On well-posedness of the first boundary-value problem within linear isotropic Toupin–Mindlin strain gradient elasticity and constraints for elastic moduli

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Within the linear Toupin–Mindlin strain gradient elasticity we discuss the well-posedness of the first boundary-value problem, that is, a boundary-value problem with Dirichlet-type boundary conditions on the whole boundary. For an isotropic material we formulate the necessary and sufficient conditions which guarantee existence and uniqueness of a weak solution. These conditions include strong ellipticity written in terms of higher-order elastic moduli and two inequalities for the Lamé moduli. The conditions are less restrictive than those followed from the positive definiteness of the deformation energy.

1 INTRODUCTION

Last few decades the interest to generalized continua models grows as a result of significant extension of applications of continuum and structural mechanics towards small scales [1–3] and new composite materials [4–6], see Maugin's comments on generalized continua [7, 8] and proceedings [9–11]. Among such models it is worth to mention the strain gradient elasticity. Within this model there exists a deformation energy as a function of deformation gradient of the first- and higher-orders. This model could be classified as a weak nonlocal model of continuum [8]. From the physical point of view, the strain gradient elasticity may describe long-range interaction, that is, interactions not only with close neighbors but also with other neighbors. Among strain gradient models the Toupin–Mindlin approach [12–15] is the most general at least from the point of view of material symmetry.

The Toupin–Mindlin strain gradient elasticity results in a linear boundary-value problem (BVP) for a system of partial differential equations (PDEs) of fourth-order. Its well-posedness, that is, existence and uniqueness of solutions, could be studied within general theory of PDEs, see for example [16–19] and [20, 21] for the particular case of strain gradient elasticity. In such an analysis the positive definiteness of the deformation energy plays a crucial role. Nevertheless, for the boundary-value problem with Dirichlet-type boundary conditions, this requirement could be relaxed. For example, in classic linear elasticity the strong ellipticity (SE) conditions are enough for the well-posedness of the first BVP. Within the strain gradient elasticity the strong ellipticity and infinitesimal stability, that is, uniqueness of solutions of a linearized problem, was discussed in [22]. A comparison of micromorphic and strain gradient continua through SE conditions was performed in [23]. For an isotropic solid the SE conditions were formulated in [24] in terms of the gradient-elastic moduli.

The aim of the paper is to establish extended conditions for uniqueness of the first BVP for the Toupin–Mindlin strain gradient elasticity. The interest to this problem relates to possibility to describe some size-dependent material instabilities. The paper is organized as follows. In Section 2 we briefly recall the basic equations of the Toupin–Mindlin strain gradient elasticity for isotropic solids. Strong ellipticity (SE) conditions are formulated. In Section 3 we discuss an admissible range of the Lamé moduli μ and λ. To this end, we formulate a basic theorem on existence and uniqueness of a weak solution of the first boundary-value problem. Then we provide a study of positive definiteness of the corresponding bilinear form, which results in a generalized two-parameter spectral problem for the Lamé moduli. Considering divergence-free and curl-free deformations we obtain one-parameter spectral problems which bring some inequalities for the Lamé moduli. Finally, using the Friedrichs inequalities we present the areas in the $\lambda-\mu$-plane related to the classic positive definiteness, strong ellipticity for classic linear elasticity, and the area where the first boundary-value problem has an unique solution.

2 | GOVERNING EQUATIONS

Let $\mathcal{B}$ be an homogeneous elastic solid body which occupies a bounded volume $V \in \mathbb{R}^3$ with a smooth enough boundary $S = \partial V$. In what follows we use the Toupin–Mindlin strain gradient elasticity of the form II [12–15]. Within the model there exists a deformation energy W given as a quadratic form of strain tensor $\boldsymbol{\varepsilon}$ and its gradient

$$W = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{E} \,\vdots\, \boldsymbol{\varkappa} + \frac{1}{2}\boldsymbol{\varkappa} \,\vdots\, \mathbb{D} \,\vdots\, \boldsymbol{\varkappa}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\right), \quad \boldsymbol{\varkappa} = \operatorname{grad}\boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbb{C}$, $\mathbb{E}$, and $\mathbb{D}$ are fourth-, fifth-, and sixth-order tensors of elastic moduli, respectively, and $\mathbf{u}$ is the displacement vector. In Cartesian coordinates x_k, $k = 1, 2, 3$, the gradient operator is defined as follows

$$\operatorname{grad}\mathbf{u} = \operatorname{grad}(u_k\mathbf{i}_k) = \partial_m u_k \mathbf{i}_k \otimes \mathbf{i}_m, \tag{2}$$

where $\partial_m = \partial/\partial x_m$, $\mathbf{i}_k$ are the unit base vectors related to x_k, $k = 1, 2, 3$, and "$\otimes$" is the dyadic product. Hereinafter "$\cdot$", "$:$" and "$\vdots$" are the dot-, double- and triple-dot products, respectively.

