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Entangled rendezvous: a possible application of Bell non-locality for mobile agents on networks

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E-mail: piotr.mironowicz@gmail.com**Keywords:** rendezvous on graphs, quantum entanglement, quantum nonlocality**Abstract**

Rendezvous is an old problem of assuring that two or more parties, initially separated, not knowing the position of each other, and not allowed to communicate, are striving to meet without pre-agreement on the meeting point. This problem has been extensively studied in classical computer science and has vivid importance to modern and future applications. Quantum non-locality, like Bell inequality violation, has shown that in many cases quantum entanglement allows for improved coordination of two, or more, separated parties compared to classical sources. The non-signaling correlations in many cases even strengthened such phenomena. In this work, we analyze, how Bell non-locality can be used by asymmetric location-aware agents trying to rendezvous on a finite network with a limited number of steps. We provide the optimal solution to this problem for both agents using quantum resources, and agents with only ‘classical’ computing power. Our results show that for cubic graphs and cycles it is possible to gain an advantage by allowing the agents to use the assistance of entangled quantum states.

Cooperation and coordination of actions are vivid factors in the achievement of common goals. Nonetheless, we all know the problems that occur in almost any case, if some communication impediments between co-working parties occur, or if the parties are supposed to coordinate their actions but cannot communicate.

Paradigmatic examples of such problems are tasks from the family of rendezvous search [1]. This covers a wide range of situations where several parties (at least two) are bested in some area, each not knowing where the other parties are located, and covet to meet together as soon as possible. In other words, rendezvous covers tasks of arranging a meeting point for two or more agents in some random situation when their communication (and possibly also other resources) are scarce, or even entirely devoided.

The problem has been first stated in 1976 by Steve Alpern [2]. It gained attention in the 1990s, and since then its popularity and importance have increased, together with the whole field of multi-agent distributed coordination, due to the rapid development of real-world implementations of autonomous agents, such as unmanned aerial, ground, or underwater, vehicles (including drones, that are assumed to be able to operate without human intervention), see [3–5] for recent reviews.

The problem of rendezvous belongs to classical computer science and, up to our knowledge, has not been investigated by quantum scientists, and those quantum information researchers who worked with related problems (see section 3) do not refer to Alpern’s and others’ works. One of the intentions of our paper is to bridge the scientists from both fields. For this reason, we provide a brief overview of them, so our results can be understood by classical computer scientists not knowing quantum information, and vice versa.

In recent years it has been shown by multiple examples [6–11] that using quantum resources can enhance solving various classical Information and Communications Technology (ICT) tasks, referred to as *quantum advantage* [12]. Our contribution is in showing that an advantage can be achieved also in the rendezvous search games, where the agents, or *parties*, or *players*, as referred to in the quantum information nomenclature, are allowed to use the assistance of quantum entangled states [13].

The organization of the paper is as follows. In section 1 we overview the current state-of-the-art of rendezvous investigations, and in section 2 provide a brief introduction to the formalism of quantum information used in this work. Then, in section 3 we place our results in the context of rendezvous nomenclature and refer to the inquiries of quantum researchers that are most relevant to the topic of the gathering of many parties. In section 4 we describe the tools specific to our work, and in section 5 provide the results of our calculations. We conclude and provide some lines of further research in section 6.

The work provides multiple new Bell inequalities [14], but it does not aim to develop the theory of quantum nonlocality. The purpose of the paper is to show a new area of its application, that, up to our knowledge, has not been yet investigated from the point of view of quantum entanglement. Thus, this work intends to show how the methods of one rich field of Bell nonlocality, can be applied to support the other rich field of research on the rendezvous problem.

1. Rendezvous search overview

Rendezvous is an example of a family of problems considered in search theory [15–17]. Modern applications of rendezvous search vary from finding a common channel of communication [18], agent recharging in persistent tasks [19], drones and other autonomous vehicles navigation and coordination [20], or finding an astray traveler by a rescue team [21]. The results possibly concern even dating and mating strategies [22].

The rendezvous problem originated from the so-called *search games* proposed by Isaacs in 1965 [23]. In those games, the *searcher* (or *minimizer*) plays a zero-sum game with a *hider* (or *maximizer*), with the payoff given to be the *capture time* required for the searcher to find the hider.

Another related problem given in 1960 by Schelling [24], was a conundrum of two parachutists who land in an area with several landmarks, and no prior agreement regarding where to meet. Schelling's game was a one-shot task, with success occurring when both parachutists decide to go to the same place. Schelling provided also a social experiment indicating that when people hope to gather in one place of a known area and without communication, they are eager to choose to head towards a location (e.g. Grand Central Station) from a small set of places named *focal points* [25–27].

In the earliest formulation of the rendezvous problems, being a cooperation game (with non-zero sum), two players were assumed not to be able to distinguish between themselves, thus the problem was to find a mixed strategy to be played by both parties, designed in such a way, that the expected time required for the parties to meet was as small as possible. Such problems, where both parties are identical, are referred to in the literature as *symmetric rendezvous* problems [2, 28]. When parties are allowed to distinguish between themselves, the problem is called *asymmetric rendezvous*, and the parties (or *agents*) are called *anonymous*, otherwise one say that parties are labeled [29, 30].

The rendezvous problem has many variants. The search space can be e.g. a graph [31] (sometimes called a *network* [32, 33]), a line, a circle, or a plane. A common assumption is that all parties know the structure of the whole search space, but cannot see other parties. Some works allow the parties to know their own location in the search space, in which case the parties are called *location-aware* [34–38], otherwise, the space is called *anonymous* [2, 30, 32].

