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REVIEW ARTICLE

A Review: Applications of the Spectral Finite Element Method

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Abstract
The Spectral Finite Element Technique (SFEM) has Several Applications in the Sciences, Engineering, and Mathematics, which will be Covered in this Review Article. The Spectral Finite Element Method (SFEM) is a Variant of the Traditional Finite Element Method FEM that Makes use of Higher Order Basis Functions (FEM). One of the most Fundamental Numerical Techniques Employed in the Numerical Simulation is the SFEM, which Outperforms Other Techniques in Terms of Faster Convergence, Reduced Diffusion and Dispersion Errors, Simplicity of the Application as well as Shorter time of Computation. The Spectral Finite Element Technique Combines the Characteristics of Approximating Polynomials of Spectral Methods. The Approach to Discretizing the Examined Region Unique to the FEM is a mix of both Approaches. Combining These Techniques Enables Quicker (Spectral) Convergence of Solutions, Higher Approximation Polynomial Order, the Removal of Geometric Constraints on the Examined Areas, and much Lower Discretization Density Requirements. Spectral Element Methods used in Different Applications are Presented Along with a Statistical Overview of Studies During 2010–2022.

1 Introduction

1.1 Numerical Techniques

In this section, several fundamental computational techniques are discussed. These methods are used to simulate and analyze complex systems that are modeled by differential equations. These differential models, mainly based on the partial differential ones, have been widely applied in a different area of research, including meteorology [1–3], mathematics [4–7], computational biology [8–10], and astronomy [11, 12]. Numerous numerical methods, including the finite element method [13, 14], the finite volume method [15, 16], the finite difference method [17, 18], the spectral method [19–22], the mesh free method [23], the domain decomposition method, and multigrid methods [24, 25] are used to solve partial differential equations. Figure 1 shows the flow chart of different numerical techniques for solving differential equations.

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A computational method for approximately solving BVP's in differential equations is the (FEM). It generates a stable solution by minimizing an error function using variational approaches. FEM covers all approaches for connecting several basic element equations across numerous little subdomains to roughly reflect a more complicated equation across a larger domain. It is similar to how joining numerous small straight lines describes a broader circle. The finite volume approach is a discretization strategy for partial differential equations, especially those originating from physical conservation rules. Using a limited partitioning set of volumes and an integral volume approach to the issue, FVM discretizes the equations. It is standard practice to discretize equations for computational fluid dynamics using FVM.

Functions are represented by their values at certain grid positions for the finite difference approach, and differences in these values roughly define derivatives. The mesh-free method discretizes the domain of the issue by generating an analytical set of equations without the need for a preset mesh. By splitting a boundary value problem down into smaller problems on different subdomains and iterating to coordinate the solutions across relevant subdomains, the domain decomposition approach may solve boundary value problems. To better coordinate the overall solution across the subdomains, a coarse problem with one or a few unknowns is used for each subdomain. The primary goal of multigrid methods is to do often global corrections that

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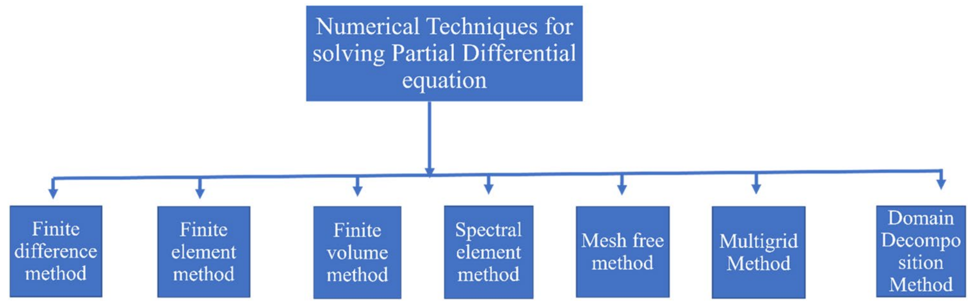


Fig. 1 Flow chart of various numerical techniques for solving PDEs.

speed up the convergence of the iterative approach, which is done by tackling the bigger problem. This notion can be compared to interpolation between coarser and finer grids. Multigrid is generally utilized in the numerical solution of elliptic PDEs in two or more dimensions [26].

1.1.1 Overview of SFEM

Partial differential equations can be numerically by spectral finite element technique (SFEM), a variation of the (FEM). Its basis functions are higher-order piecewise polynomials. In articles released in 1984 [27–34], the spectral element approach was presented.

Various differential equations are numerically solved using spectrum techniques in applied mathematics and scientific computing, usually applying the rapid Fourier transform. By combining higher-order "basis functions," one may describe the differential equation's solution by selecting the coefficients in the sum that best satisfy the differential equation.

The main distinction between spectral techniques and SFEM is the use of higher-order basis functions for faster convergence over the whole domain in spectral techniques. Theoretically, both approaches have a close bond. Finite element techniques, on the other hand, only do so on a few limited subdomains. In other words, spectral approaches adopt a global perspective, whereas FEM adopt a local one. The great error features of spectral techniques, with the quickest possible "exponential convergence" when the solution is smooth, are partly due to this [35]. Figure 2. represents the complete formulation of the SFEM.

