Ellipticity in couple-stress elasticity

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Abstract. We discuss ellipticity property within the linear couple-stress elasticity. In this theory, there exists a deformation energy density introduced as a function of strains and gradient of macrorotations, where the latter are expressed through displacements. So the couple-stress theory could be treated as a particular class of strain gradient elasticity. Within the micropolar elasticity, the model is called Cosserat pseudocontinuum or medium with constrained rotations. Applying the classic definitions of ordinary ellipticity and strong ellipticity to static equations of the couple-stress theory, we conclude that these equations are neither elliptic nor strongly elliptic. As a result, one should be aware of extending properties of full strain gradient models such as Toupin–Mindlin strain gradient elasticity to models with incomplete set of second derivatives.

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1. Introduction

Elliptic systems of partial differential equations (PDEs) constitute a wide class of the models of mathematical physics. In the literature, one can find various definitions of ellipticity which extend the property of the Poisson equation. Among the latter, it is worth mentioning strong ellipticity, ellipticity known also as ordinary or Petrovskiy's ellipticty, Douglis–Nirenberg ellipticity and some others, see, for example, [1, 22, 46]. Ellipticity brings us such properties as a regularity (smoothness) of solutions and well-posedness of the corresponding boundary value problems. On the other hand, violation of ellipticity may lead to proper description of such physical phenomena as strain localization, material instabilities, shear bands formation, softening, wave propagation, see, for example, [7, 9, 10, 12, 27, 30, 31, 37, 42, 43].

The aim of this note is to discuss ellipticity within the linear couple-stress theory [28, 36, 44]. The latter could be treated as a Cosserat continuum with constrained rotations, so-called pseudocontinuum Cosserat [19, 21, 38] or a particular class of strain gradient models [5, 6, 11, 34, 35, 45]. Within nonlinear Cosserat continuum strong ellipticity was studied in [2, 13], see also [30], whereas relation of ellipticity loss on material stability within nonlinear strain gradient elasticity was analysed in [14]. Let us note that nowadays the couple-stress theory is widely used for modelling of composite materials and structures at small scales, see, for example, [3, 23, 26, 29, 39, 41] and the reference therein.

The paper is organized as follows. First, in Sect. 2 we recall the definitions of ellipticity and strong ellipticity for linear systems of PDEs. In Sect. 3, we introduce the equilibrium equations of the couple-stress theory including the modified one [48]. Section 4 is devoted to the analysis of ellipticity of these equations. Finally, we compare the results with ellipticity properties of linear micropolar and Toupin–Mindlin strain gradient elasticity.

Note that in the following we use the index-free tensor calculus as in [16, 40, 47].

2. Mathematical preliminaries

Following [1, 22, 46], let us consider a system of linear differential equations of order m

$$\sum_{|\alpha| \leq m} A_{ij}^{(\alpha)} D^\alpha u_j = f_i, \quad i, j = 1, 2 \ldots n, \tag{2.1}$$

where $\mathbb{A}^{(\alpha)} \equiv \{A_{ij}^{(\alpha)}\}$ are $n \times n$ matrices, $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a multiindex, α_k are natural numbers, $|\alpha| = \alpha_1 + \cdots + \alpha_l$, $\mathbf{u} = (u_1, \ldots, u_n)$ is a vector of unknown functions, $u_k = u_k(x_1, \ldots, x_l)$, x_i are Cartesian coordinates, $k = 1, \ldots, n$, and $\mathbf{f} = (f_1, \ldots, f_n)$ is a vector of given functions, for example, of external loads. Differential operator D^α is defined by the formulae

$$D^\alpha = \partial_1^{\alpha_1} \ldots \partial_l^{\alpha_l}, \quad \partial_p = \frac{\partial}{\partial x_p}, \quad p = 1, \ldots l.$$

In addition, hereinafter Einstein's summation rule over repeated indices is applied.