In symmetric problems, one often tries to find a way to break the symmetry [39], e.g. by using randomness (mixed strategies) [2] or asymmetry of the search space [40] or starting time [41]. For the latter cases, the task is referred to as *deterministic rendezvous* [30, 33].

Some works consider minimization of the expected time to meet [42], or minimization of the maximum time to meet [43]. One may consider a rendezvous game when the parties have bounds on the total distance each of them can travel, like in the situation of traveling in cars with a limited amount of fuel. In case when the fuel limit is large enough that the meeting is sure to occur, then the fuel consumption is attempted to be minimized [44]. Otherwise, when there is a limit on the time or the distance allowed to be covered, some authors [45–48] have investigated the maximization of the probability that the parties will eventually meet, similarly, to the parachutists' problem. Our work is also devoted to a similar problem, with the maximization of success probability.

Other resources considered include memory capacity [40, 49] and computational power [39]. Scenarios with adversaries are also investigated [50]. In some variants more than two parties are considered [51, 52], which is sometimes called *gathering* [53].

When the search space is discrete, the parties may be allowed to move either synchronously [54] or asynchronously [30]. In some models [31] parties can meet not only in nodes, but also when they transpose their positions, i.e. if they respectively occupy nodes v_1 and v_2 in a step t , and nodes v_2 and v_1 in a step $t + 1$.

Additional features that parties are sometimes allowed to use include the possibility of leaving a mark by a party at the starting node [55], marking any node, or even leaving on nodes complex notes using whiteboards, see [33] for a review.

2. Quantum correlations

The formalism of quantum mechanics established near mid of 20th Century [56] defines quantum states as positive semi-definite operators on some Hilbert space \mathcal{H} of trace 1, and the generalized measurements (so-called positive operator-valued measures, or POVMs) as collections of positive semi-definite operators that sum to identity. A state can be shared between many parties, and each party can perform its measurement and get a result a with a probability

$$p(a) = \text{Tr}(\rho M^a), \quad (1)$$

where $\{M^a\}_a$ is the POVM used for the measurement, ρ is a quantum state, and since M^a in the multi-party case acts on a subspace of \mathcal{H} , an implicit identity operator on the rest of \mathcal{H} is used. The POVM properties can be written as $\forall_a M^a \succeq 0$ for positive semi-definiteness, and $\sum_a M^a = \mathbb{1}\mathcal{H}$ for sum to identity. Now, let us consider a scenario with two parties, Alice and Bob, each acting on a subspaces $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$, respectively, $\mathcal{H} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$. Alice can perform one of the measurements from a set X , and obtain a result from a set A^1 with POVMs $\{\{M_x^a\}_a\}_x$, and similarly Bob with settings Y and outcomes B , and POVMs $\{\{N_y^b\}_b\}_y$. Note that M_x^a and N_y^b act on $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$, respectively. The joint probabilities of Alice and Bob are given by

$$P(a, b|x, y) = \text{Tr} \left[\rho \left(M_x^a \otimes N_y^b \right) \right], \quad (2)$$

for all $a \in A$, $b \in B$, $x \in X$, and $y \in Y$. We keep the above nomenclature of *settings* and *outcomes* also for non-quantum probability distributions.

The probability distributions possible to be obtained within quantum mechanics (without communication between Alice and Bob) can be contrasted with probabilities that are possible to reach by parties that do not use quantum resources and are not allowed to communicate. This set is often called *local correlations*; if parties are given some shared random variable the probabilities are called local with a hidden variable, or *LHV correlations*.

To be more specific, a joint distribution $P(a, b|x, y)$ is local if there exist a pair of conditional probability distributions $P^{(A)}(a|x)$ and $P^{(B)}(b|y)$ such that

$$P(a, b|x, y) = P^{(A)}(a|x)P^{(B)}(b|y). \quad (3)$$

The joint distribution is local hidden variable (LHV) if there exist a set Λ , a probability distribution $p(\lambda)$ over $\lambda \in \Lambda$, and a pair of sets of conditional probability distributions $\{P_\lambda^{(A)}(a|x)\}$ and $\{P_\lambda^{(B)}(b|y)\}$ such that

$$P(a, b|x, y) = \sum_\lambda p(\lambda)P_\lambda^{(A)}(a|x)P_\lambda^{(B)}(b|y). \quad (4)$$

Another set of probability distributions that are often considered is the set of non-signaling (NS) probabilities [57]. They are required to satisfy a condition that their local marginals are well defined, i.e.

$$P(a|x, y) \equiv \sum_b P(a, b|x, y) \quad (5)$$

does not depend on y , and the same with the roles of Alice and Bob swapped. This set is the largest allowed by physical theories where there is no instantaneous communication, meaning that there is no way that the choice of the setting of Alice is detected by the probability distribution of outcomes of Bob, and vice versa. On the other hand, the probability of the outcome of one party can depend on the outcome of the other party; this phenomenon is called *steering* [58]. Steering is present also in quantum theory but to a weaker extent [59–62].

Denote by \mathfrak{L} , \mathfrak{S} , \mathfrak{Q} and \mathfrak{N} the sets of all possible joint probability distributions that are local, LHV, quantum, or NS, respectively. Since we usually abstract from the description of how a particular joint probability is realized, as long as it belongs to the set we consider, we refer to it as a *box*. One part of the box is possessed by Alice, the other by Bob, and each party has access to her or his part of the outcome.