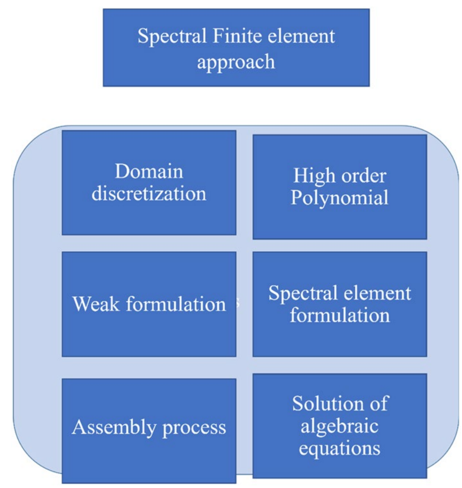


Fig. 2 Flow chart shows the formulation of the spectral element method

1.2 Why SFEM?

The finite element approach is used more intricately with the spectral element technique. The solution across each element is described in terms of discrete values that are known in advance at a small number of spectral nodes.

The SFEM formulation is based on the precise solution of the governing (PDEs) in the spectral domain. This identical solution serves as the interpolating function for formulating spectral elements. The mass and stiffness matrix distribution is precise because of adopting correct solutions in the element formulation. The element, therefore, immediately produces an accurate dynamic stiffness matrix. Regarding equation construction and solution, the SFEM program architecture is similar to the c-FE approach. The only variation is how the temporal component is handled. It is generally regarded as the best practice to produce high-quality discretization and enhance the conditioning of the equation system. In [36] discusses the reasons for ill-conditioning in the context of the conventional FEM and suggests remedies to raise the condition number. In addition to these valuable factors, time and effort are put into designing specialized pre-conditioners for specific uses.

2 Applications

The benefit of using spectral elements is that they may be used in various scenarios to produce steady computational solutions and high accuracy for SFEM applications [37–41]. The spectral methods that the authors in [42] developed over a protracted period beginning in 1969 include pseudo-spectral methods for highly nonlinear problems. To execute the spectrum approach, either collocation or a Galerkin methodology is frequently utilized. The spectral technique is unique because it allows for the symbolic expression of answers to tiny problems, offering a viable alternative to series solutions for differential equations. Compared to finite element methods, spectral techniques could be cheaper and simpler to compute. They perform well when great precision is required in straightforward problems with easy solutions. However, the matrices involved in step computation are dense due to their global nature, and when there are numerous degrees of freedom, computational performance will quickly decrease in the applications.

Numerous researchers have investigated the SFEM and its application to the solution of PDEs during the recent twelve years. To summarize prior research on SFEM, we retrieved some data from "Scopus." Figs. 3 and 4 show the data of yearly publications. Moreover, Fig. 5 shows the applications of the spectral element method within specific research areas. In Sect. 3, work on various strategies will be analyzed along with a comparison of the spectrum finite techniques with other numerical issues. In Sect. 5, recommendations for more study will be made (Fig. 6).

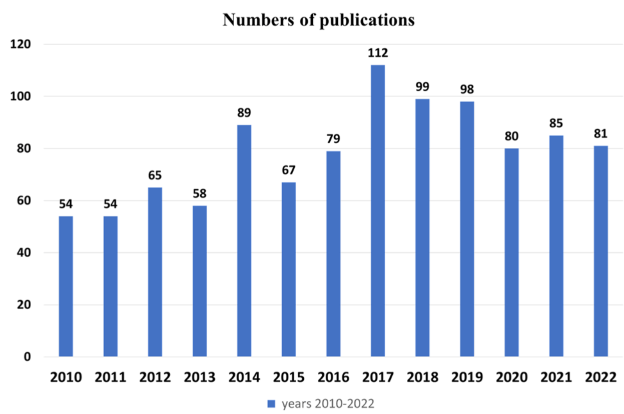


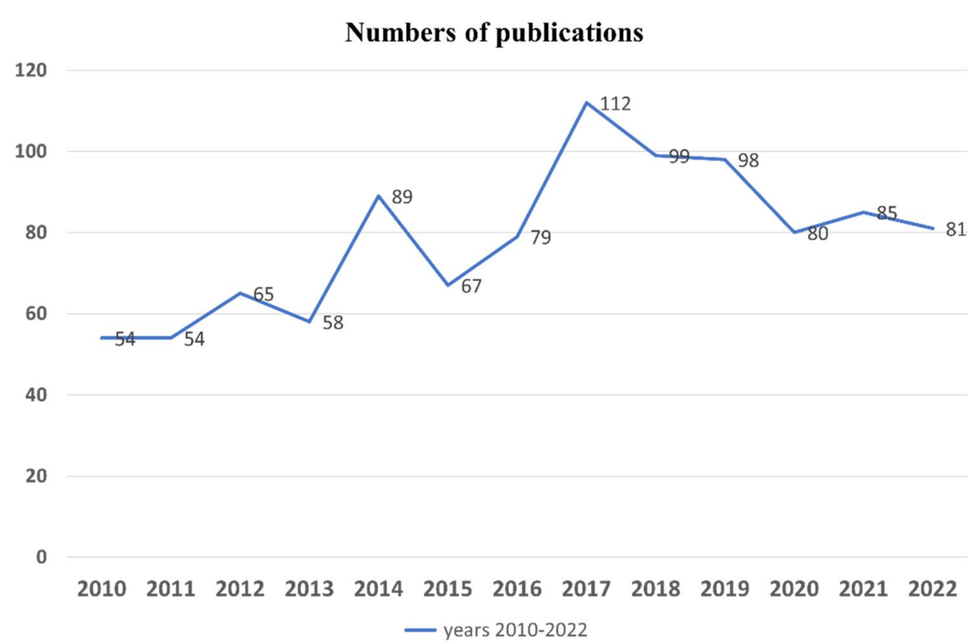
Fig. 3 Number of articles published on SEM (2010–2022)