In what follows we restrict ourselves to an isotropic behavior, so $\mathbb{E}$ is zero, whereas $\mathbb{C}$ and $\mathbb{D}$ have the following representation [3, 25]

$$\mathbb{C} = \mathbb{C}_{ijkl}\mathbf{i}_i \otimes \mathbf{i}_j \otimes \mathbf{i}_k \otimes \mathbf{i}_l, \quad \mathbb{D} = \mathbb{D}_{ijmkln}\mathbf{i}_i \otimes \mathbf{i}_j \otimes \mathbf{i}_m \otimes \mathbf{i}_k \otimes \mathbf{i}_l \otimes \mathbf{i}_n, \tag{3}$$

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4}$$

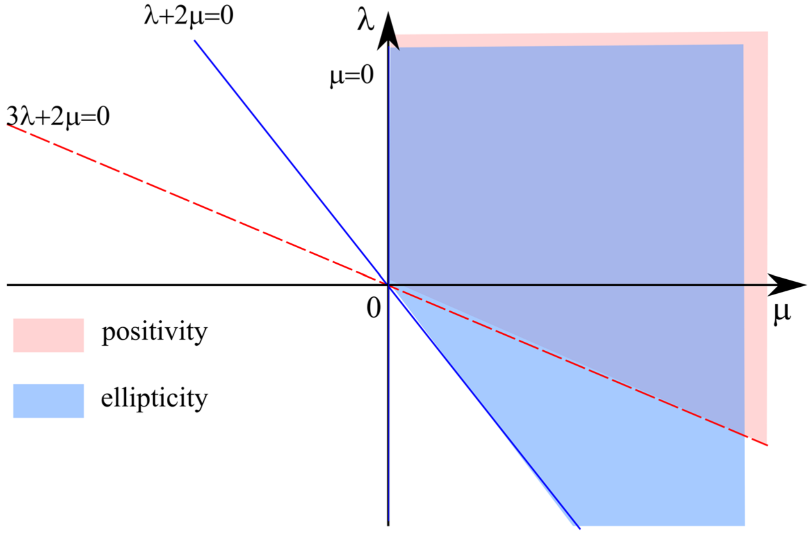
$$\begin{aligned}\mathbb{D}_{ijmkln} = {}& \frac{a_1}{2}\left(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lmn} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{kl}\delta_{in}\delta_{jm}\right) + 2a_2\delta_{ij}\delta_{kl}\delta_{mn} \\ & + \frac{a_3}{2}\left(\delta_{jk}\delta_{im}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{jl}\delta_{im}\delta_{kn}\right) + a_4\left(\delta_{il}\delta_{jk}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn}\right) \\ & + \frac{a_5}{2}\left(\delta_{jk}\delta_{in}\delta_{lm} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{jl}\delta_{km}\delta_{in} + \delta_{il}\delta_{km}\delta_{jn}\right),\end{aligned} \tag{5}$$

where λ and μ are the Lamé elastic moduli, $a_1, \dots, a_5$ are elastic moduli of higher order, and δ_{ij} is the Kronecker symbol.

Equilibrium equations expressed in displacements are

$$\begin{aligned}&(\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} - (2\alpha - \beta)\Delta\operatorname{grad}\operatorname{div}\mathbf{u} - \beta\Delta\Delta\mathbf{u} + \mathbf{f} = \mathbf{0}, \\ &\alpha = a_1 + a_2 + a_3 + a_4 + a_5, \quad \beta = \frac{1}{2}(a_3 + 2a_4 + a_5),\end{aligned} \tag{6}$$

FIGURE 1 Lamé moduli plane: positive definiteness area (red) and strong ellipticity area (blue)



where div is the divergence operator, Δ is the 3D Laplace operator, and $\mathbf{f}$ is the body force vector. In what follows we consider the first boundary-value problem with Dirichlet boundary conditions

$$\mathbf{u}\Big|_S = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial n}\Big|_S = \mathbf{0}, \tag{7}$$

where $\partial/\partial n$ is the external normal derivative.