A set of coefficient $\mathbb{B} \equiv \{B_{a,b,x,y}\} \subset \mathbb{R}$ applied to a joint distribution as

$$\mathbb{B}[P] \equiv \sum_{a,b,x,y} B_{a,b,x,y} P(a, b|x, y) \quad (6)$$

¹ One may think of it as Alice's measuring device has a knob with positions labeled by elements of X , and display able to print a single character from the set A .

is called a *two-partite game*. One often considers the maximal value of (6) within a given set. In particular, let

$$L \equiv \max_{P \in \mathfrak{P}} \mathbb{B}[P] \quad (7)$$

with $\mathfrak{P} = \mathcal{L}$. It is possible to find games in which there exist $P_Q \in \mathcal{Q}$ such that $\mathbb{B}[P_Q] > L$. Such games are referred to as *Bell operators*, the formula (7) is called a *Bell inequality* and P_Q is said to *violate* the Bell inequality [63]. The upper bound (7) on the value of a game (6) within a given set of joint probabilities \mathfrak{P} is called a *Tsirelon's bound* [64].

A necessary condition for a state ρ shared by parties to allow obtaining a probability distribution that violates a Bell inequality is to be *entangled* [13], that is a quantum property not possible to be reproduced with classical resources no matter with how powerful are computational capabilities of the parties [65].

Optimization of values of games over \mathcal{L} , \mathcal{S} , and \mathcal{Q} is very difficult [66, 67]. For \mathcal{Q} one often consider a superset of joint distributions called macroscopic locality (ML) [68], for that the maximization of (6) can be expressed as a semi-definite program [69], and efficiently computed; in this work, we denote this set as \mathfrak{M} . One can show that

$$\mathcal{L} \subset \mathcal{S} \subset \mathcal{Q} \subset \mathfrak{M} \subset \mathfrak{N}. \quad (8)$$

On the other hand, one can easily see that the maximal value of (7) does not change if maximization is performed over \mathcal{S} instead of \mathcal{L} .

The fact that $\mathcal{S} \subset \mathcal{Q}$ allows for the manifestation of so-called *quantum non-locality*, or *Bell non-locality* [70]. It was first observed in 1935 by Einstein, Podolsky, and Rosen [71], and considered as a caveat of the quantum theory, and later reconsidered by Bell in 1964 [14], who derived operational consequences of this phenomenon, first shown experimentally in the 1980s by Aspect [72, 73], later in a loophole-free realization in series of works in 2015 [74, 75], and in a recent spectacular experiment with entangled photon distributed over satellites [76].

3. Related works and scope

The problem of coordinating actions without communication but with the assistance of entangled quantum states in multi-partite games falls into the general category of quantum pseudo-telepathy [77, 78].

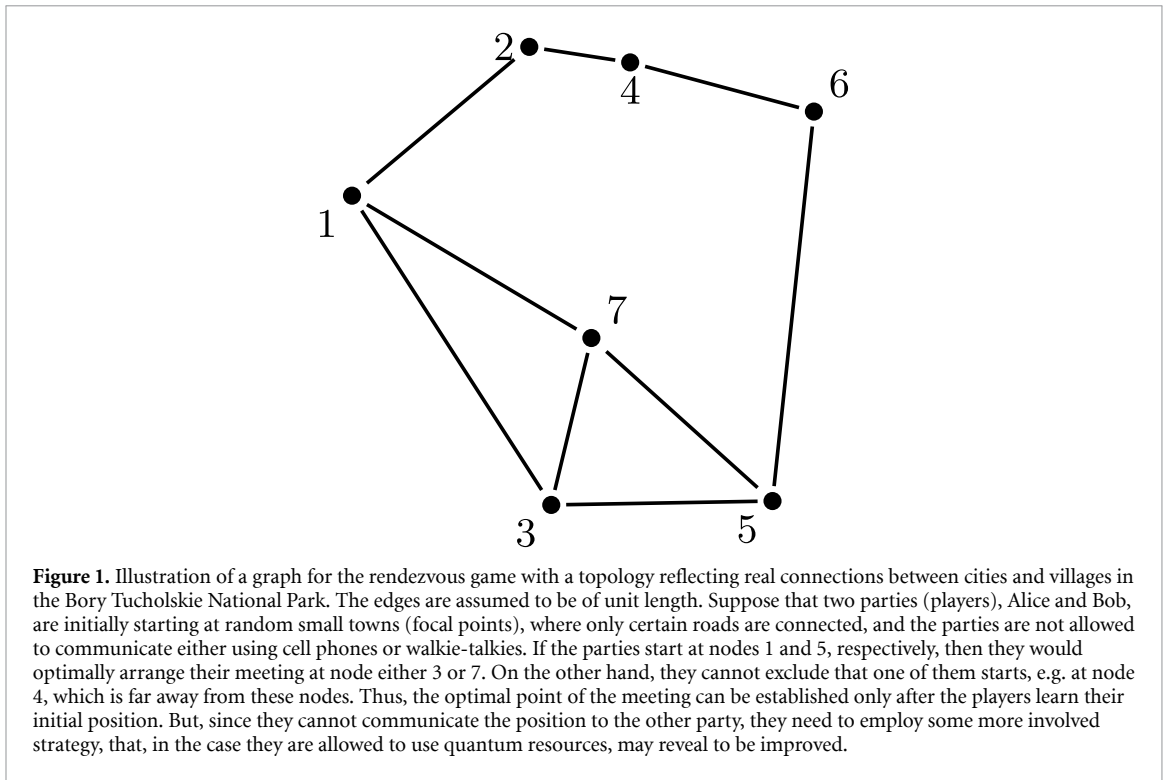
The closest to our work is the paper [79], where Bell-type inequalities were used in the context of arranging the meeting of two parties that were not able to communicate. In the paper Alice and Bob were located on the North and the South Pole and had 6 possible paths to reach the Equator, each path separated by a latitude of 60° . Their task was to use an entangled state to choose their paths so that the latitude difference after reaching the Equator was not greater than 60° . The authors assumed that the parties could not pre-agree on the chosen paths, and they both knew their starting locations of both of them. Thus this was not a rendezvous scenario, and the limit that the parties had fixed known initial locations reduces the applicability of that scenario.