2.1 Computational Fluid Dynamics (CFD)

A branch of fluid mechanics known as computational fluid dynamics (CFD) utilizes numerical methods and analysis to solve and evaluate issues involving fluid flows that are represented by the widely used Navier-Stokes (NS) equations. Most of the spectral methods used in direct simulations of turbulent and transitional flows during the past few years have directly evaluated fluid flows [43]. In [44–51], authors briefly investigate CFD with different parameters.

The accuracy of spectrum approaches and the geometric adaptability of finite elements are combined in the spectral element approach, which utilizes high-order finite elements. It has been successfully used to model and simulate technical difficulties in the automotive, oil & gas, and aerospace/aeronautics domains. The spectral element approach has much potential for CFD because of the advent of previously unheard-of supercomputer capability. In [52], the authors present the applications of SFEM in the laminar fluid flow channel. A one-dimensional inflow-outflow advection-diffusion equation is used to validate the approach before it is used to study laminar two-dimensional (separated) circulation in a channel expansion. Comparisons are made using an experiment as well as prior numerical work. The

Fig. 4 The line graph shows the number of articles published on SEM (2010–2022)



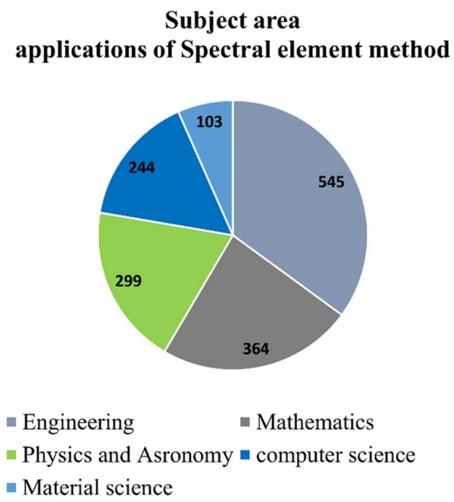


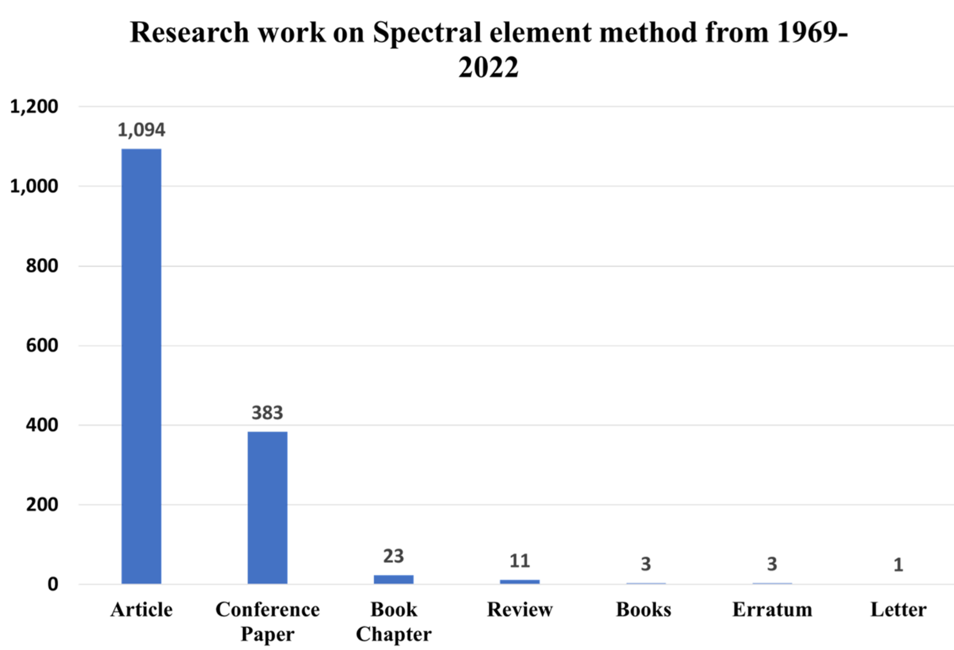
Fig. 5 Number of publications on SFEM that have been published in that particular field

Navier-Stokes and transport equations could be solved using spectral element methods and high-order weighted residual techniques based on the spectral expansions of variables and geometries provided in [53]. Results for linear and secondary spatial stability of flat Poiseuille flow, as well as steady and unstable separated channel flow at Reynolds numbers of several thousand, are shown in [54] using a time-splitting method to solve the Navier-Stokes equations. Recent studies have looked at spectral element techniques in fluid dynamics and materials science [55].

