary arithmetic and calculus. Accordingly, the limit "non-Newtonian to Newtonian" becomes analogous to the classical limit of a quantum theory.

Keywords Hidden variables · Bell's inequality · Non-Diophantine arithmetic · Non-Newtonian calculus · Correspondence principle · Spin

1 Introduction

The problem of hidden variables in quantum mechanics (Einstein et al., 1935; von Neumann, 1995; Bohm, 1952a, b; Bohm & Aharonov, 1957) has been turned by Bell into an experimental criterion for classicality of probabilities (Bell-type inequalities (Bell, 1964; Clauser et al., 1969; Clauser & Horne, 1974; Clauser & Shimony, 1978; Aspect, 1982). One of the most spectacular applications of the criterion is in tests for security in quantum cryptography (Ekert, 1991). It is generally believed that a system that exhibits two-spin-1/2 singlet state probabilities cannot be mimicked by a classical one, while by a classical system one means the one where Einstein-Podolsky-Rosen-type elements of reality exist. The belief is at the very heart of various proofs of the fundamental security of quantum cryptography, but it has been recently challenged in a series of papers (Czachor, 2021; 2020a, b; Czachor & Nalikowski, 2021). The claim is that it is possible to reconstruct singlet-state probabilities in a local hidden-variable way if one assumes that the hidden-variable model is constructed by means of a non-Newtonian calculus (Grossman & Katz, 1972; Pap, 1993; Burgin & Czachor, 2020), a possibility not taken into account so far. Non-Newtonian calculus involves derivatives and integrals that are linear with respect to projective arithmetics of real numbers (Burgin & Czachor, 2020; Burgin, 2010), but typically are not linear with

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respect to the usual (Diophantine) arithmetic of reals (Mesiar, 1995; Pap, 2002). This is the technical reason why various standard proofs of Bell-type inequalities are not applicable in the new framework. A physical meaning of constructions based on non-Newtonian calculus is not very clear, however. The generalized type of linearity one encounters here is analogous to the one known from fuzzy calculus (Zimmermann, 1996), so perhaps classical systems based on fuzzy logic should be reexamined from the point of view of security in quantum cryptography (Pykacz & D'Hooghe, 2001). Another open issue is related to correspondence principles, relating non-Newtonian calculus with the standard Newtonian one.

The goal of the present paper is to generalize the discussion from Czachor (2021) to systems involving pairs of particles with higher spins *s*. What we regard as a trademark of a particle with spin *s* is the form of the Malus law (eg. $\cos^2 \theta$ for photons and $\cos^2(\theta/2)$ for electrons). One begins with some space of uniformly distributed classical hidden variables (a circle), which is decomposed into subsets corresponding to different spin values. With each subset, one associates a characteristic function. The probability is computed in an ordinary way, as an integral over the space of hidden variables. The integrand is given by a product of a probability density times a characteristic function. All these formulas have the classical, local Clauser-Horne form (Clauser & Horne, 1974). The only difference is in the form of the integral and the exact meaning of the multiplication of functions under the integral. The integral is non-Newtonian, and the product is from a projective arithmetic.

Each spin system is described by a different arithmetic. It is shown that in the limit of infinite spins, $s \to \infty$, the arithmetic becomes Diophantine (in the terminology of Burgin (Burgin and Czachor 2020; Burgin 2010) we would say that it is characterized by a trivial projector), and Bell-type inequalities are no longer violated. The limit of infinite spin thus plays here a role of a correspondence principle with the usual arithmetic.

2 Mathematical Prerequisites

2.1 Arithmetic

Let us consider a bijection $f : \mathbb{X} \to \mathbb{R}$ which allows us to define addition, subtraction, multiplication and division in the set $\mathbb{X}$:

$$x \oplus y = f^{-1}(f(x) + f(y)), \tag{1}$$

$$x \ominus y = f^{-1}(f(x) - f(y)), \tag{2}$$

$$x \odot y = f^{-1}(f(x) \cdot f(y)), \tag{3}$$

$$x \oslash y = f^{-1}(f(x)/f(y)), \quad (f(y) \neq 0), \tag{4}$$

where $x, y \in \mathbb{X}$. f is an isomorphism of the arithmetic in $\mathbb{X}$ with the one in $\mathbb{R}$. In the terminology of Burgin the arithmetic so defined is projective, while the bijection *f* is called a projector Burgin (2010). A well-known physical example of projective arithmetic is given by the relativistic addition of velocities in $(1 + 1)$-dimensional Minkowski space Czachor (2020b). Although the arithmetic we will use in the paper is isomorphic to the usual addition, subtraction, multiplication, and division, it should be kept in mind that many

non-isomorphic arithmetics are possible as well, including non-Diophantine arithmetics of natural and whole numbers introduced by Burgin (1977, 2020).

The ordering relation in $\mathbb{X}$ is defined as follows: $x <' y$ if and only if $f(x) < f(y)$, where $<'$ denotes the less-than symbol in the projective arithmetic. $<'$ and $<$ are equivalent if f is strictly increasing. We can also define the neutral element of addition, $0' = f^{-1}(0)$, and the neutral element of multiplication, $1' = f^{-1}(1)$, satisfying

$$x \oplus 0' = x, \tag{5}$$

$$x \odot 1' = x, \tag{6}$$

$$x \ominus x = 0', \tag{7}$$

$$x \oslash x = 1'. \tag{8}$$

$0'$ and $1'$ can be termed the projective bits. Projective bits are analogous to quantum bits, but are equipped with Einstein-Podolsky-Rosen-type elements of reality (Czachor, 2020a, b, 2021; Czachor & Nalikowski, 2021). Additive inverse of a number $x \in \mathbb{X}$ is defined as $\ominus x = 0' \ominus x = f^{-1}(-f(x))$, which implies $f(\ominus x) = -f(x)$.