Quantum cooperation between animals has been hypothesized in [80, 81]. In those works, insects are assumed to share multiple copies of maximally entangled states to correlate their probabilistic moves in order to find each other. The insects performed short flights with a support of a presence of a weak scent from the other insect to provide an approximate orientation of where to fly. Their decision to either wait or move was based on the result of a quantum measurement. A similar approach was applied to ants coordinating their moves when pushing a pebble [82–84].

In [85, 86] a quantum communication between robots, where a leading robot transmits entangled states to follower robots to communicate information about cooperative tasks, is considered. In [85] collaborating robots perform subsequent quantum measurements on multiple shared entangled states, where measurements depend on the direction of the move to be chosen, and the result is used to determine whether to perform the action, but the current location is not taken into account in the measurement, and the direction prediction is based on classical computation. In [86] quantum channels were used to enhance security using a variant of BB84 cryptographic protocol [87] and speed of communication, but no Bell-type nonlocality was analyzed; see [88] for the experimental realization of that concept. In [89, 90] quantum teleportation protocol [91] for safe communication between robots was implemented.

In [92] an extensive survey of the current application of quantum technologies for drones is given.

In this work, we consider a rendezvous problem over graphs. The two parties are asymmetric, move synchronously, are location-aware, and have a step number limit. An example of such a game is given in figure 1.



4. Methods

The figure of merit considered in this paper is the success probability that two parties, Alice and Bob, will meet on a known graph \mathcal{G} in a given number of steps N_{\max} when their initial positions are not known *a priori* and are uniformly randomized before the first step. This refers to a real-world situation when the graph models a structure of focal points in a given area, and parties pre-agreed that they wait for each other at full hours. Let N denote the number of nodes in the graph.

We consider this problem with the following attributes:

- The parties may have a different maximal allowed number of steps,
- The parties may or may not be allowed to choose not to move,
- The parties may or may not be allowed to meet on edges (upon nodes transposition),
- The parties may or may not start at the same node.

In the models that we consider, the two parties move synchronously, so the first attribute effectively means a time limit, so parties have to meet before the deadline.

4.1. Rendezvous problems as Bell operators

Suppose we are given a graph \mathcal{G} with N nodes, and a positive integer N_{\max} specifying the number of time steps allowed for the parties to be taken to win the game.

For the sake of simplicity, let us consider a rendezvous game involving only two parties (players), Alice and Bob. The considerations can be easily extended to more parties. The players are placed in two randomly selected vertices of \mathcal{G} , with distribution $p_{XY}(x, y)$, where x and y are vertices of \mathcal{G} .

During the game, the players proceed as follows. They change their vertex position in \mathcal{G} , in a synchronous manner, to another vertex, if the corresponding vertices are adjacent to each other (a player can stay at the same vertex if there is a self-loop; this is also considered as a move) and consume a time step for each movement. Players win if they occupy the same vertex within the possible N_{\max} moves in the same time step. Let us call the conditions of winning the *meeting conditions*.

Within the paper, we consider some possible variations of the game. One of them is if we add a rule that the players win not only if they are located at the same node, but as well if they exchange (permute) their position in the same time step. This reflects a situation in which they meet in the edge that connects the neighboring vertices. We refer this as a rule of *meet on edges*. In case, when this rule is applied, we consider it as a part of the meeting conditions.

Obviously, the rendezvous problem would be trivially solved if players could communicate and inform the other side of their starting position. However, any form of communication is forbidden by the rules of the rendezvous game, when the players are placed on the graph \mathcal{G} . Thus, they only have to decide their strategy based on their original position on the graph, without knowing which vertex is the starting point of the other player.

It can be observed, that the rendezvous game is a particular example of a non-local game [70, 93, 94]. Indeed, for a given graph \mathcal{G} and step number limit N_{\max} , each player receives as input from the referee a random element in $\{1, \dots, N\}$ specifying his initial vertex position. Afterward, each player attempts at deriving a path $\mathcal{P}^{(p)} = (v_0, \dots, v_{N_{\max}})$, $p = A, B$, in \mathcal{G} , such that node v_0 in the path is the assigned starting vertex. The players win if their exit paths meet the above meeting conditions, otherwise, they lose.

Players are aware of \mathcal{G} , N_{\max} and game rules and may communicate in advance to coordinate strategy, but communication is prohibited once the game has started, *viz.* after the players received their inputs (i.e. learn their locations). As a non-local game, the average probability of success in the game can then be formulated as a Bell-type functional of the joint probabilities. Thus, the optimal such probability is determined by the maximum value of this function in the set of allowed strategies, like the sets \mathcal{L} , \mathcal{S} , \mathcal{Q} , \mathcal{M} , and \mathcal{N} defined in section 2.

For instance, when the allowed strategies are given by the set \mathcal{L} , the ‘classical’ players are playing the rendezvous game, i.e. they choose their strategies based solely on their inputs and given set of instructions, and are bound by local hidden variable theories. On the other hand, the quantum players decide their moves (or, the paths $\mathcal{P}^{(p)}$) after obtaining some outcomes from a local measurement on an entangled quantum state, giving the set of strategies \mathcal{Q} . Since \mathcal{Q} is strictly larger than \mathcal{L} , one may expect that the average winning probability the quantum players achieve can be larger than the one possible for the classical players.