Strong ellipticity (SE) conditions for (6) are given by two inequalities [24]

$$2\alpha \equiv a_1 + a_2 + a_3 + a_4 + a_5 > 0, \quad 2\beta \equiv a_3 + 2a_4 + a_5 > 0. \tag{8}$$

So they do not imply constraints to λ and μ within the strain gradient elasticity.

In addition to the strain gradient elasticity let us consider a linearly elastic isotropic material with a strain energy density

$$W_0 = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \frac{1}{2}\lambda(\operatorname{div}\mathbf{u})^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, \tag{9}$$

which we will call the base material. One can treat constitutive equation (1) as a gradient regularization of (9). For the base material, the first boundary-value problem takes the form

$$(\lambda + \mu)\text{grad div}\,\mathbf{u} + \mu\Delta\mathbf{u} + \mathbf{f} = \mathbf{0}, \quad \mathbf{u}\Big|_S = \mathbf{0}. \tag{10}$$

The strong ellipticity conditions are given by

$$\lambda + 2\mu > 0, \quad \mu > 0. \tag{11}$$

Note that (11) are less restrictive than the positive definiteness of W_0 which results in

$$3\lambda + 2\mu > 0, \quad \mu > 0. \tag{12}$$

In order to distinguish (8) and (11) which constitute SE conditions for two elastic models, we call (8) and (11) the strong ellipticity of second- and first-order, respectively, see [22] for more details. In $\lambda - \mu$-plane inequalities (12) and (11) form two open areas related to the positive definiteness of the strain energy of the base material (in Figure 1 it is shown in red), and strong ellipticity (shown in Figure 1 in blue).

Let us note that SE conditions are less restrictive than the conditions followed from positive definiteness of the deformation energy, see [14, 15, 26–28] for the details. One can prove that if both SE conditions are fulfilled, for problem (6) and (7) there exists a weak solution in $H_0^2(V)$ and it is unique. Here $H_0^2(V)$ is a standard notation for a Sobolev's space [29]. So (8) and (11) are sufficient conditions for well-posedness of the corresponding boundary-value problems. On the other hand, these conditions could be weakened. In the next section we discuss this matter in more detail.

3 | SPECTRAL PROBLEM FOR LAMÉ MODULI

3.1 | Mathematical preliminaries: Existence and uniqueness of weak solutions

In what follows we use basic properties of Lebesgue's and Sobolev's spaces [29, 30]. Lebesgue's space $L^p(V)$, $p \geq 1, V \in \mathbb{R}^3$, is a Banach space of measurable functions with the norm

$$\|u\|_{L^p} = \left(\iiint\limits_V |u|^p \, dV \right)^{1/p}. \tag{13}$$

Sobolev's space $W^{m,p}(V)$ consists of elements such that

$$W^{m,p}(V) = \left\{ u : u \in L^p(V), D^k u \in L^p(V) \right\} \tag{14}$$

with the norm

$$\|u\|_{W^{m,p}} = \left\{ \iiint\limits_V \left[|u|^p + \sum_{1 \leq |k| \leq m} \left| D^k u \right|^p \right] dV \right\}^{1/p}, \tag{15}$$

where

$$D^k u = \frac{\partial^{|k|} u}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}, \quad k = (k_1, k_2, k_3), \quad |k| = k_1 + k_2 + k_3. \tag{16}$$

Finally, $W_0^{m,p}(V)$ is the closure (completion) of $C_0^\infty(V)$-functions in the norm (15). For brevity, we use notation $H_0^m(V) = W_0^{m,2}(V)$ and $H^m(V) = W^{m,2}(V)$. In the case of $W_0^{m,p}(V)$ there are no smoothness requirements to $S = \partial V$, see [29, p. 86].

For a vector function $\mathbf{u}$ we use the notation $\mathbf{u} \in \mathbf{L}^p(V)$ or $\mathbf{u} \in \mathbf{W}^{m,p}(V)$ if each Cartesian component of $\mathbf{u}$ belongs to $L^p(V)$ or $W^{m,p}(V)$, respectively.