2.2 Calculus

Having all the basic properties defined, one can define non-Newtonian calculus. The derivative of function $A : \mathbb{X} \to \mathbb{X}$ is defined by the difference quotient,

$$\frac{\mathrm{D}A(x)}{\mathrm{D}x} = \lim_{h \to 0'} (A(x \oplus h) \ominus A(x)) \oslash h. \tag{9}$$

satisfying

$$\frac{\mathrm{D}(A(x) \oplus B(x))}{\mathrm{D}x} = \frac{\mathrm{D}A(x)}{\mathrm{D}x} \oplus \frac{\mathrm{D}B(x)}{\mathrm{D}x}, \tag{10}$$

$$\frac{\mathrm{D}(A(x) \odot B(x))}{\mathrm{D}x} = \frac{\mathrm{D}A(x)}{\mathrm{D}x} \odot B(x) \oplus A(x) \odot \frac{\mathrm{D}B(x)}{\mathrm{D}x}, \tag{11}$$

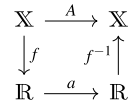$$\frac{\mathrm{D}A[B(x)]}{\mathrm{D}x} = \frac{\mathrm{D}A[B(x)]}{\mathrm{D}B(x)} \odot \frac{\mathrm{D}B(x)}{\mathrm{D}x}. \tag{12}$$

For a given function $A : \mathbb{X} \to \mathbb{X}$ there exists a function $a : \mathbb{R} \to \mathbb{R}$ defined by $A = f^{-1} \circ a \circ f$. It can be shown that

$$\frac{\mathrm{D}A(x)}{\mathrm{D}x} = f^{-1}(a'(f(x))), \tag{13}$$

where a' means a derivative of function a in the Diophantine arithmetic (which uses $+,-,\cdot$, and $/$). It can be represented by a commutative diagram (Fig. 1).

Integral non-Newtonian calculus is based on the fundamental theorem of calculus,

Fig. 1 Relation between A and a

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\ A\ } & \mathbb{X} \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^{-1}} \\ \mathbb{R} & \xrightarrow{\ a\ } & \mathbb{R} \end{array}$$

$$\frac{\mathrm{D}}{\mathrm{D}X} \int_{Y}^{X} A(x) \odot \mathrm{D}x = A(X), \tag{14}$$

$$\int_{Y}^{X} \frac{\mathrm{D}A(x)}{\mathrm{D}(x)} \odot \mathrm{D}x = A(X) \ominus A(Y), \tag{15}$$

which is guaranteed by

$$\int_{x_1}^{x_2} A(x) \odot \mathrm{D}x = f^{-1}\left(\int_{f(x_1)}^{f(x_2)} f \circ A \circ f^{-1}(x)\mathrm{d}x\right) = f^{-1}\left(\int_{f(x_1)}^{f(x_2)} a(x)\mathrm{d}x\right), \tag{16}$$

where $\int a(x)\mathrm{d}x$ is standard (i.e. Riemann or Lebesgue) integral in $\mathbb{R}$.

2.3 Characteristic Functions

In classical probability theory the characteristic function $\chi_+(x) = 1$, if x corresponds to "yes", and $\chi_+(x) = 0$, if x corresponds to "no". We can also define the orthogonal characteristic function $\chi_-(x) = 1 - \chi_+(x)$. An analogous object is defined in non-Newtonian calculus Czachor (2021),

$$\chi_\pm(x) = 1' \ominus \chi_\mp(x), \tag{17}$$

$$\chi_\pm(x) \odot \chi_\pm(x) = \chi_\pm(x), \tag{18}$$

$$\chi_\pm(x) \odot \chi_\mp(x) = 0'. \tag{19}$$

Knowing the bijection f, connecting non-Diophantine arithmetic in $\mathbb{X}$ with the Diophantine arithmetic in $\mathbb{R}$, a commutative diagram can be drawn (Fig. 2),
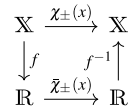
where

$$\bar{\chi}_+(r) = \begin{cases} 1 & \text{if } r' = f^{-1}(r) \text{ means "yes",} \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

$$\bar{\chi}_-(x) = 1 - \bar{\chi}_+(x). \tag{21}$$

Reduction of probability density to conditional probability density is defined in the usual way,

$$\rho(\lambda) \Rightarrow \rho_\pm(\lambda) = \chi_\pm(\lambda) \odot \rho(\lambda) \oslash \int \chi_\pm(x) \odot \rho(x) \odot \mathrm{D}x. \tag{22}$$

Fig. 2 Commutative diagram for characteristic functions

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\chi_{\pm}(x)} & \mathbb{X} \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^{-1}} \\ \mathbb{R} & \xrightarrow{\bar{\chi}_{\pm}(x)} & \mathbb{R} \end{array}$$

3 Bell Experiment with Arbitrary Spins (A Non-Newtonian Model)

The space of hidden variables for the spin s Bell experiment will be presented as a disc (or a circle), in which specific subsets represent spin values. The characteristic functions $\bar{\chi}_{+}(x)$ correspond to these subsets. The model is similar to the "wheel of fortune", where each sector represents some cash value. For each particle with a different spin number, the disc will look differently, but all the basic properties will be the same.