In the rest of this paper, we consider the distribution p_{XY} to be either uniform over all pairs of x and y , or uniformly with $p_{XY}(x, x) = 0$. In the former, the case when the players are placed on the same vertex results in a trivial win.

4.2. Construction of Bell operators

We consider cases when a party is allowed to decide to wait (i.e. to not move); this is modeled by considering a reflexive graph, and by an anti-reflexive graph in the opposite case. This refers to a situation when it is physically not possible for a party to stop for a longer time, e.g. when there are no parking places, or when high inertia forces the vehicle to move continuously.

Meeting on the edges reflex possibility that there is only one physical way directly connecting the pair of neighboring focal points, thus when the parties transpose these nodes, then they will meet on their way. Further, we denote the boolean variable indicating if the parties can meet on edges by E .

The last attribute, regarding the possibility of starting in the same node, is to be understood in a way that parties do not perform checking if the other party is present at the same place in step 0. The real-world interpretation is that parties may start with a slight time shift, so they are not present at exactly the same time 0, or that they start in a vicinity of a focal point, so may not notice each other. Note that if the parties were allowed to start at the same place and check rendezvous immediately, then all success probabilities would simply be rescaled, and this would not differentiate different rendezvous strategies and resources. The boolean variable to indicate if the parties can be randomized to start in the same node is denoted further as S .

If the number N_{\max} is equal to or greater than the diameter of the considered graph, and the parties are either allowed to wait or meet on edges, then the success probability is always 1. On the other hand, if the parties do not possess these capabilities, then for some starting locations, when the distance between them is odd, it may not be possible to rendezvous, no matter how many steps are performed.

We model the strategy to be used by Alice and Bob using a joint probability distribution $P(a, b|x, y)$, with $x, y \in \{1, \dots, N\}$ and $a \in A$ and $b \in B$ for some A and B . Here A and B may depend on x and y , respectively. This joint distribution should be understood in the following way. In their time 0 both parties input as settings their starting locations to their parts of the box, and then they proceed in a way resulting from some deterministic function of their respective outcomes.

The evaluation of the success probability of a rendezvous problem is a particular case of the evaluation of a Bell game (6) for the used box. Let us denote by $\mathcal{M}[\mathcal{P}, \mathcal{P}']$ a predicate function that for given two paths, \mathcal{P} , and \mathcal{P}' , has value 1 if the paths satisfy the meeting conditions for the game, and value 0 otherwise.

To be more precise, for given $\mathcal{P} = (v_0, \dots, v_{N_{\max}})$ and $\mathcal{P}' = (v'_0, \dots, v'_{N_{\max}})$ we have $\mathcal{M}[\mathcal{P}, \mathcal{P}'] = 1$ if one of the following hold:

- (a) there exist $i \in \{0, \dots, N_{\max}\}$ such that: $v_i = v'_i$;
- (b) the parties can meet on edges ($E = 1$), and there exist $i \in \{1, \dots, N_{\max}\}$ such that: $v_i = v'_{i-1}$ and $v_{i-1} = v'_i$.

Algorithm 1. Get a rendezvous game (a Bell operator) from a graph adjacency list

Input: graph adjacency list L , N_{\max} , boolean for allowed edge mitting E , boolean for allowed same setting S
Output: $\{B_{a,b,x,y}\}$ (coefficients of a Bell-type game)

1. **function** GETGAME (L , N_{\max} , E , S)
2. $N \leftarrow$ number of vertices in the graph
3. $R \leftarrow$ degree of vertices in the graph
4. **if** S **then** \triangleright average over settings
5. $p \leftarrow 1/N^2$
6. **else**
7. $p \leftarrow 1/(N \cdot (N - 1))$
8. **end if**
9. **for** $a \leftarrow 1$ to $R^{N_{\max}}$ **do**
10. **for** $b \leftarrow 1$ to $R^{N_{\max}}$ **do**
11. **for** $x \leftarrow 1$ to N **do**
12. **if** S **then**
13. $\tilde{Y} \leftarrow \{1, \dots, N\}$
14. **else**
15. $\tilde{Y} \leftarrow \{1, \dots, N\} \setminus x$
16. **end if**
17. **for** $y \in \tilde{Y}$ **do**
18. $p_A \leftarrow x$ \triangleright current positions
19. $p_B \leftarrow y$
20. **for** $s \leftarrow 1$ to N_{\max} **do** \triangleright steps
21. $n_A \leftarrow L(p_A, a_s)$ $\triangleright a_s$ -th edge of p_A
22. $n_B \leftarrow L(p_B, b_s)$ $\triangleright b_s$ -th edge of p_B
23. **if** $n_A = n_B$ or (E and $p_A = n_B$ and $p_B = n_A$) **then** \triangleright meet or transpose
24. $B_{a,b,x,y} \leftarrow p$
25. **end if**
26. $p_A \leftarrow n_A$ \triangleright move to next positions
27. $p_B \leftarrow n_B$
28. **end for**
29. **end for**
30. **end for**
31. **end for**
32. **end for**
33. **return** $\{B_{a,b,x,y}\}$
34. **end function**

Otherwise, we have $\mathcal{M}[\mathcal{P}, \mathcal{P}'] = 0$.