Let us multiply the both parts of (6) by a function $\mathbf{v} \in \mathbf{C}_0^2(V)$ and integrate the result over V. Then, integrating by parts with use of (7), we come to the equation

$$B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}), \tag{17}$$

where

$$\begin{aligned}
B(\mathbf{u}, \mathbf{v}) &= B_1(\mathbf{u}, \mathbf{v}) + B_2(\mathbf{u}, \mathbf{v}), \\
B_1(\mathbf{u}, \mathbf{v}) &= \iiint\limits_V [(\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + \mu \operatorname{grad} \mathbf{u} : \operatorname{grad} \mathbf{v}] \, dV, \\
B_2(\mathbf{u}, \mathbf{v}) &= \iiint\limits_V [\beta \Delta \mathbf{u} \, \Delta \mathbf{v} + (2\alpha - \beta) \operatorname{grad} \operatorname{div} \mathbf{u} \cdot \operatorname{grad} \operatorname{div} \mathbf{v}] \, dV, \\
L(\mathbf{v}) &= \iiint\limits_V \mathbf{f} \cdot \mathbf{v} \, dV,
\end{aligned} \tag{18}$$

Following [16–19] we use the weak solution approach as a basis of the well-posedness analysis of the problem under consideration. For this we introduce two definitions and formulate the main theorem on well-posedness of the problem under consideration.

Definition 1. The energy space E is a completion of $C_0^2(V)$-functions in the norm

$$\|\mathbf{u}\|_E = [B_2(\mathbf{u}, \mathbf{u})]^{1/2}. \tag{19}$$

E is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_E = B_2(\mathbf{u}, \mathbf{v}). \tag{20}$$

One can prove that the norms of E and $H_0^2(V)$ are equivalent in E.

Definition 2. $\mathbf{u} \in E$ is a weak solution of (6) and (7) if Equation (17) holds for any $\mathbf{v} \in \mathbf{C}_0^2(V)$.

Existence and uniqueness of a weak solution depends on the properties of $B(\mathbf{u}, \mathbf{v})$ and $L(\mathbf{v})$. It is known that $L(\mathbf{v})$ is a linear continuous functional if $\mathbf{f} \in \mathbf{L}^2(V)$, see for example [30] for a general representation of linear continuous functionals in $W^{m,p}(V)$. Using Sobolev's embedding theorems we can see that $B(\mathbf{u}, \mathbf{v})$ has the property

$$B(\mathbf{u}, \mathbf{v}) \le C_1 \|\mathbf{u}\|_E \|\mathbf{v}\|_E, \tag{21}$$

where C_1 is a positive constant independent of $\mathbf{u}$ and $\mathbf{v} \in E$.

So one can formulate a standard theorem, see for example [16–19]

Theorem 1. *Let $B(\mathbf{u}, \mathbf{v})$ be a positive definite, that is,*

$$B(\mathbf{u}, \mathbf{u}) \ge C_2 \|\mathbf{u}\|_E^2, \quad \forall \mathbf{u} \in E, \tag{22}$$

where C_2 is a positive constant independent on $\mathbf{u}$ and $\mathbf{f} \in \mathbf{L}^2(V)$. Then there exists a weak solution in the sense of Definition 2. This solution is unique.

3.2 | Positive definiteness of the bilinear form

As it is seen, for (6) and (7) the property of positive definiteness of $B(\mathbf{u}, \mathbf{v})$ is essential. It is easy to show that if the both systems of SE inequalities are fulfilled, then B is positive definite, see for example [22] for details and further references. It could be also demonstrated that $\alpha \ge 0$ and $\beta \ge 0$ are necessary conditions of the positive definiteness [22]. Particular cases when $\alpha = 0$ or $\beta = 0$ are briefly discussed in [24]. So in what follows we assume that SE conditions (8) are fulfilled and we came to the problem of determination of an admissible range of Lamé moduli λ and μ. By the admissible range we mean the range where (11) could be violated but B is still positive definite.

Non-uniqueness of a solution of (6) and (7) means that there exists a nontrivial (nonzero) solution of the homogeneous boundary-value problem

$$(2\alpha - \beta)\Delta \operatorname{grad} \operatorname{div} \mathbf{u} + \beta \Delta\Delta \mathbf{u} = (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \Delta \mathbf{u}, \quad \mathbf{u}\Big|_S = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial n}\Big|_S = \mathbf{0}. \tag{23}$$

This problem could be treated as a generalized eigenvalue problem where μ and λ play a role of spectral parameters. So we have a two-parameter spectral problem, see for example [31, 32]. For finite dimensional operators eigen-values form an algebraic curve in the $\lambda - \mu$-plane.