When we rotate a particle by a certain small angle and then measure the orientation of the spin, the obtained value cannot be much different from the previous one. As we know, a particle with the spin number s can have $2s + 1$ different values of spin projection on a certain axis. The values are $\{-s, -s + 1, \ldots, s - 1, s\}$. When the particle is in state m, we assume the state can change only into $m + \frac{1}{2}$ or $m - \frac{1}{2}$. For example, let us consider a particle with $s = \frac{3}{2}$ (such as Ω^-). The measured spin of the particle must be an element of the set $\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$ (which will be written as "$--$","$-$", "$+$", "$++$"). Imagine that we have two measuring apparatuses placed one after another, where the second is slightly tilted with respect to the first one. If the first device measures state $-$, the second device can register state $-$, $--$ or $+$ but not $++$. This means that in our model the difference between neighbouring sectors must be equal to 1. We can say that the change from one section to another is equivalent to the action of an annihilation or creation operator.

We assume that the probability of getting one of the $2s + 1$ states is equal to $\frac{1}{2s+1}$. In this model, the probability is represented by the length of an arc or an area of a particular sector. So, for every possible state, the sum of all arc lengths (sector areas) describing one particular state is the same and equals $\frac{1}{2s+1}$ of the whole circumference (area of the disc).

One of the most important points of this model is a definition of Bell-type measurements for particles with spin greater than $\frac{1}{2}$. Electrons as particles with a spin $\frac{1}{2}$ have only two possible states. Photons as massless bosons also have only two states: $+1$ and -1. For fermions, we will use the convention that if the measured spin is oriented towards our chosen axis, it will be counted as "+", in the opposite case as "−", regardless of the value of the projection of the spin on our chosen axis. For the Stern–Gerlach experiment, every inclination towards the up direction would be counted as "+", no matter how big the spin value is. Similarly with an inclination towards the down direction. For instance, for the Ω^- particle, the states "$--$" and "$-$" will be counted as "−", while "$++$" and "$+$" will be counted as "+".

In the case of bosons, there is a probability that the projection of spin on our chosen axis is equal to 0. When our apparatus gives 0, we randomly decide if the result is "+" or "−" (for instance, by tossing a fair coin). The construction differs from the one by Pearle Pearle (1970) where the results 0 are neglected (which effectively introduces undetected signals). In our model, no data are rejected.

In order for our model to be consistent with the experiments, the Malus law or the Malus-like laws must be fulfilled. The Malus law, originally formulated for photons (massless particles with spin 1), states that conditional probability is $p(+|+) = \cos^2 \theta$, where θ

Fig. 3 Model for particle with a spin $s = \frac{1}{2}$, for example an electron. Sign "+" means spin "up", sign "−" means spin "down"
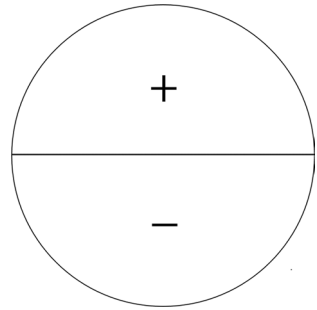
is the angle by which the polarizers are tilted. For a particle with other spin numbers, the Malus-like law is stated as $p(+|+) = \cos^2(s\theta)$, for instance for electron (spin 1/2) the conditional probability is equal to $p(+|+) = \cos^2(\theta/2)$

To sum up, our model is a circular disc divided into sections representing each spin orientation. The sections are designed according to the following rules:

1. Neighbours of the sector describing one particular state of spin can only be those, which can be obtained by applying the creation or annihilation operator.
2. Sum of the areas of all sections describing one particular state must be equal.
3. We neglect the exact value of spin, we consider only its sign ("+" or "−").
4. When 0 occurs for a boson, we randomly select "+" or "−" with equal probability.
5. Malus-like law is fulfilled.

What we have described is the space of hidden variables. The next step is to define an appropriate arithmetic and calculus so that a non-Newtonian integral over the hidden variables will give a joint probability analogous to those occurring in quantum mechanical correlation experiments involving pairs of spin-s particles with anti-correlated spins.

4 Explicit Model for Any s

Let us begin with spin $\frac{1}{2}$ e.g. electron. It has two possible states, which leads to the construction presented in Fig. 3 and maximal violation of the Clauser-Horne inequality, as described in detail in Czachor (2021).

Now consider a spin-1 boson, massive or massless. A photon has only two possible polarisations. For two polarizers tilted by θ the Malus law (rule 5) gives the conditional probability $p(+|+) = \cos^2 \theta$, which should be contrasted with the spin-1/2 result $p(+|+) = \cos^2(\theta/2)$. Thus we conclude that a model following our rules must be similar to that presented in Fig. 4.

For a massive spin-1 particle, e.g J/ψ, the reasoning is similar, but we have to keep in mind the possibility of finding 0. It must be fitted between "+" and "−" regions (rule 1), and total area of sections "0", "+" and "−" must be equal. In this manner, we get a model presented in Fig. 5.