2.2 Dynamic Analysis

The Spectral Element Method is used in [56] to compare planar frame designs considering soil-structure interaction (SFEM). It covers the creation of spectral element matrices based on higher-order element theories as well as the combining of members with various geometries. It is demonstrated that, particularly at high frequencies, SFEM provides better accurate findings at a substantially lower computational cost. As the authors show in [57], the purpose of this study is to use a spectral element approach to evaluate the dynamic behavior of periodic plate structures. For the plate components with two parallel supported sides, spectral equations were constructed. Consider using the lumped mass Chebyshev spectral element method described in [58] (frequency domain) to address problems with dynamic structures. A spectral domain-modified spectral element approach (SEM) is expanded in the study [59] from single-span beams to multi-span beams subject to dynamic point forces. In order to create the multi-layer smart composite structure shown in [60], wafer-type piezoelectric transducers are frequently placed on the surface of laminated composite structures. This structure is used to stimulate or monitor the dynamic responses of the structure for the active control of vibrations or sounds. The dynamic study of soil structure interaction using the spectral element approach is shown in [61]. The spectral finite element method and cubic spline layer-wise theory are used in [62, 63] to analyze low-velocity impacts on composite sandwich plates. An effective computational approach for simulating low energy impacts on composite laminated plate structures is provided in [64] by combining a contact law with a unique time domain spectral

Fig. 6 Number of articles published in SEM on the particular topic



shear plate finite element. A modified Fourier spectral element technique (SEM) is created to analyze the vibrational behaviors of these structures. Based on the type of fundamental structure that is present in [65]. In [66] introduces the spectral element technique as a precise and practical design tool for static and dynamic simulations of cantilever-based MEMS devices. Using the spectral element approach, the microcantilever is discretized while accounting for the fringing field and the nonlinearity brought on by the electrostatic driving force. A Timoshenko beam is used to simulate the microcantilever. Several articles have explored the spectral element approach for dynamic analysis, including [67–72].

2.3 Wave Propagation

Any method of wave transmission is considered wave propagation. Either the first-order one-way wave equation or the standing wavefield wave equation can be used to determine the propagation of a single wave. It is shown that spectral and finite element techniques are particularly useful for the numerical modeling of seismic body wave propagation problems. In [73–75] describe how a three-dimensional Piezo-Enabled Spectral Element Analysis tool was developed to simulate piezo-induced ultrasonic wave propagation in composite structures. Piezo-Enabled Spectral Element Analysis solves the associated electromechanical governing equations for a given arbitrary voltage input to a piezoelectric actuator and outputs the voltage response of the piezoelectric sensors. One-dimensional (1D) elastic wave propagation issues are addressed using spectral finite elements (SFEM). In [76–78], investigations have been done on the waves that move through an isotropic rod and a Timoshenko beam. A 1D SFE was used to represent the rod, and a 1D and 2D SFEM were used to simulate the beam. In comparison to the outcomes of the traditional FEM, numerical results have been achieved. Authors in [79, 80] provide an overview of the spectral element technique, which offers a novel numerical method for synthetic computing seismograms in 3-D earth models. The technique combines spectral precision with the adaptability of a finite element approach. In [81–83], the authors present the simulation and validation of the spectral finite element method through the global transmission of seismic waves. An example of a high-order finite element program that does numerical simulations of seismic wave propagation, such as that caused by active seismic acquisition operations in the oil sector or earthquakes on a continental scale, is presented in [84].

2.4 Fractional Calculus

Several researchers employ various numerical techniques to solve fractional differential equations. The application of fractional calculus to the heat equation recently created an issue brought on by a non-local operator [85]. A class of fractional variational problems with a generic finite element formulation is presented in [86].

Compared to other numerical approaches, using SFEM to solve fractional differential equations sets effective stability requirements and offers greater flexibility when addressing inhomogeneity and complicated geometries. SEM is a numerical technique to obtain approximative solutions to differential equations in which the domain of interest is divided into different elements. It is effective for resolving complicated physical events, especially those that display geometrical and material non-linearities (such as those that are often seen in engineering and the sciences) [87, 88].

The nonlinear fractional evolution problems stability and convergence utilizing the spectral element approach are analyzed in [89]. The main goal of this work [90] is to present a novel numerical method for solving the neutrality duration distributed-order fractional damped diffusion-wave problem. By using the spectral finite element approach, several additional scholars are creating scientific breakthroughs in fractional calculus, including [91–95] (Fig. 7).

3 Comparison

3.1 A comparative Study of Spectral Elements with Different Problems

In [96], the authors used three numerical algorithms: the spectral Quasi linearization technique, spectral local linearization method, and spectral relaxation methods are contrasted for precision and convergence rate. Researching the linearization method's effects on convergence and accuracy is crucial. The techniques are used on various differential equations describing engineering-relevant fluid flow, and their linearization schemes are thoroughly explored. The authors of [97] contrast and compare the direct numerical modeling of artificial jets using spectral-element and finite-volume solvers. A comparative analysis of the Lattice Boltzmann and SEM for the numerical modeling of restricted flows past barriers describe in [98]. Accurate solutions to wave propagation issues under impact loading using high-order finite elements of the conventional, spectral, and geometric types and discusses a contrast between the Spectral-Element and Pseudo-Spectral Methods. Table 1 in [99] contrasts the finite element and spectral element approaches using various orders of polynomials.