For a given node v , let $\mathbf{P}(v) = \{\mathcal{P}_1, \dots, \mathcal{P}_{n(v)}\}$ denote the set of all paths of length N_{\max} starting at that node, where $n(v)$ is the number of such paths for the node v . In the boxes $P(a, b|x, y)$ that we consider for the graph \mathcal{G} the values of x and y belongs to $\{1, \dots, N\}$, and the values of a and b for given x and y belongs to $A = \mathbf{P}(x)$ and $B = \mathbf{P}(y)$, respectively. The game success probability (6) is then given by

$$\sum_{x,y} \sum_{a \in \mathbf{P}(x)} \sum_{b \in \mathbf{P}(y)} p(x, y) \cdot \mathcal{M}[a, b] \cdot P(a, b|x, y). \quad (9)$$

For instance, Alice starting at the node 1 of the graph given in figure 1 and $N_{\max} = 0$ would obtain from her measurement one of the five outcomes from the set $A = \mathbf{P}(1)$, viz. $a = \mathcal{P}^A$ equal to either (1, 3, 5), or (1, 3, 7), or (1, 7, 5), or (1, 7, 3), or (1, 2, 4).

Note, that the boxes are here outputting the whole paths. For the sake of simplicity, further, in this paper, we consider only regular graphs of degree R , and $A \cong B \cong \{1, \dots, R^{N_{\max}}\}$ for all x and y , and the outcome of Alice a directly determines the sequence of N_{\max} numbers $(a_s)_s$, each in $\{1, \dots, R\}$, and each a_s determines towards which edge of the currently occupied node in the step s to head in the next $s + 1$ step; the same for Bob. The formalism easily generalizes to non-regular graphs.

We also note, that the input of Alice is her starting position, and similarly the input of Bob is his starting position. The form of the measurement chosen by each party depends on the input provided and since the parties are spatially separated their measurements commute. To avoid confusion, we stress that the position is the input, not the result of the measurement.

The game (9) for the regular graphs can be equivalently obtained using the algorithm 1.

4.3. Evaluation of Bell operators within different theories

To evaluate the gain possible to be obtained when the parties are assisted with a quantum entangled state we calculated the Tsirelon's bound for Alice and Bob using different classes of resources, *viz.* with boxes from sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} . To this end, we wrote MATLAB scripts with crucial parts implemented in C++ using the MEX technology.

For maximization over \mathfrak{S} we enumerated all possible deterministic strategies. For a single party there are $R^{N_{\max}}$ outcomes and N settings, meaning $R^{N \cdot N_{\max}}$ deterministic functions $\{1, \dots, N\} \rightarrow \{1, \dots, R^{N_{\max}}\}$. Thus, for two asymmetric parties, we needed to consider the game value for $R^{2N \cdot N_{\max}}$ deterministic boxes.

A lower bound for what can be obtained using quantum resources, we searched for explicit state ρ , and measurements $\{\{M_x^a\}_a\}_x$, and $\{\{N_y^b\}_b\}_y$ on fixed Hilbert spaces $\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$. We employed the see-saw optimization technique [95], where one intertwines linear optimizations separately over each variable in the non-linear formulae (2). This provides only a lower bound, as the method is not guaranteed to reach the global optimum, yet it suffices to show the advantage in many cases. Still, one should note that the more complicated the case, *i.e.* the more settings and outcomes and the larger dimension used, the smaller are chances that the method yields the maximal value.

An upper bound for the gain from using the quantum resources is got from the maximization over the set \mathfrak{M} of the ML correlations mentioned above. A direct numerical method to perform this optimization is the Navascués–Pironio–Acin [66, 67] semi-definite programming method. These correlations may not be possible to be realized in practical implementations, but determines the scope of what possibly could be reached.

Finally, we also derived the NS bound of the set \mathfrak{N} using linear programming. This value has (almost [96]) no practical application, but provides insight regarding the meaning of quantum limits on the rendezvous tasks.

Below we use the following notation for box outcomes. We assume the edges of the graphs to be labeled from 1 to the vertex's degree in such a way that the labels of the vertices towards the relevant edges are connecting increase with the edge's number. For instance, if the vertex 3 has edges leading to vertices 1 and 5, and an edge towards itself (*e.g.* in reflexive graphs), then these edges will have labels 1, 3, and 2, respectively.

5. Results

We performed the analysis over three sets of graphs of small size: cubic (3-regular) graphs with 6 or 8 nodes, see section 5.1 and Listing 1; cycles over at most 9 nodes with step number limit 1, see section 5.2, and 2, see section 5.3; and directed cycles, see section 5.4. In the cases when the parties are allowed to choose not to move, we used reflexive versions of these graphs. In section 5.5 we analyze the robustness and experimental feasibility of our results.

5.1. Cubic graphs with single step

Listing 1. Adjacency lists of cubic graphs with 6 or 8 nodes

```

cubic-2 = [2 3 4; 1 3 5; 1 2 6; 1 5 6;
           2 4 6; 3 4 5]
cubic-3 = [4 5 6; 4 5 6; 4 5 6; 1 2 3;
           1 2 3; 1 2 3]
cubic-4 = [3 5 7; 4 6 8; 1 5 7; 2 6 8;
           1 3 7; 2 4 8; 1 3 5; 2 4 6]
cubic-5 = [2 5 6; 1 3 6; 2 4 7; 3 5 8;
           1 4 8; 1 2 7; 3 6 8; 4 5 7]
cubic-6 = [2 3 4; 1 3 6; 1 2 8; 1 5 7;
           4 6 8; 2 5 7; 4 6 8; 3 5 7]
cubic-7 = [2 4 5; 1 3 6; 2 4 7; 1 3 8;
           1 6 8; 2 5 7; 3 6 8; 4 5 7]
cubic-8 = [2 6 7; 1 3 7; 2 4 7; 3 5 8;
           4 6 8; 1 5 8; 1 2 3; 4 5 6]
cubic-9 = [2 5 8; 1 3 6; 2 4 7; 3 5 8;
           1 4 6; 2 5 7; 3 6 8; 1 4 7]

```

Let us consider the case with single step restriction, *i.e.* $N_{\max} = 1$. For cubic graphs, we consider only a single-step case due to numerical complexity for cases with more vertices.