Fig. 4 Model for a massless particle with spin number $s = 1$ e.g. photon. The angular width of each piece is $\frac{\pi}{2}$
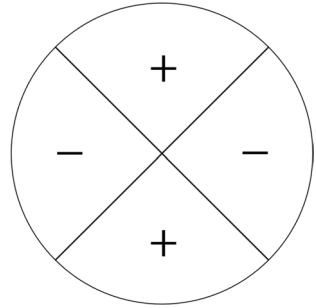

Fig. 5 Model for massive particle with spin number $s = 1$ e.g. J/ψ. Angular width of bigger sections is $\frac{\pi}{3}$, and smaller is $\frac{\pi}{6}$
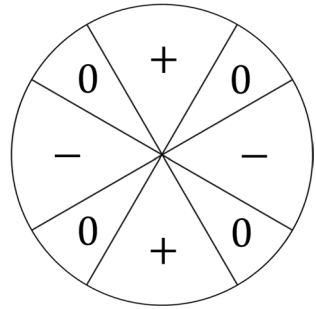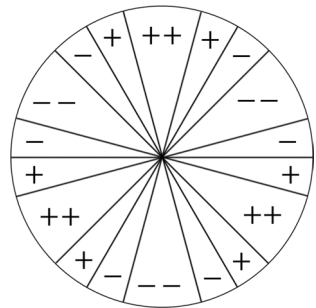

Fig. 6 Model for particle with spin number $s = \frac{3}{2}$. Angle width of greater sectors is $\frac{\pi}{6}$, and lesser sectors is $\frac{\pi}{12}$

As a clear presentation of rule 3, we consider a fermion with spin $\frac{3}{2}$, such as Ω^-. There are two distinguishable "up" states and "down" states. An appropriate spin-$\frac{3}{2}$ Malus-type law leads to the model presented in Fig. 6.

According to rule 3, however, we neglect the exact value of the spin projection on a chosen axis but consider only the sign. This leads to the model presented in Fig. 7.

Now we are ready to develop the model of probabilities typical of a generalized Bell-type scenario, where for $\theta = 0$ we obtain the perfect anti-correlation $p_{++} = p_{--} = 0$. Axes of spin measurements performed by Alice and Bob are denoted by $\hat{a}$ and $\hat{b}$. The directions do not have to be parallel. In Bell-type inequalities, the relevant joint probabilities are p_{++}, p_{+-}, p_{-+} and p_{--}. Our system should be rotationally symmetric, so $p_{++} = p_{--}$ and $p_{+-} = p_{-+}$. To calculate the classical probability one has to layer one circular model described before on another so that their centres overlay. Probability p_{++} is defined as a ratio between arc length of pieces

Fig. 7 Model for particle with spin number $s = \frac{3}{2}$ after applying the third rule. Angle width of each sector is $\frac{\pi}{3}$
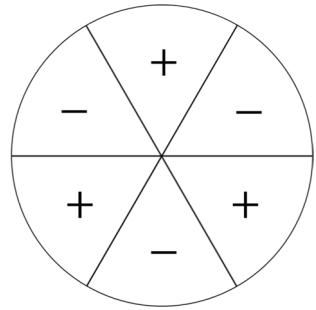

Fig. 8 Presentation of the probability model for $s = \frac{1}{2}$. Directions chosen by Alice and Bob are shown. We assume the singlet state, so the anti-correlation exists. Definition of angle θ is also shown
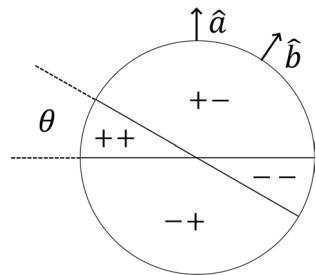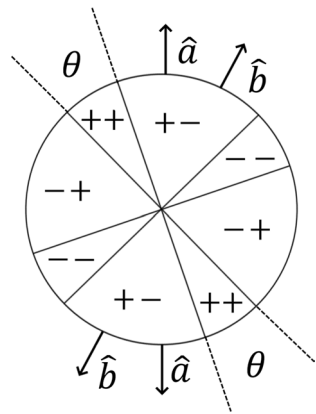

Fig. 9 Presentation of the probability model for $s = \frac{1}{2}$. Directions chosen by Alice and Bob are shown. Due to the multiplicity of regions with different states, all possible Alice's and Bob's directions are displayed. Definition of angle θ is also shown

where both sectors from both discs have a "+" symbol (denoted by "++"), to the full length of the circle. We use the same procedure for the other zones p_{+-}, p_{-+} and p_{--}, as shown in Fig. 8. The arc width of one piece marked as "++" will be also denoted by θ. Due to the fact that both particles are in a singlet-type state, their spins are anti-correlated. To indicate this fact, the direction chosen by Alice is marked as an arrow starting in the middle of the arc representing the "+" state and Bob's direction is an arrow starting from the "−" state. When there are more than one regions with described state, an arrow is placed on every appropriate arc. Simple examples of this are presented in Figs. 8 and 9. Angle θ can be easily correlated with the orientations of axes chosen by Alice and Bob. It can be shown that,

$$\theta = \min\left(|\hat{a} - \hat{b}|_{\mathrm{mod}\frac{\pi}{2s}}, \frac{\pi}{s} - |\hat{a} - \hat{b}|_{\mathrm{mod}\frac{\pi}{2s}}\right). \tag{23}$$

It is clear that $\theta \leq \frac{\pi}{2s}$.