3.1.1 Accuracy

In real-world circumstances, we prefer to use high-order expansions, where the solution is expected to exhibit dramatic spatial variations, and low-order expansions, like

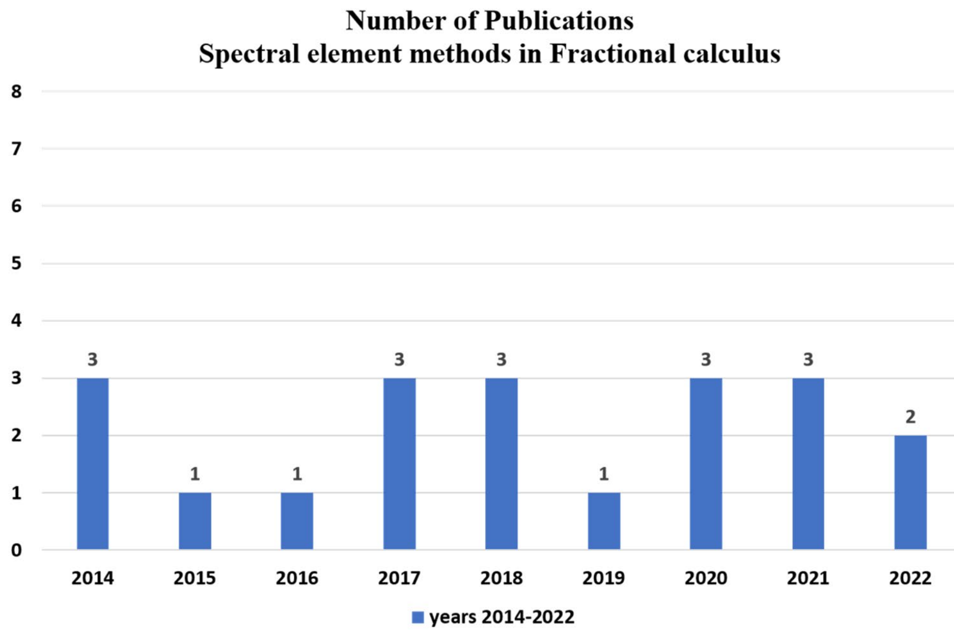
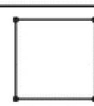
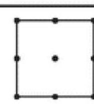
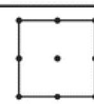
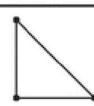
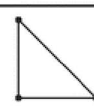
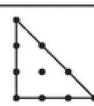
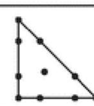


Fig. 7 Number of articles published in s-EM on the particular topic

Table 1 The distribution of the reference elements interpolant points for various polynomial orders [100]

Type of mesh	Order of polynomials	FEM Diagrams	FEM Number of nodes	SEM Diagrams	SEM Number of nodes	Coincide?
Quadrilateral elements	First-order		4		4	Y
	Second-order		9		9	Y
Triangular elements	First-order		3		3	Y
	Third-order		10		10	N
Mass-matrix handling strategies	/	Lumped mass techniques		Quadrature rule		/

c-FEM, where the solution is supposed to fluctuate gradually. We also want the best accuracy for a given number of interpolation nodes. In the case of linear and homogeneous differential equations, the node distribution merely influences the structure and standing of the global matrix as assessed by the condition number. It has no impact on the numerical solution. For non - homogeneous or complex linear equations, the node distribution may have a significant influence on the precision and convergence of the solution.

Theoretical analysis of the interpolation error, which depends on how many interpolation nodes should be distributed over an element, demonstrates that the interior nodes should be distributed at locations that correspond to specific families of orthogonal polynomials' zeros to achieve the highest interpolation accuracy. As a result, we will have an extended spectral element and a method for extending spectral elements.

The spatial resolution SEM is determined by the standard element size and the degree of the polynomial used to represent functions on an element, with each element holding a point in each direction. In this sense, SEMs are analogous to the "h-p version of FEM" or c-FEMs with high polynomial degrees.

4 The Spectral Element System

4.1 Element Nodal Sets in Spectral Finite Element Method SFEM

Consider the case when we have chosen to map the answer over the lth element as a rough estimate mediated by the function:

$$x(\xi) = \frac{1}{2}\left(x_2^{(l)} + x_1^{(l)}\right) + \frac{1}{2}\left(x_2^{(l)} - x_1^{(l)}\right)\xi, \tag{1}$$

where, $x_1^{(l)}$ is the first element end-node and $x_2^{(l)}$ is the second element end-mode by increasing the η from -1 to 1. Figure 8 The element nodes are developed at the positions ξ_i for $i = 1{,}2{,}3\ldots, m+1$ along the$\xi - axis$.

4.2 Shape Functions in the Spectral Finite Element Method

Shape functions that describe the area of the desired physical qualities must ensure that they are continuous inside the elements and adhere to the borders of the elements up to an order lower than the maximum derivative found in the DEq's describing the phenomena.

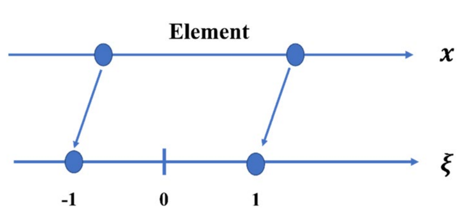


Fig. 8 Mapping of an element from the $x - axis$ to the standard interval [− 1,1] parametric ξ-axis

Shape functions must be capable of representing constant values of the necessary physical features or their derivatives within the component to an order lower than the largest derivative found in the differential equation describing the phenomena.