For the 6 node cubic graphs, labeled in Listing 1 as 2 and 3, we observed no gain if the parties were not allowed to meet, and some gain in the opposite case when quantum resources were used. The gains for ML and NS correlation were, on contrary, very strong, up to 28.6%. A similar gain in the case with allowed

Table 1. Comparison of the efficiency of sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} , for selected cubic graphs, when parties are allowed to wait and $N_{\max} = 1$. In one case the see-saw method failed to find a reliable result.

Set	E	S	cubic-2	cubic-3	cubic-6	cubic-9
\mathfrak{S}	0	0	0.466 67	0.466 67	0.321 43	0.321 43
\mathfrak{Q}	0	0	0.466 76	0.466 76	0.336 56	0.351 01
\mathfrak{M}	0	0	0.500 14	0.500 14	0.3587	0.3585
\mathfrak{N}	0	0	0.6	0.6	0.428 57	0.428 57
\mathfrak{S}	1	0	0.466 67	0.466 67	0.357 14	0.321 43
\mathfrak{Q}	1	0	0.470 72	0.469 78	fail	0.35101
\mathfrak{M}	1	0	0.503 56	0.512 87	0.362 68	0.361 08
\mathfrak{N}	1	0	0.6	0.6	0.428 57	0.428 57
\mathfrak{S}	0	1	0.555 56	0.555 56	0.406 25	0.406 25
\mathfrak{Q}	0	1	0.555 64	0.555 64	0.418 31	0.432 14
\mathfrak{M}	0	1	0.575 79	0.575 79	0.436 25	0.436 51
\mathfrak{N}	0	1	0.666 67	0.666 67	0.5	0.5
\mathfrak{S}	1	1	0.555 56	0.555 56	0.406 25	0.406 25
\mathfrak{Q}	1	1	0.558 57	0.558 19	0.417 26	0.432 14
\mathfrak{M}	1	1	0.577 43	0.583 52	0.437 12	0.436 65
\mathfrak{N}	1	1	0.666 67	0.666 67	0.5	0.5

Table 2. Comparison of the efficiency of sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} , for cubic graphs with 8 nodes when parties are not allowed to wait and $N_{\max} = 1$. In two cases the see-saw method failed to find a reliable result.

Set	E	S	cubic-4	cubic-5	cubic-6	cubic-7	cubic-8	cubic-9
\mathfrak{S}	0	0	0.214 29	0.25	0.25	0.214 29	0.214 29	0.25
\mathfrak{Q}	0	0	0.228 57	fail	0.253 03	0.228 57	0.228 57	0.265 46
\mathfrak{M}	0	0	0.2381	0.254 62	0.258 93	0.2381	0.244 78	0.267 49
\mathfrak{N}	0	0	0.285 71	0.285 71	0.285 71	0.285 71	0.285 71	0.285 71
\mathfrak{S}	1	0	0.285 71	0.285 71	0.285 71	0.285 71	0.285 71	0.285 71
\mathfrak{Q}	1	0	0.333 33	0.320 87	0.313 38	0.333 33	0.333 33	0.307 64
\mathfrak{M}	1	0	0.333 33	0.330 63	0.326 51	0.333 33	0.333 33	0.329 51
\mathfrak{N}	1	0	0.428 57	0.428 57	0.428 57	0.428 57	0.428 57	0.428 57
\mathfrak{S}	0	1	0.3125	0.343 75	0.343 75	0.3125	0.3125	0.343 75
\mathfrak{Q}	0	1	0.322 53	fail	0.346 04	0.322 53	0.322 52	0.357 28
\mathfrak{M}	0	1	0.325	0.347 34	0.351 56	0.325	0.330 98	0.358 98
\mathfrak{N}	0	1	0.375	0.375	0.375	0.375	0.375	0.375
\mathfrak{S}	1	1	0.375	0.375	0.375	0.375	0.375	0.375
\mathfrak{Q}	1	1	0.398 15	0.388 84	0.375 68	0.398 15	0.398 15	0.382 99
\mathfrak{M}	1	1	0.4	0.396 79	0.383 32	0.4	0.4	0.385 35
\mathfrak{N}	1	1	0.5	0.5	0.5	0.5	0.5	0.5

waiting was present also for cubic graphs labeled 6 and 9, and here the gain with quantum resources was up to 9.2%, and with NS was up to 33.3%; see table 1. The cubic graphs 6 and 9 showed also a quantum gain when waiting is not allowed.

In contrast to 6 node cubic graphs, for $N_{\max} = 1$ all 8 node cubic graphs showed an advantage when the parties were not allowed to wait, see table 2. The gains are largest both in quantum and NS cases when parties were not possible to be randomized to start at the same position and were allowed to meet on edges ($S = 0$ and $E = 1$); the gains were smallest in the opposite case. The quantum gain varied between tiny 0.2% and 16.7%, whereas the NS gain was between 9.1% and 50%.

5.2. Cycle graphs with single step

The cycle graphs when $N_{\max} = 1$, there was no gain in most of the cases when $S = 0$, with exceptions for the cycle over 4, 5, 7, and 8 nodes with the possibility of waiting and without edge meeting, see table 3.