5 Classical and Non-Classical (Non-Newtonian/Non-Diophantine) Probabilities

Probabilities are constructed in analogy to those from Czachor (2021). There are two levels of description. Probabilities written without a prime are the ones following from ratios described in the previous section. Probabilities denoted with a prime are obtained from the unprimed ones by means of the bijection that defines the arithmetic. We demand the consistency condition,

$$1 = p_{++} + p_{+-} + p_{-+} + p_{--} = p'_{++} + p'_{+-} + p'_{-+} + p'_{--}. \tag{24}$$

For a particle with spin s the above construction leads to

$$p_{\pm\pm} = \frac{2s\theta}{2\pi}, \tag{25}$$

$$p_{\pm\mp} = \frac{\pi - 2s\theta}{2\pi}, \tag{26}$$

$$p'_{\pm\pm} = \frac{n}{2 \cdot 2s} + \frac{1}{4s}\sin^2\left(2s^2\theta - \frac{n\pi}{2}\right), \tag{27}$$

$$p'_{\pm\mp} = \frac{2s - 1}{2 \cdot 2s} - \frac{n}{2 \cdot 2s} + \frac{1}{4s}\cos^2\left(2s^2\theta - \frac{n\pi}{2}\right), \tag{28}$$

$$n = [\frac{4s^2\theta}{\pi}], \tag{29}$$

where $[x]$ is the integral part of x. It can be checked that (24) is fulfilled. The two sets of probabilities can be mapped by the bijection

$$p'_{\pm\pm} = f^{-1}(p_{\pm\pm}) \tag{30}$$

$$p'_{\pm\mp} = f^{-1}(p_{\pm\mp}), \tag{31}$$

where

$$f(x) = \frac{[4sx]}{2 \cdot 2s} + \frac{1}{2s \cdot \pi}\sin^{-1}\sqrt{2s \cdot 2x - [4sx]} \tag{32}$$

$$f^{-1}(x) = \frac{[4sx]}{2 \cdot 2s} + \frac{1}{2 \cdot 2s}\sin^2\pi(2s \cdot x - \frac{[4sx]}{2}). \tag{33}$$

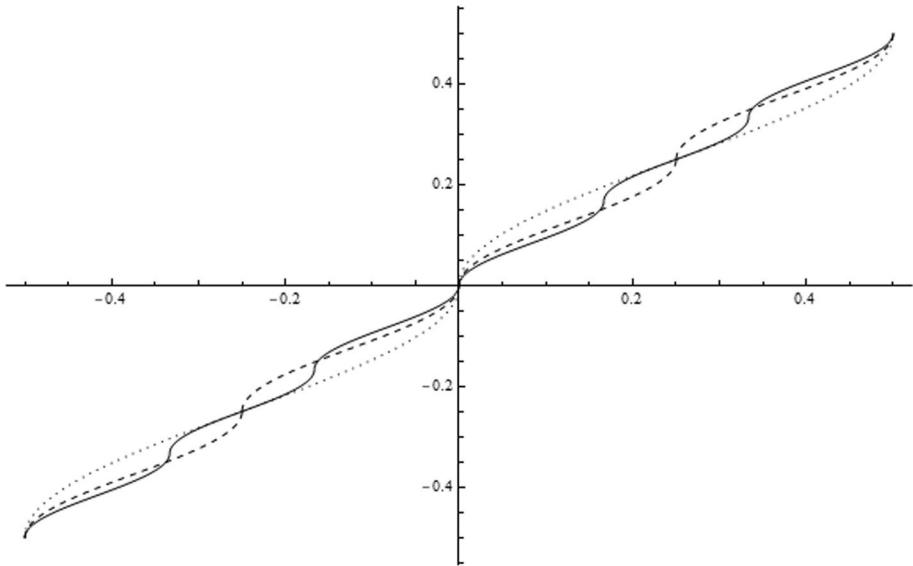Examples of the bijections are illustrated in Figs. 10 and 11

Fig. 10 Function f for different spin numbers. Dotted line presents function f for $s = \frac{1}{2}$, dashed line for $s = 1$, continuous line for $s = \frac{3}{2}$
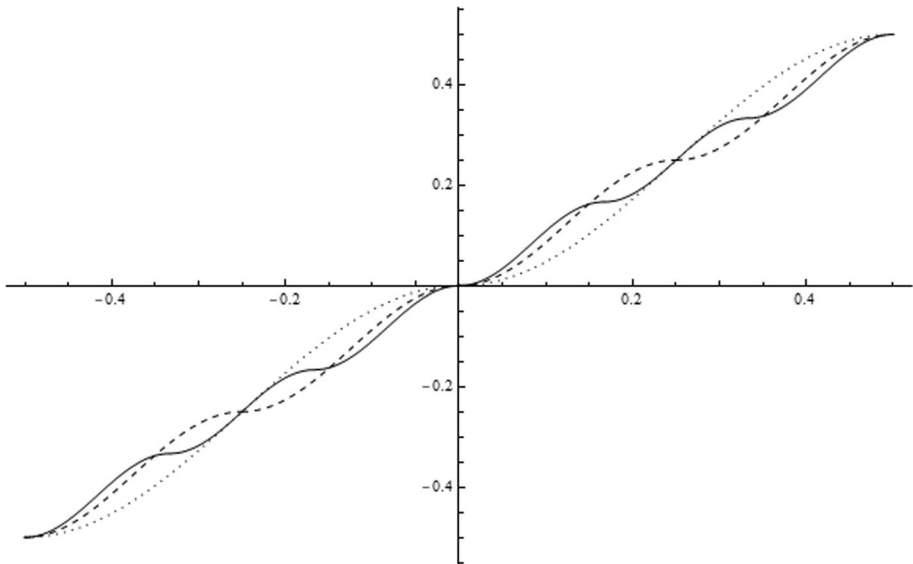