The approximate approximation of a continuous function is a classic example of such a process, and differentiable function $f(x)$ in the range $x \in [a, b]$ by n initial terms of its expansion into a Taylor series is:

$$f(x) \approx f(a) + \frac{x-a}{1!}f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a). \tag{2}$$

Error of this approximation $R_n(x, a)$ can be expressed in the other form of the Lagrange residual:

$$R_n(x,a) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \tag{3}$$

When a further circumstance is:

$$\lim_{x\to a}\frac{R_n(x,a)}{(x-a)^n} = 0 \tag{4}$$

4.3 Lagrange Interpolation

The Lagrange interpolating polynomial is the only polynomial of the lowest degree that interpolates a specific set of information in numerical analysis. The element interpolation can be conveniently identified with the mth-degree Lagrange interpolating polynomials.

$$\sum_{i=1}^{N+1} M_{ji}^{(e)}\ddot{u}_i^{(e)}(t) + \sum_{i=1}^{N+1} K_{ji}^{(e)}u_i^{(e)}(t), e = 1, 2, \ldots, n_e \tag{5}$$

$$M_{ji}^{(e)} = w_j\rho'(\xi)\frac{dx}{d\xi}\delta_{ij}|_{\xi=\xi_j}, \tag{6}$$

$$K_{ji}^{(e)} = \sum_{k=1}^{N+1} w_k u'(\xi) l_j'(\xi)\dot{l}_i(\xi)\left(\frac{d\xi}{dx}\right)^2\frac{dx}{d\xi}|_{\xi=\xi_k}, \tag{7}$$

and,

$$f_j^{(e)}(t) = w_j f'(\xi, t)\frac{dx}{d\xi}|_{\xi=\xi_j}. \tag{8}$$

For stiffness matrix K, and diagonal mass matrix M.

4.4 Numerical quadrature for calculating the mass and stiffness matrices

Quadrature node points = GLL points.

Since the mass matrix is diagonal, inverting it is simple. The advantages of using the spectral element technique is this.

4.5 Time extrapolation

$$u \approx \frac{u(t+\Delta t) - 2\ddot{u}(t) + u(t-\Delta t)}{\Delta t^2}, \tag{9}$$

$$\Delta(t+\Delta t) = 2u(t) - u(t-\Delta t) + \Delta t^2 M^{-1}\left[f(t) - k - Ku(t)\right], \tag{10}$$

Representation in terms of polynomials

$$u(x,t) \approx \sum_{i=0}^{N} u_i(t) l_i^{(N)}(x) \tag{11}$$

$l_i^{(N)}(x)$: N -degree Lagrange polynomials

When the polynomial coefficients are solved for the PDEs may be transformed into an ODEs

$$M_{ki}\ddot{u}_i - K_{ki}u_i = f_k \tag{12}$$

where,M_{ki} :Mass matrix and K_{ki} :Stiffness matrix.

4.6 Hexahedral Components are used to Divide the Computational Domain

In this section, different hexahedral elements are represented from [101] in Fig. 9a–c. An overview of the mesh at the surface reveals that there are 150 total slices, with 25 slices on each of the six sides of the cubed sphere, each representing a distinct color. The calculation of the wave field in each of these slices is carried out by one of the 150 needed processors.

4.6.1 Mapping to the unit cube

Mapping to unit cube is shown in Fig. 10a–d, [101].

4.6.2 Collocation Points Chosen

Interpolation of Runge's function $R(x)$ using **Sixth -order** polynomials and **equidistant** collocation points.

We are choosing interpolant as

$$R(x) = \frac{1}{1 + ax^2},$$

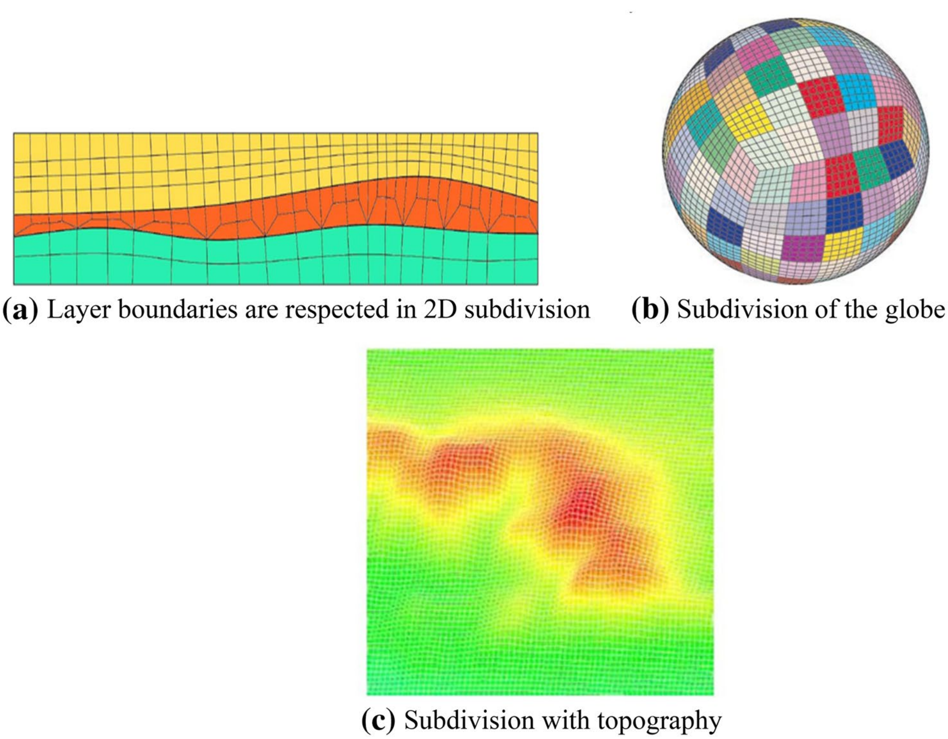
We should use the GLL points as collocation points in above Fig. 11a and b for the Lagrange polynomials.