It is worth to note that a spectral problem for elastic moduli was first formulated by Cosserat brothers, see the review by Mikhlin [33] on the Cosserat spectrum. Here we have an essential difference with Cosserats' problem as here differential operators have forth- and second order.

The weak form of (23) can be rewritten as follows

$$B_2(\mathbf{u},\mathbf{v}) = -\lambda B_3(\mathbf{u},\mathbf{v}) - \mu B_4(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{v} \in E, \tag{24}$$

$$B_3(\mathbf{u},\mathbf{v}) = \iiint\limits_V \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, dV, \quad B_4(\mathbf{u},\mathbf{v}) = \iiint\limits_V [\operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v} + \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{v}]\, dV.$$

Using the Rayleigh–Ritz procedure for (24) one can estimate values of λ and μ which form an area in the $\lambda - \mu$-plane of positive-definiteness of B, that is, the area of uniqueness of solutions.

3.2.1 | Anti-plane deformations

In order to demonstrate the peculiarities of the spectral problem let us restrict ourselves to a more simple case. First we consider a relatively simple case of anti-plane deformations. In this case $\mathbf{u}$ is

$$\mathbf{u} = u(x_1, x_2)\mathbf{i}_3, \tag{25}$$

where $(x_1, x_2) \in \Omega \subset \mathbb{R}^2$. So (23) reduces to an eigenvalue problem

$$\Delta\Delta u = \omega \Delta u, \quad u\Big|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, \quad \omega = \frac{\mu}{\beta}. \tag{26}$$

Note that in this section ∇ and $\Delta = \nabla \cdot \nabla$ are two-dimensional nabla and Laplace operators, respectively. In Equation (26) one can easily recognize a buckling problem for a clamped Kirchhoff plate [34], where $-\omega$ plays a role of an uniform compressing load normalized with bending stiffness. It is well-known that (26) has an infinite series of eigen-values ω_k^*. So (26) has an unique solution until ω is larger than the minimal value of $|\omega_k|$, $\omega > -\omega^*$, $\omega^* \equiv \min\limits_k |\omega_k^*|$. The value ω^* could be calculated through the minimization of the Rayleigh quotient

$$\omega^* = \inf_{u \in H_0^2(\Omega)} \frac{\iint\limits_\Omega (\Delta u)^2\, d\Omega}{\iint\limits_\Omega \nabla u \cdot \nabla u\, d\Omega}. \tag{27}$$

The weak statement for (26) has the form

$$\iint\limits_\Omega (\Delta u)(\Delta v)\, d\Omega = \omega \iint\limits_\Omega \nabla u \cdot \nabla v\, d\Omega \quad \forall\ v \in H_0^2(\Omega). \tag{28}$$

ω^* relates to the Friedrichs inequality. Let us recall that for any function $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ the inequality could be formulated as follows

$$\iint\limits_\Omega \nabla\mathbf{w} : \nabla\mathbf{w}\, d\Omega \geq C_P \iint\limits_\Omega \mathbf{w} \cdot \mathbf{w}\, d\Omega \tag{29}$$

with a positive constant C_P independent on $\mathbf{w}$. Considering $\mathbf{w} = \nabla u$ and comparing (27) and (29) we can conclude that ω^* relates to the exact value of the constant C_P: $\omega^* = C_P$. Indeed, we have

$$\frac{\iint\limits_\Omega (\Delta u)^2\, d\Omega}{\iint\limits_\Omega \nabla u \cdot \nabla u\, d\Omega} = \frac{\iint\limits_\Omega \nabla\mathbf{w} : \nabla\mathbf{w}\, d\Omega}{\iint\limits_\Omega \mathbf{w} \cdot \mathbf{w}\, d\Omega}. \tag{30}$$

TABLE 1 Critical coefficient $\varkappa^*$ *versus* aspect ratio a/b

a/b	1	2	3	4
$\varkappa^*$	5.30	3.92	3.86	3.83

Taking infimum of the latter expression we came to

$$\omega^* = \inf_{u\in H_0^2(\Omega)} \frac{\iint\limits_\Omega (\Delta u)^2\, d\Omega}{\iint\limits_\Omega \nabla u\cdot\nabla u\, d\Omega} = \inf_{\mathbf{w}\in \mathbf{H}_0^1(\Omega)} \frac{\iint\limits_\Omega \nabla\mathbf{w} : \nabla\mathbf{w}\, d\Omega}{\iint\limits_\Omega \mathbf{w}\cdot\mathbf{w}\, d\Omega} = C_P. \tag{31}$$