Fig. 11 Function f^{-1} for different spin numbers. Dotted line presents function f for $s = \frac{1}{2}$, dashed line for $s = 1$, continuous line for $s = \frac{3}{2}$

f and f^{-1} are strictly monotonically increasing, so ordering relations are the same in both arithmetics. Functions shown in (32) and (33) have infinitely many fixed points,
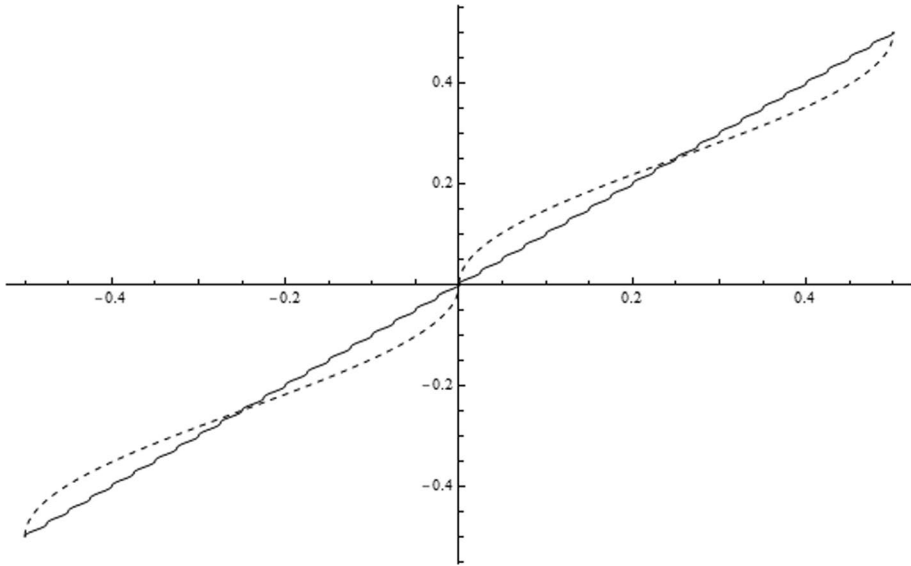
Fig. 12 Comparison of f function for $s = \frac{1}{2}$ (dashed line) and $s = 10$ (continuous line). It is visible, that for large s function f is very similar to $g(x) = x$

$$f\left(\frac{k}{8s}\right) = f^{-1}\left(\frac{k}{8s}\right) = \frac{k}{8s}, \qquad k \in \mathbb{Z}. \tag{34}$$

In particular, $f(n) = f^{-1}(n) = n$ for $n \in \mathbb{N}$, so the change of arithmetic does not influence number theory. For $s \to \infty$ the bijection converges to $f(x) = x$, because

$$
\begin{aligned}
\lim_{s\to\infty} f(x) &= \lim_{s\to\infty} \left(\frac{[4sx]}{2 \cdot 2s} + \frac{1}{2s \cdot \pi} \sin^{-1} \sqrt{2s \cdot 2x - [4sx]} \right) \\
&= \lim_{s\to\infty} \left(\frac{4sx - \{4sx\}}{4s} + \frac{\sin^{-1} \sqrt{4sx - (4sx - \{4sx\})}}{2s\pi} \right) \\
&= \lim_{s\to\infty} \left(x - \frac{\{4sx\}}{4s} + \frac{\sin^{-1} \sqrt{\{4sx\}}}{2s\pi} \right) \\
&= x,
\end{aligned}
\tag{35}
$$

where $\{x\}$ is the fractional part of x.

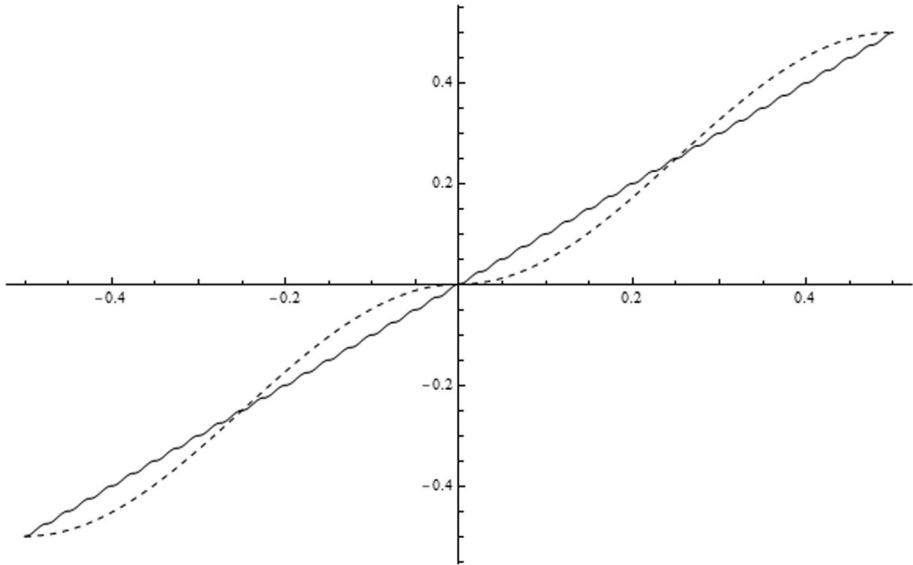Analogously, the limit of the inverse function also converges to $f(x) = x$:

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Fig. 13 Comparison of f^{-1} function for $s = \frac{1}{2}$ (dashed line) and $s = 10$ (continuous line). It is visible, that for large s function f is very similar to $g(x) = x$

$$
\begin{aligned}
\lim_{s \to \infty} f^{-1}(x) &= \frac{[4sx]}{2 \cdot 2s} + \frac{1}{2 \cdot 2s} \sin^2 \pi \left(2s \cdot x - \frac{[4sx]}{2} \right) \\
&= \lim_{s \to \infty} \left(\frac{4sx - \{4sx\}}{4s} + \frac{\sin^2 \pi (2s \cdot x - (\frac{4sx - \{4sx\}}{2}))}{2 \cdot 2s} \right) \\
&= \lim_{s \to \infty} \left(x - \frac{\{4sx\}}{4s} + \frac{\sin^2 \pi \left(\frac{\{4sx\}}{2} \right)}{4s} \right) \\
&= x.
\end{aligned}
\tag{36}
$$

For large spins the non-Newtonian/non-Diophantine probabilities are indistinguishable from the classical ones. It is clearly visible in Figs. 12 and 13 where functions with higher spin parameters are shown. The effect resembles a classical limit of a quantum theory.

6 Clauser-Horne Inequality

The inequality by Clauser and Horne (1974) employs the following lemma (proof in appendix A):

Lemma *If* $0 \leq x, x' \leq X$ *and* $0 \leq y, y' \leq Y$, *then*

$$
-XY \leq x'y - x'y' + xy + xy' - Yx - Xy \leq 0.
\tag{37}
$$

Suppose $p_1(\lambda, \hat{a})$ is the probability that the first electron has spin in direction $\hat{a}$ chosen by Alice, and $p_2(\lambda, \hat{b})$ corresponds to Bob. Lambda is a classical hidden variable. $p_{12}(\lambda, \hat{a}, \hat{b})$ is the joint probability. The locality assumption means that

$$p_{12}(\lambda, \hat{a}, \hat{b}) = p_1(\lambda, \hat{a})p_2(\lambda, \hat{b}). \tag{38}$$

Measured probabilities are then defined by the integrals,

$$p_1(\hat{a}) = \int_\Lambda p_1(\lambda, \hat{a})\rho(\lambda)\,\mathrm{d}\lambda, \tag{39}$$

$$p_2(\hat{b}) = \int_\Lambda p_2(\lambda, \hat{b})\rho(\lambda)\,\mathrm{d}\lambda, \tag{40}$$

$$p_{12}(\hat{a}, \hat{b}) = \int_\Lambda p_{12}(\lambda, \hat{a}, \hat{b})\rho(\lambda)\,\mathrm{d}\lambda. \tag{41}$$

Using the inequality (37) and the definition of locality (38), it can be rewritten as:

$$-1 \le p_{12}(\lambda, \hat{a}', \hat{b}) - p_{12}(\lambda, \hat{a}', \hat{b}') + p_{12}(\lambda, \hat{a}, \hat{b}) + p_{12}(\lambda, \hat{a}, \hat{b}') - p_1(\lambda, \hat{a}) - p_2(\lambda, \hat{b}) \le 0. \tag{42}$$

Multiplying by $\rho(\lambda)$ and integrating over the space Λ of the hidden variables, we get

$$-1 \le p_{12}(\hat{a}', \hat{b}) - p_{12}(\hat{a}', \hat{b}') + p_{12}(\hat{a}, \hat{b}) + p_{12}(\hat{a}, \hat{b}') - p_1(\hat{a}) - p_2(\hat{b}) \le 0. \tag{43}$$

Assuming rotational invariance, we conclude that $p_1(\hat{a})$ i $p_2(\hat{b})$ are constants, and that $p_{12}(\hat{a}, \hat{b}) = p_{12}(\varphi)$, where φ is an angle between directions $\hat{a}$ i $\hat{b}$. The inequality can be rewritten as

$$-1 \le 3p_{12}(\varphi) - p_{12}(3\varphi) - p_1(\hat{a}) - p_2(\hat{b}) \le 0, \tag{44}$$

where the directions $\hat{a}, \hat{a}', \hat{b}$ and $\hat{b}'$ are chosen so that

$$|\hat{a} - \hat{b}| = |\hat{a}' - \hat{b}| = |\hat{a} - \hat{b}'| = \frac{1}{3}|\hat{a}' - \hat{b}'| = \varphi. \tag{45}$$