So we have interpolation of Runge‘s function $R(x)$ using **Sixth -order** polynomials and **Gauss-Lobatto-Legendre** collocation points.

[roots of$(1 - x2)LoN - 1$ = completed Lobatto polynomial]

Example : GLL Lagrange polynomials of degree 6 presented in Fig. 12.

Global maximum at the collocation points equals collocation points for GLL points.



(a) Layer boundaries are respected in 2D subdivision **(b)** Subdivision of the globe **(c)** Subdivision with topography

Fig. 9 Different hexahedral elements. **a** Layer boundaries are respected by the 2D, **b** Subdivision of the globe, **c** Subdivision with topography [101]

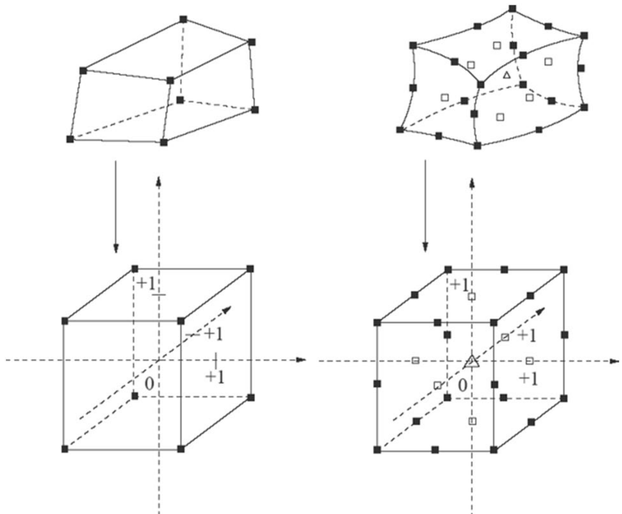


Fig. 10 Mapping to the unit cube (mapping like in isoparametric FEM—in SFEM we have more nodes we can applied subparametric mapping)

5 Future Work

- For future work, we will use the spectral element methods in the heat transfer problems, such as heat transfer enhancement in nanoparticle vis Spectral element method.
- We already have many scientific publications on numerical studies in nanoparticle heat transfer enhancement using the (c-FEM). So, for future recommendations, we will focus on a comparative study of the classical FEM (c-FEM) with the (SFEM) for heat transfer problems.
- We will investigate which numerical method is more suitable for our problems, showing temperature, velocity, and concentrations profile.

6 Conclusion

A more comprehensive description of the SFEM is provided in this advance review. We were able to draw the following findings, understandings, and suggestions for further study as a result.

Statistics over the previous 12 years (2010–2022) support the widespread application of spectral element approaches to a range of scientific issues. The above data clearly shows that there is no more research work on this numerical technique.

- Spectral techniques are discretization strategies for the weakly expressed approximation of partial differential equations. They are founded on specific quadrature criteria and high-order Lagrangian interpolants. A collection of polynomial basis functions that, as the polynomial degree approaches infinity, can perfectly approximate the solution in some norm might be used to represent the solution using spectral techniques. The finite element method is particularly used for spectral elements SFEM.
- As a result, there is no need to carry out a (sparse) matrix inverse inversion, leading to a completely explicit approach.
- Every location inside the elements has a different set of material properties.