As an example let us consider a circular area of a radius a, $\Omega = \{(x_1,x_2) : x_1^2 + x_2^2 \le a^2\}$. The minimal eigen-value ω^* is $\omega^* = \kappa^2/a^2$, where $\kappa \approx 3.832$ is the first nontrivial root of the equation $J_1(x) = 0$, J_1 is the Bessel function of first kind. As a result, we get an inequality for μ

$$\mu > -\mu^*, \quad \mu^* \equiv \frac{\kappa^2\beta}{a^2}. \tag{32}$$

It is interesting to note that μ^* depends on a, that is, it is size-dependent. Obviously, $\mu^* \to 0$ at $a \to \infty$ and vice versa, $\mu^* \to \infty$ at $a \to 0$.

In order to demonstrate the dependence of ω^* on a domain shape, let us consider a rectangle $\Omega = \{(x_1,x_2) : 0 \le x_1 \le, a, 0 \le x_2 \le b\}$ with two lengths a and b. Here we define μ^* as follows

$$\mu^* \equiv \frac{\varkappa^*\beta}{b^2}, \tag{33}$$

where $\varkappa^*$ is a coefficient. Using the results on the buckling of a rectangular clamped plate [35] we present $\varkappa^*$ as a function of the aspect ratio a/b in Table 1.

These examples show that if μ is negative the uniqueness depends on Ω. More precisely, for any given $\beta > 0$ and $\mu < 0$ there is an area $\Omega \subset \mathbb{R}^2$ such that positive definiteness required for Theorem 1 is violated. For example, in case of circular area there is a critical radius a^*,

$$a^* = \kappa\sqrt{\frac{\beta}{|\mu|}}, \tag{34}$$

such that there exists a nontrivial solution of (26). In other words, for a small area we can meet non-uniqueness whereas for a larger area a solution is unique. So we can conclude that for $\mu > 0$ the uniqueness is *absolute*, that is, does not depend on Ω, whereas for $\mu < 0$ it is *relative* due to dependence on Ω.

3.2.2 | Curl-free deformations

Since $\operatorname{div}\mathbf{u} = 0$, anti-plane deformation represents an example of so-called divergence-free (solenoidal) deformations having the form $\mathbf{u} = \operatorname{curl}\boldsymbol{\Psi}$ with a vectorial potential $\boldsymbol{\Psi}$. Let us now consider another case, called curl-free (irrotational) deformation. In this case $\operatorname{curl}\mathbf{u} = \mathbf{0}$, where curl is the curl differential operator. So $\mathbf{u} = \operatorname{grad}\Phi$, where Φ is a scalar potential. Using the Laplace operator decomposition

$$\Delta\mathbf{u} = \operatorname{grad}\operatorname{div}\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u}, \tag{35}$$

instead of (6) we get

$$\Delta\operatorname{grad}\operatorname{div}\mathbf{u} = \varpi\operatorname{grad}\operatorname{div}\mathbf{u}, \quad \mathbf{u}\Big|_S = \mathbf{0}, \quad \frac{\partial\mathbf{u}}{\partial n}\Big|_S = \mathbf{0}, \quad \varpi = \frac{\lambda + 2\mu}{2\alpha}. \tag{36}$$

So we again have came to an one-parameter generalized spectral problem. Replacing $\mathbf{u}$ by grad Φ, instead of $(36)_1$ we again get a plate-buckling-type equation

$$\Delta\Delta\Phi = \varpi\Delta\Phi \tag{37}$$

with another boundary conditions

$$\left.\frac{\partial\Phi}{\partial n}\right|_S = 0, \quad \left.\frac{\partial^2\Phi}{\partial n^2}\right|_S = 0. \tag{38}$$

For uniqueness this problem should be complemented by an additional constraint $\iiint_V \Phi\, dV = 0$ as for the Neumann boundary conditions. The latter BVP brings us another inequality $\varpi > -\varpi^*$, where $\varpi^* \equiv \min_k |\varpi_k^*|$ relates to the first non-zero eigen-value. As a result, we have an inequality

$$\lambda + 2\mu > -\gamma^*, \quad \gamma^* = 2\varpi^*\alpha, \tag{39}$$

which guaranties the uniqueness of curl-free deformations.