System (2.1) defines the differential operator $\mathcal{A}$ given by

$$\mathcal{A} = \sum_{|\alpha| \leq m} \mathbb{A}^{(\alpha)} D^\alpha. \tag{2.2}$$

Motivated by strain gradient elasticity applications in the following, we consider $n = 3$, $l = 3$ and $m = 4$. For simplicity, we also assume that $\mathbb{A}^{(\alpha)}$, $|\alpha| = m$, does not depend on x_i.

Following [1, 22, 46], we call system (2.1) or operator $\mathcal{A}$ *elliptic* or *Petrovskiy's elliptic* or *ordinary elliptic* if

$$\det \mathbb{A}_0(\mathbf{k}) \neq 0, \quad \forall \mathbf{k} \neq \mathbf{0}, \quad \mathbf{k} = (k_1, \ldots, k_l), \tag{2.3}$$

where $\mathbb{A}_0(\mathbf{k})$ is the principal symbol given by the formula

$$\mathbb{A}_0(\mathbf{k}) = \sum_{|\alpha| = m} \mathbb{A}^{(\alpha)} \mathbf{k}^\alpha, \quad \mathbf{k}^\alpha = k_1^{\alpha_1} \ldots k_l^{\alpha_l}. \tag{2.4}$$

Note that operator $\mathcal{A}$ could be represented symbolically through the polynomial $\mathbb{A}(\mathbf{k}) = \sum_{|\alpha| \leq m} \mathbb{A}^{(\alpha)} \mathbf{k}^\alpha$ called symbol of $\mathcal{A}$. Symbolically $\mathbb{A}(\mathbf{k})$ could be obtained by formal replacement $\partial_p \to k_p$. So $\mathbb{A}_0(\mathbf{k})$ is a homogeneous polynomial of degree m in $\mathbf{k}$.

We call (2.1) or $\mathcal{A}$ *strongly elliptic* if there is an inequality

$$(\mathbb{A}_0(\mathbf{k})\mathbf{a}, \mathbf{a}) \equiv \sum_{|\alpha| = m} A_{ij}^{(\alpha)} \mathbf{k}^\alpha a_i a_j \geq C \|\mathbf{k}\|^m \|\mathbf{a}\|^2 \tag{2.5}$$

for any vector $\mathbf{k}$ and any vector $\mathbf{a} = (a_1, \ldots, a_n)$, where C is a positive constant independent on $\mathbf{k}$ and $\mathbf{a}$ and

$$\|\mathbf{k}\|^2 = k_1^2 + \cdots + k_l^2, \quad \|\mathbf{a}\|^2 = a_1^2 + \cdots + a_n^2.$$

Obviously, strong ellipticity is more restrictive than ordinary ellipticity. Indeed, Eq. (2.5) means that matrix $\mathbb{A}_0(\mathbf{k})$ is positive definite, whereas (2.3) requires that $\mathbb{A}_0(\mathbf{k})$ does not have zero eigenvalues for any $\mathbf{k}$. So, strong ellipticity implies ellipticity. Let us note that the positive definiteness could be replaced by negative definiteness requirement since in this case one can get the positive definiteness by multiplication of (2.1) by -1, see [1, 46] for more details.

3. Couple-stress theory

In what follows, we restrict ourselves to small deformations. So let $\mathbf{u} = \mathbf{u}(\mathbf{x})$ be a vector of displacements and $\mathbf{x}$ be a position vector. We introduce the linear strain tensor ε and rotation vector ϕ as follows [36]

$$\varepsilon = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right), \quad \phi = \frac{1}{2} \nabla \times \mathbf{u}, \tag{3.1}$$

where ∇ is the 3D nabla-operator, the superscript T means transpose and $\times$ denotes the cross product.