By symmetry,

$$p_1(\hat{a}) = p_2(\hat{b}) = \frac{1}{2} \tag{46}$$

and finally

$$0 \le 3p_{12}(\varphi) - p_{12}(3\varphi) \le 1. \tag{47}$$

This derivation can be similarly performed in non-Newtonian calculus. Replacing in (39)–(41) products, characteristic functions, and integrals by their non-Newtonian analogues, one arrives at Czachor (2021)

$$0 \le 3 \odot p'_{+-}(\theta) \ominus p'_{+-}(3\theta) \le 1. \tag{48}$$

Expression in the middle can be reformulated as

$$3 \odot p'_{+-}(\theta) \ominus p'_{+-}(3\theta) = f^{-1}[f(3)f(p'_{+-}(\theta)) - p'_{+-}(3\theta)]$$
$$= f^{-1}[3\frac{\pi - 2s\theta}{2\pi} - \frac{\pi - 6s\theta}{2\pi}] = f^{-1}(1) = 1. \tag{49}$$

The inequality is not violated because it is expressed in terms of the correct arithmetic, namely the one which makes non-Newtonian integrals linear. However, if one employs the "wrong" arithmetic, then one arrives at

$$0 \le 3p'_{+-}(\theta) - p'_{+-}(3\theta) \le 1 \tag{50}$$

which is not satisfied, of course (because it was derived according to the rules that are not valid in the non-Newtonian formalism). Alternatively, the standard CH inequality can be written as

$$0 \le 3f^{-1}\left(\frac{\pi - \theta}{2\pi}\right) - f^{-1}\left(\frac{\pi - 3\theta}{2\pi}\right) \le 1, \tag{51}$$

which is true only for $f(x) = x$. Substituting $\theta = \frac{\pi}{16s^2}$, we get

$$3p'_{+-}(\theta) - p'_{+-}(3\theta) = 3f^{-1}\left(\frac{\pi - \theta}{2\pi}\right) - f^{-1}\left(\frac{\pi - 3\theta}{2\pi}\right)$$
$$= \frac{3(2s - 1)}{4s} + \frac{3}{4s}\cos^2\left(\frac{\pi}{8}\right) - \frac{2s - 1}{4s} - \frac{1}{4s}\cos^2\left(\frac{3\pi}{8}\right)$$
$$= \frac{2(2s - 1)}{4s} + \frac{3}{4s}\frac{2 + \sqrt{2}}{4} - \frac{1}{4s}\frac{2 - \sqrt{2}}{4} \tag{52}$$
$$= 1 + \frac{1 + 2\sqrt{2}}{2s} \ge 1.$$

The inequality is violated for any finite s, but the degree of violation decreases with growing s.

7 Conclusions

We have generalized the construction from Czachor (2021) in two ways. First of all, conditional and joint probabilities typical of a single spin-1/2 or two-spin-1/2 singlet states have been generalized in a local hidden-variable way to arbitrary spins. Violation of the Clauser-Horne inequality was found for any s, but the degree of violation was decreasing with growing s. The latter is consistent with the intuition that larger spins are "more classical" than the small ones. As a by-product, an arithmetic and calculus have been constructed which reconstruct the standard formalism in the limit $s \to \infty$, which supports the idea that arithmetic is as physical as geometry. As stressed in Czachor and Nalikowski (2021), what we call a violation of Bell's inequality looks like a confusion of languages problem, where arithmetic rules applicable to macroscopic systems are incorrectly applied to the microscopic ones (quantum, or subquantum). The example of dark energy (Czachor, 2021) shows that the same may be true if one thinks of cosmological scales.

The examples we have discussed are of a toy-model type so cannot be treated as alternatives to the formalism of quantum mechanics. However, what they show is that one has to

be very cautious while formulating any claims about the nonexistence of Einstein-Podolsky-Rosen elements of reality just on the basis of statistics of correlations.

Appendix

A Lemma used in CH inequality

We will prove a theorem Clauser and Horne (1974).

Theorem *If* $0 \leq x, x' \leq X$ *i* $0 \leq y, y' \leq Y$, *then*

$$-XY \leq W \leq 0, \quad W = x'y - x'y' + xy + xy' - Yx - Xy. \tag{53}$$

Proof We will firstly focus on the right inequality. When we suppose that $x \leq x'$ the inequality can be rewritten as

$$W = y(x' - X) + x(y - Y) + y'(x - x') \leq 0, \tag{54}$$

which is true, because every expression in brackets is negative. If $x \geq x'$ then it can be modified as

$$\begin{aligned} W &= x'(y - y') + y(x - X) + x(y' - Y) \\ &\leq x'(y - y') + y(x - X) + x'(y' - Y) = y(x - X) - x'(y - Y) \\ &\leq 0. \end{aligned} \tag{55}$$

Now, we prove the left inequality. Supposing $x' \leq x$ the expression can be rewritten as

$$W + XY = (X - x)(Y - y) + x'y + y'(x - x'). \tag{56}$$

All parts of this expression are positive, so $W + XY \geq 0$. Similarly, if $y' \leq y$ then it can be rewritten as

$$W + XY = (X - x)(Y - y) + xy' + x'(y - y') \tag{57}$$

and following the previous reasoning it is clear that $W + XY \geq 0$. The last case, which has to be checked, is when $x \leq x'$ i $y \leq y'$. Then the expression $W + XY$ can be rewritten as

$$W + XY = (X - x)(Y - y) - (x' - x)(y' - y) + xy. \tag{58}$$

Sum of the two first expressions is non-negative, because

$$(X - x) \geq (x' - x) \geq 0 \tag{59}$$

and

$$(Y - y) \geq (y' - y) \geq 0. \tag{60}$$

After considering every possibility, we conclude that the theorem is ☐

Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

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