3.2.3 | General deformations

Let us now consider a general case. Using the Helmholtz decomposition we represent $\mathbf{u}$ as a sum

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{curl}\boldsymbol{\Psi}, \quad \operatorname{div}\boldsymbol{\Psi} = 0, \tag{40}$$

where Φ and $\boldsymbol{\Psi}$ are scalar and vector potentials, respectively. Equation (40) presents an orthogonal decompositions of generalized functions in $\mathbf{L}^2(V)$ and in $\mathbf{H}_0^1(V)$, see [19] for more details. In other words, any element of these spaces could be represented as a sum of curl-free and divergence-free elements

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'', \quad \operatorname{curl}\mathbf{u}' = \mathbf{0}, \quad \operatorname{div}\mathbf{u}'' = 0. \tag{41}$$

Using (35) and $(23)_2$ we get the identities

$$\begin{aligned}\iiint_V \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u}\, dV &= -\iiint_V (\Delta\mathbf{u})\cdot\mathbf{u}\, dV \\ &= -\iiint_V \left[\operatorname{grad}\operatorname{div}\mathbf{u}' - \operatorname{curl}\operatorname{curl}\mathbf{u}''\right]\cdot\mathbf{u}\, dV \\ &= \iiint_V \left[(\operatorname{div}\mathbf{u}')^2 + \operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\mathbf{u}''\right] dV. \end{aligned} \tag{42}$$

So $B_1(\mathbf{u},\mathbf{u})$ takes the form

$$\begin{aligned} B_1(\mathbf{u},\mathbf{u}) =& (\lambda+2\mu)\iiint_V (\operatorname{div}\mathbf{u}')^2\, dV + \mu\iiint_V \operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\mathbf{u}''\, dV \\ =& (\lambda+2\mu)\iiint_V (\operatorname{div}\mathbf{u})^2\, dV + \mu\iiint_V \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{u}\, dV. \end{aligned} \tag{43}$$

Similarly, we transform $B_2(\mathbf{u},\mathbf{u})$ as follows

$$
\begin{aligned}
B_2(\mathbf{u},\mathbf{u}) =&2\alpha \iiint\limits_V \operatorname{grad}\operatorname{div}\mathbf{u}'\cdot\operatorname{grad}\operatorname{div}\mathbf{u}'\,dV+\beta\iiint\limits_V \operatorname{curl}\operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\operatorname{curl}\mathbf{u}''\,dV\\
=&2\alpha \iiint\limits_V \operatorname{grad}\operatorname{div}\mathbf{u}\cdot\operatorname{grad}\operatorname{div}\mathbf{u}\,dV+\beta\iiint\limits_V \operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\operatorname{curl}\mathbf{u}\,dV.
\end{aligned}\tag{44}
$$

As a result, we get

$$
\begin{aligned}
B(\mathbf{u},\mathbf{u}) =&2\alpha \iiint\limits_V \operatorname{grad}\operatorname{div}\mathbf{u}\cdot\operatorname{grad}\operatorname{div}\mathbf{u}\,dV+(\lambda+2\mu)\iiint\limits_V (\operatorname{div}\mathbf{u})^2\,dV\\
&+\beta\iiint\limits_V \operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\operatorname{curl}\mathbf{u}\,dV+\mu\iiint\limits_V \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{u}\,dV.
\end{aligned}\tag{45}
$$

Using Fridrichs inequality (29) and the identity

$$
\iiint\limits_V \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u}\,dV = \iiint\limits_V (\operatorname{div}\mathbf{u}')^2\,dV + \iiint\limits_V \operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\mathbf{u}''\,dV \tag{46}
$$

we get two inequalities

$$
\iiint\limits_V \operatorname{grad}\operatorname{div}\mathbf{u}'\cdot\operatorname{grad}\operatorname{div}\mathbf{u}'\,dV \geq C_{P1}\iiint\limits_V (\operatorname{div}\mathbf{u}')^2\,dV, \tag{47}
$$

$$
\iiint\limits_V \operatorname{curl}\operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\operatorname{curl}\mathbf{u}''\,dV \geq C_{P2}\iiint\limits_V \operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\mathbf{u}''\,dV \tag{48}
$$

with positive constants C_{P1} and C_{P2} independent on $\mathbf{u}'$ and $\mathbf{u}''$.