For a hyper-elastic solid there exists a deformation energy density W introduced as a function of strains and gradient of rotations [36]

$$W = W(\varepsilon, \kappa), \quad \kappa = \nabla \phi. \tag{3.2}$$

So we have two stress measures, i.e. the classic symmetric stress tensor σ and the non-symmetric couple-stress tensor μ both defined as follows

$$\sigma = \frac{\partial W}{\partial \varepsilon}, \quad \mu = \frac{\partial W}{\partial \kappa}.$$

Note that as ϕ is introduced through Eq. $(3.1)_2$, there is no difference between ε and the strain tensor $\mathbf{e} = \nabla \mathbf{u} + \phi \times \mathbf{I}$ usually used in the linear micropolar elasticity [16, 19, 21]. Hereinafter $\mathbf{I}$ is the 3D unit tensor.

For an isotropic material, W takes the form [36]

$$W = \frac{1}{2} \lambda \text{tr}^2 \varepsilon + \mu \varepsilon : \varepsilon + 2\eta \kappa : \kappa + 2\zeta \kappa : \kappa^T, \tag{3.3}$$

where λ, μ, η and ζ are elastic moduli, and : stands for the double dot product [16, 19, 21]. As a result, σ and μ become

$$\sigma = \lambda \mathbf{I} \text{tr}\, \varepsilon + 2\mu \varepsilon, \quad \mu = 4\eta \kappa + 4\zeta \kappa^T. \tag{3.4}$$

Positive definiteness of W leads to inequalities [36]

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \eta > 0, \quad -\eta < \zeta < \eta. \tag{3.5}$$

Equilibrium equations have the following form [36]

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \eta \Delta \nabla \times (\nabla \times \mathbf{u}) + \mathbf{f} = \mathbf{0}. \tag{3.6}$$

where $\Delta = \nabla \cdot \nabla$ is the Laplace operator and $\mathbf{f}$ is a vector of external loads. Note that ζ is not included into (3.6), it appears in static boundary conditions. Equation (3.6) is a particular form of general equations given also in [34, 35, 44].

Yang et al. [48] proposed to consider another symmetric strain measure χ given by

$$\chi = \frac{1}{2} \left(\nabla \phi + \nabla \phi^T \right)$$

with the constitutive relation in the form

$$W_m = \frac{1}{2} \lambda \text{tr}^2 \varepsilon + \mu \varepsilon : \varepsilon + 2\eta \chi : \chi \tag{3.7}$$

and symmetric couple-stress tensor $\mathbf{m} = 4\eta \chi$. Equation (3.7) also results in (3.6).

4. Ellipticity

Calculation of the principal symbol of Eq. (3.6) is straightforward. It is determined by the highest order term of (3.6), that is, $\eta\Delta\nabla\times(\nabla\times\mathbf{u})$. Using formal substitution $\nabla\to\mathbf{k}$, we come to the formula

$$\mathbb{A}_0(\mathbf{k}) = \eta(\mathbf{k}\cdot\mathbf{k})\mathbf{k}\times(\mathbf{k}\times\mathbf{I}). \tag{4.1}$$

The determinant of $\mathbb{A}_0(\mathbf{k})$ is equal to zero: $\det\mathbb{A}_0(\mathbf{k}) = 0$. Indeed, we have

$$\det\mathbb{A}_0(\mathbf{k}) = \det\left[\eta(\mathbf{k}\cdot\mathbf{k})\mathbf{k}\times(\mathbf{k}\times\mathbf{I})\right] = \eta^3(\mathbf{k}\cdot\mathbf{k})^3\det\left[\mathbf{k}\times(\mathbf{k}\times\mathbf{I})\right] = \eta^3(\mathbf{k}\cdot\mathbf{k})^3\det\left[\mathbf{k}\times\mathbf{I}\times\mathbf{k}\right].$$

Using the identity

$$\mathbf{i}\times\mathbf{I}\times\mathbf{i} = \mathbf{i}\otimes\mathbf{i}-\mathbf{I},$$

valid for any unit vector $\mathbf{i}$, see, for example, [16, p. 104], we came to

$$\det\mathbb{A}_0(\mathbf{k}) = \eta^3(\mathbf{k}\cdot\mathbf{k})^6\det\left[\mathbf{i}\otimes\mathbf{i}-\mathbf{I}\right] = 0, \quad \mathbf{i} = \mathbf{k}/\|\mathbf{k}\|,$$

as $\mathbf{i}\otimes\mathbf{i}-\mathbf{I}$ is a singular tensor. Hereinafter $\otimes$ is the dyadic product.