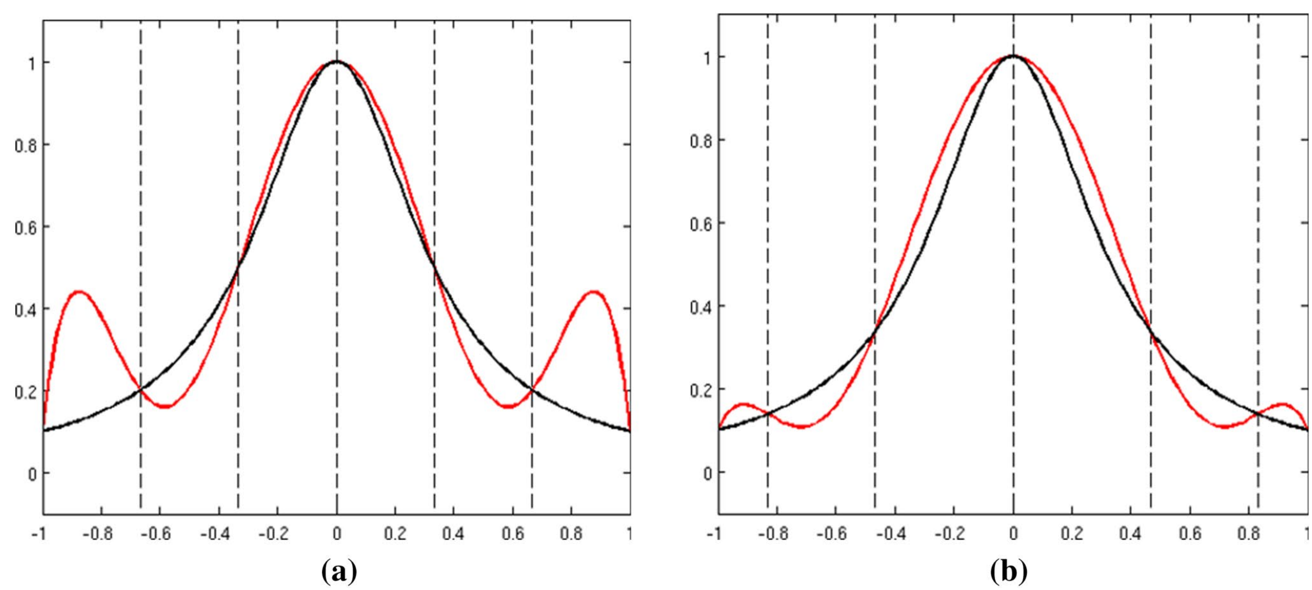


Fig. 11 GLL points as collocation points

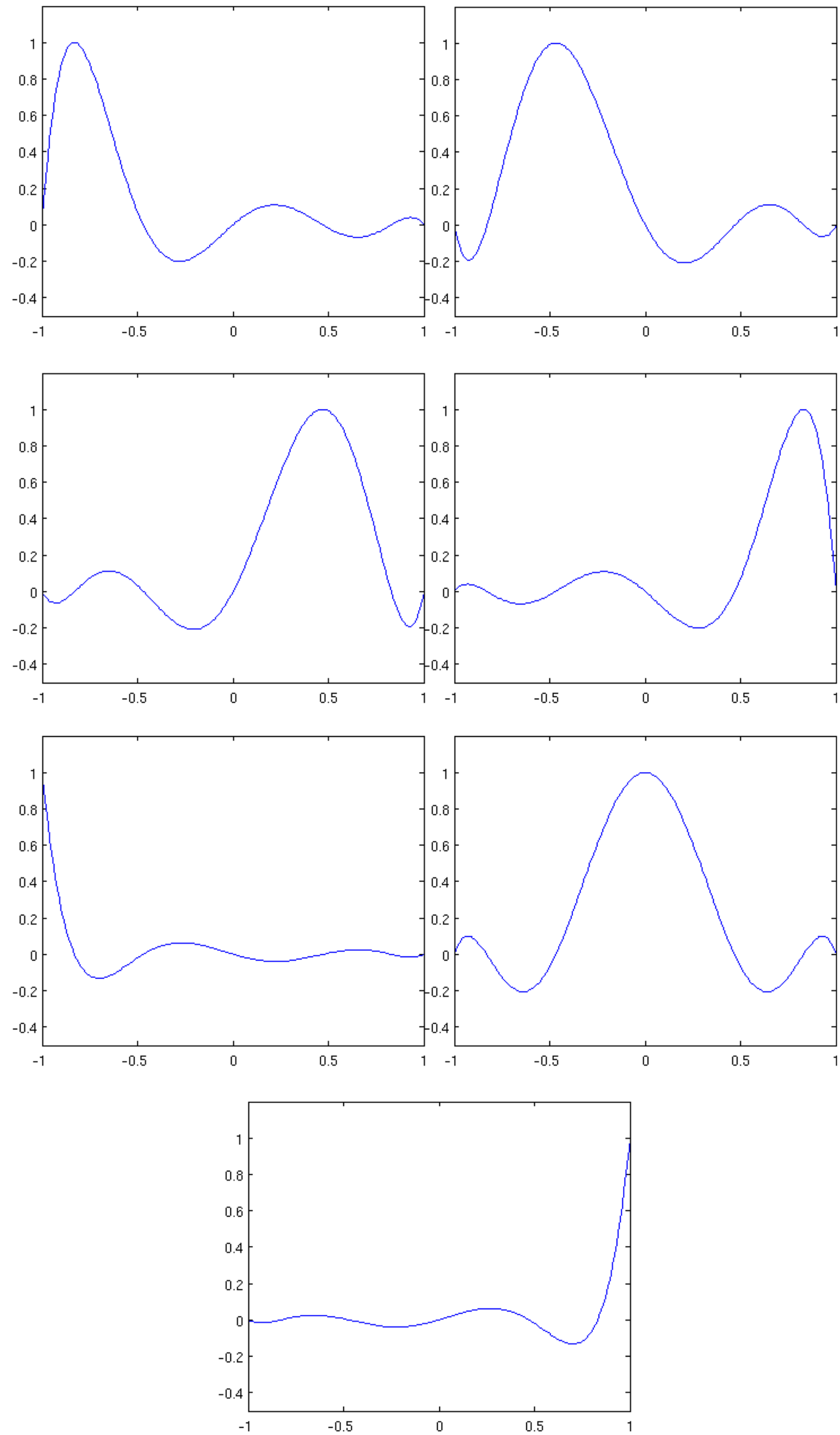


Fig. 12 GLL Lagrange polynomials of degree 6

- SFEM typically utilizes hexahedral grids.
- The mass matrix diagonal is built using Lagrange polynomials with Gauss-Lobato-Legendre (GLL) collocation points.

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