Finally, we come to the formulae

$$
\begin{aligned}
B(\mathbf{u},\mathbf{u}) &= \varepsilon B_2(\mathbf{u},\mathbf{u}) + (1-\varepsilon)B_2(\mathbf{u},\mathbf{u}) + B_1(\mathbf{u},\mathbf{u})\\
&\geq \varepsilon B_2(\mathbf{u},\mathbf{u}) + [2\alpha(1-\varepsilon)C_{P1}+(\lambda+2\mu)]\iiint\limits_V (\operatorname{div}\mathbf{u}')^2\,dV + [\beta(1-\varepsilon)C_{P2}+\mu]\iiint\limits_V \operatorname{curl}\mathbf{u}''\cdot\operatorname{curl}\mathbf{u}''\,dV
\end{aligned}\tag{49}
$$

for any number $\varepsilon\in[0,1]$. So the lines $\lambda+2\mu=-2\alpha C_{P1}\equiv-\gamma^*$ and $\mu=-\beta C_{P2}\equiv-\mu^*$ form the boundary in $\lambda-\mu$–plane separating the uniqueness and non-uniqueness areas, see Figure 2. Here one can see ranges of Lamé moduli related to positive definiteness of a strain energy density of the base material, its strong ellipticity and the uniqueness of the solutions, respectively. Obviously, the boundary lines of the uniqueness ares coincides with particular cases (32) and (39) studied above.

4 | CONCLUSIONS

Within the linear Toupin–Mindlin strain gradient elasticity of isotropic solids we discussed existence and uniqueness of solutions of the first boundary-value problem with Dirichlet boundary conditions. Violation of uniqueness of such a solution is closely related to a material instability. Indeed, for a finite solid body with clamped surface non-uniqueness means infinitesimal instability induced only by material behaviour. We formulate inequalities (32), (39), and strong ellipticity conditions (8), which result in the positive definiteness of the energy functional $\mathcal{E}$. Let us underline, that for negative values of the first-order elastic moduli the uniqueness property became *relative* as it depends on the size and shape of domain occupied by a solid body. In particular, in this case non-uniqueness could be observed for relatively small domains. If

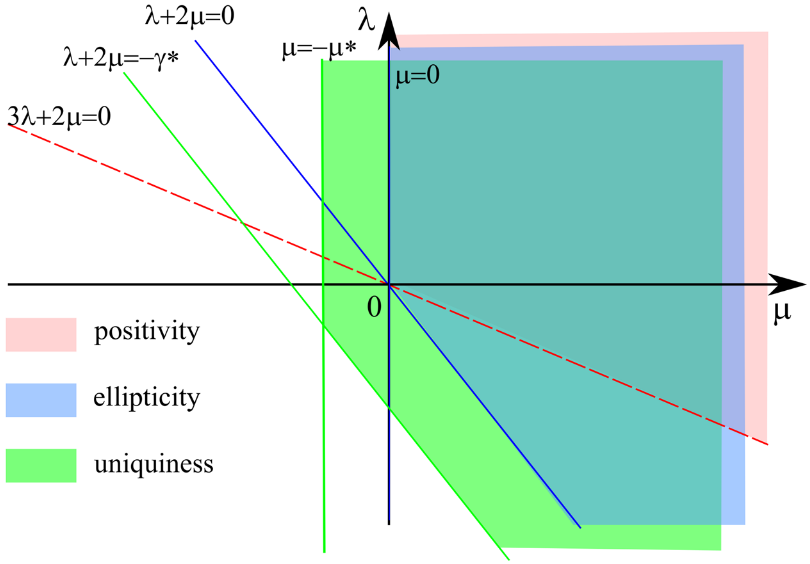


FIGURE 2 Lamé moduli plane: positive definiteness (red), strong ellipticity (blue), uniqueness of the first BVP (green)

inequalities (11) are fulfilled, one has uniqueness of solutions for any domain. The presented results extend the uniqueness analysis provided in [15, 28].

Note, that in this paper we restrict ourselves to linear strain gradient elasticity. Nevertheless, a similar approach could be applied to the linearized strain-gradient elasticity and to study of some material instabilities. More precisely, certain inequalities could be formulated for tangent first-order moduli similar to (32) and (39) which could guarantee uniqueness of solution under strong ellipticity conditions.

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CONFLICT OF INTEREST

The author declares no potential conflict of interests.

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