Thus, despite positive definiteness of W equilibrium equations does not constitute an elliptic system. Since ordinary ellipticity is a necessary condition of the strong ellipticity, the latter is also violated. This statement could be also proved using (2.5). Indeed, we have that

$$\mathbf{a}\cdot\mathbb{A}_0(\mathbf{k})\cdot\mathbf{a} = \eta(\mathbf{k}\cdot\mathbf{k})\left[(\mathbf{a}\cdot\mathbf{k})^2-(\mathbf{k}\cdot\mathbf{k})(\mathbf{a}\cdot\mathbf{a})\right],$$

which is zero if $\mathbf{a}$ is collinear to $\mathbf{k}$.

5. Comparison with strain gradient and micropolar elasticity

5.1. Toupin–Mindlin strain gradient elasticity

Within the Toupin–Mindlin strain gradient elasticity, the deformation energy W_{TM} depends on strains $\boldsymbol{\varepsilon}$ and on its gradient (or on $\boldsymbol{\varepsilon}$ and $\nabla\nabla\mathbf{u}$) [34,35]. For an isotropic solid, W_{TM} takes the form

$$W_{\mathrm{TM}} = \frac{1}{2}\boldsymbol{\varepsilon}:\mathbf{C}:\boldsymbol{\varepsilon} + +\frac{1}{2}\nabla\boldsymbol{\varepsilon}\vdots\mathbf{D}\vdots\nabla\boldsymbol{\varepsilon}, \tag{5.1}$$

where ":" and "$\vdots$" are the double and triple dot products, respectively, and fourth- and sixth-order tensors of elastic moduli $\mathbf{C}$ and $\mathbf{D}$ have components

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{5.2}$$

$$\begin{aligned}
\mathbb{D}_{ijmkln} &= \frac{a_1}{2}\left(\delta_{ij}\delta_{km}\delta_{ln}+\delta_{ij}\delta_{kn}\delta_{lmn}+\delta_{kl}\delta_{im}\delta_{jn}+\delta_{kl}\delta_{in}\delta_{jm}\right)+2a_2\delta_{ij}\delta_{kl}\delta_{mn}\\
&\quad+\frac{a_3}{2}\left(\delta_{jk}\delta_{im}\delta_{ln}+\delta_{ik}\delta_{jm}\delta_{ln}+\delta_{il}\delta_{jm}\delta_{kn}+\delta_{jl}\delta_{im}\delta_{kn}\right)\\
&\quad+a_4\left(\delta_{il}\delta_{jk}\delta_{mn}+\delta_{il}\delta_{jk}\delta_{mn}\right)\\
&\quad+\frac{a_5}{2}\left(\delta_{jk}\delta_{in}\delta_{lm}+\delta_{ik}\delta_{jn}\delta_{lm}+\delta_{jl}\delta_{km}\delta_{in}+\delta_{il}\delta_{km}\delta_{jn}\right),
\end{aligned} \tag{5.3}$$

where δ_{ij} is Kronecker's symbol, and a_1, a_2, a_3, a_4 and a_5 are five elastic moduli of higher order [34].

In this case, the equilibrium equations take the form

$$(\lambda+2\mu)\nabla\nabla\cdot\mathbf{u}-\mu\nabla\times(\nabla\times\mathbf{u})-\gamma_1\Delta\nabla\nabla\cdot\mathbf{u}+\gamma_2\Delta\nabla\times(\nabla\times\mathbf{u})+\mathbf{f} = \mathbf{0}, \tag{5.4}$$

where $\gamma_1 = (\lambda+2\mu)\ell_1^2$, $\gamma_2 = \mu\ell_2^2$, and ℓ_1, ℓ_2 are characteristic lengths given by

$$\ell_1 = \sqrt{\frac{a_1+a_2+a_3+a_4+a_5}{\lambda+2\mu}}, \quad \ell_2 = \sqrt{\frac{a_3+2a_4+a_5}{2\mu}}, \tag{5.5}$$

see [32,34,35] for more details.

Here the principal symbol takes the value

$$\mathbb{A}_{\mathrm{TM}}(\mathbf{k}) = -\gamma_1(\mathbf{k} \cdot \mathbf{k})\mathbf{k} \otimes \mathbf{k} + \gamma_2(\mathbf{k} \cdot \mathbf{k})\mathbf{k} \times (\mathbf{k} \times \mathbf{I}). \tag{5.6}$$

It could be transformed into the more simple form

$$\mathbb{A}_{\mathrm{TM}}(\mathbf{k}) = -(\mathbf{k} \cdot \mathbf{k})^2 \left[\gamma_1 \mathbf{i} \otimes \mathbf{i} + \gamma_2(\mathbf{i}_2 \otimes \mathbf{i}_2 + \mathbf{i}_3 \otimes \mathbf{i}_3) \right], \tag{5.7}$$

where $\mathbf{i}_2$ and $\mathbf{i}_3$ are two unit vectors orthogonal to $\mathbf{i} \equiv \mathbf{k}/\|\mathbf{k}\|$.

Since

$$\det \mathbb{A}_{\mathrm{TM}}(\mathbf{k}) = -(\mathbf{k} \cdot \mathbf{k})^6 \gamma_1 \gamma_2,$$

the ordinary ellipticity is fulfilled if and only if $\gamma_1 \gamma_2 \neq 0$ or the two following inequalities are fulfilled

$$a_1 + a_2 + a_3 + a_4 + a_5 \neq 0, \quad a_3 + 2a_4 + a_5 \neq 0. \tag{5.8}$$

In other words, ordinary ellipticity requires that both characteristic lengths are nonzero.

Similarly, we get that

$$-\mathbf{a} \cdot \mathbb{A}_{\mathrm{TM}}(\mathbf{k}) \cdot \mathbf{a} = (\mathbf{k} \cdot \mathbf{k}) \left[\gamma_1(\mathbf{a} \cdot \mathbf{k})^2 + \gamma_2 \left((\mathbf{k} \cdot \mathbf{k})(\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{k})^2 \right) \right]$$

So strong ellipticity condition (2.5) is satisfied if and only if $\gamma_1 > 0$ and $\gamma_2 > 0$ that is equivalent to [18]

$$a_1 + a_2 + a_3 + a_4 + a_5 > 0, \quad a_3 + 2a_4 + a_5 > 0. \tag{5.9}$$

Comparing equations (3.6) and (5.4), one can easily observe that (5.4) transforms into (3.6) if one assume that $\gamma_2 = \eta$ while $\gamma_1 = 0$. So the lack of ellipticity of (3.6) follows from the violation of the positivity of one characteristic length, that is, $\ell_1 = 0$, for the couple-stress theory.

Similar situation with ellipticity can be observed in another incomplete strain gradient model called dilatational strain gradient elasticity [15,33]. Here a deformation energy is given by

$$W_{\mathrm{dsg}} = \frac{\lambda}{2}\mathrm{tr}^2\boldsymbol{\varepsilon} + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \frac{\alpha}{2}\mathbf{k} \cdot \mathbf{k}, \tag{5.10}$$

where λ and μ are Lamé moduli and α is an additional dilatational elastic modulus. Corresponding equilibrium equation is given by

$$\mu\Delta\mathbf{u} + (\mu + \lambda)\nabla\nabla \cdot \mathbf{u} - \alpha\nabla\Delta(\nabla \cdot \mathbf{u}) + \mathbf{f} = \mathbf{0}. \tag{5.11}$$

Its principal symbol is given by the formula

$$\mathbb{A}_{\mathrm{dsg}}(\mathbf{k}) = -\alpha(\mathbf{k} \cdot \mathbf{k})\mathbf{k}\mathbf{k}.$$

Obviously, $\mathbb{A}_{\mathrm{dsg}}(\mathbf{k})$ is singular, and so, (5.11) is not elliptic. Comparing (5.11) with (5.4), one can see that here $\ell_2 = 0$.

5.2. Cosserat continuum

Within the linear micropolar elasticity translations $\mathbf{u}$ and rotations $\boldsymbol{\varphi}$ are kinematically independent and $\boldsymbol{\varphi} \neq \boldsymbol{\phi}$, in general. A deformation energy W_{me} is a function of two strain measures defined as follows [19,21]

$$\mathbf{e} = \nabla\mathbf{u} + \mathbf{I} \times \boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \nabla\boldsymbol{\varphi}. \tag{5.12}$$

For an isotropic material, W_{me} is given by

$$W_{\mathrm{me}} = \frac{1}{2}\lambda\mathrm{tr}^2\mathbf{e} + (\mu + \kappa)\mathbf{e} : \mathbf{e} + \mu\mathbf{e} : \mathbf{e}^T + \frac{1}{2}\eta_1\mathrm{tr}^2\boldsymbol{\kappa} + \frac{1}{2}\eta_2\boldsymbol{\kappa} : \boldsymbol{\kappa} + \frac{1}{2}\eta_3\boldsymbol{\kappa} : \boldsymbol{\kappa}^T, \tag{5.13}$$

where λ, μ, κ, η_1, η_2 and η_3 are elastic moduli. Equilibrium equations take the form

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + (\mu + \kappa)\Delta\mathbf{u} + \kappa\nabla \times \boldsymbol{\varphi} + \mathbf{f} = \mathbf{0}, \tag{5.14}$$

$$(\eta_1 + \eta_2)\nabla\nabla \cdot \boldsymbol{\varphi} + \eta_2\Delta\boldsymbol{\varphi} + \kappa(\nabla \times \mathbf{u} - 2\boldsymbol{\varphi}) + \mathbf{c} = \mathbf{0}, \tag{5.15}$$

where $\mathbf{f}$ and $\mathbf{c}$ are vectors of volumetric forces and couples, respectively.

The principal symbol of system (5.14) and (5.15) has the form

$$\mathbb{A}_{\mathrm{me}}(\mathbf{k}) = \begin{pmatrix} \mathbb{A}_{\mathrm{me}}^{(1)}(\mathbf{k}) & \mathbf{0} \\ \mathbf{0} & \mathbb{A}_{\mathrm{me}}^{(2)}(\mathbf{k}) \end{pmatrix},$$

$$\mathbb{A}_{\mathrm{me}}^{(1)}(\mathbf{k}) = (\lambda + \mu)\mathbf{k} \otimes \mathbf{k} + (\mu + \kappa)(\mathbf{k} \cdot \mathbf{k})\mathbf{I}, \quad \mathbb{A}_{\mathrm{me}}^{(2)}(\mathbf{k}) = (\eta_1 + \eta_3)\mathbf{k} \otimes \mathbf{k} + \eta_2(\mathbf{k} \cdot \mathbf{k})\mathbf{I}. \tag{5.16}$$

So ellipticity condition reduces to the inequality

$$\det \mathbb{A}_{\mathrm{me}}^{(1)}(\mathbf{k}) \det \mathbb{A}_{\mathrm{me}}^{(2)}(\mathbf{k}) \neq 0$$

that results in the following inequalities for the elastic moduli

$$\lambda + 2\mu + \kappa \neq 0, \quad \mu + \kappa \neq 0, \quad \eta_1 + \eta_2 + \eta_3 \neq 0, \quad \eta_3 \neq 0.$$

The strong ellipticity conditions have the form [2,13,19]

$$\lambda + 2\mu + \kappa > 0, \quad \mu + \kappa > 0, \quad \eta_1 + \eta_2 + \eta_3 > 0, \quad \eta_2 > 0. \tag{5.17}$$

Comparing W_{me} and W, one can see that they are coincide to each other if $\kappa = 0$, $\eta_1 = 0$, $\eta_2 = \eta$, $\eta_3 = \zeta$. Even in this case, (5.14) and (5.15) are strongly elliptic if and only if

$$\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta_2 > 0,$$

whereas (3.6) is not elliptic. So despite the obvious similarity between micropolar elasticity and couple-stress theory, the corresponding ellipticity conditions are different.

6. Conclusions and discussion

Within the linear couple-stress theory, we discussed the ellipticity and strong ellipticity conditions. It was shown that both conditions are violated. So equilibrium equation does not constitute nor strongly elliptic neither elliptic system of PDEs. This relates to a certain degeneration of a deformation energy within the Toupin–Mindlin strain gradient elasticity. Nevertheless, in order to get advantages of ellipticity one can consider more general conditions of ellipticity or transform somehow the problem under consideration. For example, let us consider the following transformation of (3.6). Let us represent displacement using the Helmholtz decomposition

$$\mathbf{u} = \nabla\psi - \nabla \times \boldsymbol{\Psi}, \quad \nabla \cdot \boldsymbol{\Psi} = 0, \tag{6.1}$$

where ψ and $\boldsymbol{\Psi}$ are potential of dilatation and rotation, respectively. Similarly we represent $\mathbf{f}$ as a sum

$$\mathbf{f} = \nabla g - \nabla \times \mathbf{F}, \quad \nabla \cdot \mathbf{F} = 0. \tag{6.2}$$

Upon substituting (6.1) and (6.2) into (3.6), we obtain

$$\mu\Delta\nabla\psi + (\lambda + \mu)\nabla\nabla \cdot \nabla\psi + \nabla g - \mu\Delta\nabla \times \boldsymbol{\Psi} - \eta\Delta\nabla \times (\nabla \times (\nabla \times \boldsymbol{\Psi})) - \nabla \times \mathbf{F}$$

$$= \nabla\left[(\lambda + 2\mu)\Delta\psi + g\right] + \nabla \times \left[-\mu\Delta\boldsymbol{\Psi} + \eta\Delta^2\boldsymbol{\Psi} - \mathbf{F}\right] = \mathbf{0}.$$

Thus, the problem under consideration can be reduced to the two equations

$$(\lambda + 2\mu)\Delta\psi + g = 0, \quad \eta\Delta^2\boldsymbol{\Psi} - \mu\Delta\boldsymbol{\Psi} - \mathbf{F} = \mathbf{0};$$

both of them are elliptic if and only if $\lambda + 2\mu \neq 0$ and $\eta \neq 0$, and strongly elliptic if $\lambda + 2\mu > 0$ and $\eta > 0$. Let us note that these inequalities constitute a part of ellipticity conditions for a basic simple material (without gradients of strain) and a general strain gradient elastic material.

So one should be aware to transmit ellipticity properties from a general model to a reduced one. For example, similarities between micropolar elasticity and couple-stress theory are obvious. Nevertheless, the ellipticity conditions are different since for micropolar elasticity equilibrium equations constitute a system of six scalar differential equations of second order, whereas within the couple-stress theory we have three differential equations of fourth order. Couple-stress theory could be also derived as a Cosserat continuum with constraint $\phi = \varphi$ [19, 38] that brings another mathematical form of equilibrium conditions, see also [4] for Lagrange multiplier technique to strain gradient and media with microstructure. On the other hand, even reduced model may become elliptic after certain transformations as was shown above.

In the literature, one can also find another gradient incomplete models [11], which are also result in non-elliptic but hypoelliptic differential operators [17]. Let us also note that for nonlinear models relations between ellipticity properties of different models could be less straightforward. For example, similar to micropolar elasticity and couple-stress theory, for a micromorphic medium and strain gradient material the ellipticity conditions are different, in general, but these models can demonstrate some similarities in behaviour, see [20]. The provided comparison could be also extended to more complex models of continua such as discussed in [8, 24, 25].

Acknowledgements

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