Let us take a look at strategies optimal for the cycle over 4 nodes in this scenario. It reveals that the optimal strategy is for both parties to take the edge that leads to the node with the smallest number, i.e. to move from nodes 1, 2, 3, and 4 towards nodes 1 (wait), 1, 2 and 1, respectively, cf [34]. Direct calculation shows that they succeed in 50%, viz. whenever their starting positions were given by pairs (1, 2), (1, 4), (2, 1),

Table 3. Comparison of the efficiency of sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} , for cycle graphs with 4, 5, 7, and 8 nodes, when the parties are allowed to wait, no edge meeting is allowed ($E = 0$), parties are always randomized at different initial positions ($S = 0$), and $N_{\max} = 1$.

Set	Cycle 4	Cycle 5	Cycle 7	Cycle 8
\mathfrak{S}	0.5	0.4	0.285 71	0.25
\mathfrak{Q}	0.533 33	0.4117	0.307 09	0.265 46
\mathfrak{M}	0.555 56	0.428 88	0.308 96	0.267 48
\mathfrak{N}	0.666 67	0.5	0.333 33	0.285 71

Table 4. Comparison of the efficiency of sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} , for cycle graphs, when parties are not allowed to wait and are possibly randomized at the same initial positions ($S = 1$), and $N_{\max} = 1$.

Set	E	Cycle 3	Cycle 4	Cycle 5	Cycle 6	Cycle 7	Cycle 8	Cycle 9
\mathfrak{S}	0	0.555 56	0.5	0.36	0.277 78	0.265 31	0.25	0.209 88
\mathfrak{Q}	0	0.583 33	0.5	0.3809	0.291 67	0.278 64	0.25	0.218 87
\mathfrak{M}	0	0.583 33	0.5	0.3809	0.291 67	0.278 64	0.25	0.218 87
\mathfrak{N}	0	0.666 67	0.5	0.4	0.333 33	0.285 71	0.25	0.222 22
\mathfrak{S}	1	0.777 78	0.625	0.44	0.388 89	0.346 94	0.3125	0.259 26
\mathfrak{Q}	1	0.833 33	0.625	0.45	0.416 67	0.365 96	0.3125	0.277 78
\mathfrak{M}	1	0.833 33	0.625	0.45	0.416 67	0.365 96	0.3125	0.277 78
\mathfrak{N}	1	1	0.75	0.6	0.5	0.428 57	0.375	0.333 33

(2, 4), (4, 1), or (4, 2), and that they fail if these were (1, 3), (2, 3), (3, 1), (3, 2), (3, 4), or (4, 3). The quantum box is tedious to be written explicitly, and the NS is given by

$$\forall_{a,b,x} P(a, b, x, x) = 1/9, \tag{10a}$$

$$\begin{aligned} \forall_a P(a, a, 2, 1) = P(a, a, 4, 1) = P(a, a, 1, 2) = \\ P(a, a, 4, 3) = P(a, a, 1, 4) = P(a, a, 3, 4) = 1/3, \end{aligned} \tag{10b}$$

$$\forall_{a,b} P(a, b, 3, 1) = P(a, b, 1, 3) = B_1(a, b), \tag{10c}$$

$$\forall_{a,b} P(a, b, 2, 4) = P(a, b, 4, 2) = B_2(a, b), \tag{10d}$$

$$\begin{aligned} P(3, 1, 3, 2) = P(1, 2, 3, 2) = P(2, 3, 3, 2) = \\ P(2, 1, 2, 3) = P(3, 2, 2, 3) = P(1, 3, 2, 3) = 1/3, \end{aligned} \tag{10e}$$

where $B_1(2, 1) = B_1(1, 2) = B_1(3, 3) = 1/3$ and $B_2(1, 1) = B_2(3, 2) = B_2(2, 3) = 1/3$, and all remaining values are equal 0.

Now, we consider the case when the parties may be initially located at the same initial position, i.e. in a vicinity of the same focal point, $S = 1$. In this case, we observed a gain in the majority of the cases with cycle graphs.

The results for the case with anti-reflexive graphs, i.e. when parties are not allowed to wait, are shown in table 4. For cycles of length 4 and 8, there was no quantum gain, and for the other case, the quantum gain varies between 2.3% and 7.1%, whereas NS gains are within the range from 5.9% to 36.4%.

The cases of length 4 and 8 are interesting, as the NS advantage is possible there even though quantum and ML sets yield the same value as LHV. Similarly, as in the case $S = 0$ described above, the optimal LHV is for both parties to move toward the nodes with a smaller number; anti-reflexivity implies that a party from node 1 should move to node 2. For the NS set the optimal strategy, cf (10), is to use a box with $P(a, \neg a|x, y) = 0.5$ for $(x, y) \in \{(1, 4), (4, 1), (2, 3), (3, 2)\}$, and with $P(a, a|x, y) = 0.5$ for other pairs of settings.

For the case when the parties are allowed to wait and are possibly starting at the same position, the results are shown in table 5, and the gain is observed, similarly as for the case when the parties always start at a different position, cf table 3, only for cycles over 4, 5, 7 and 9 nodes. The quantum gain is between 1.8% and 4.8%, and the NS gain is between 9.1% and 20%. Note that for the cycle over 4 nodes and $E = 1$ the LHV and NS values are the same no matter if the parties are not allowed to wait or not, whereas quantum and ML gains become possible in this case when the parties are allowed to wait; this means that the quantum advantage comes from a possibility to better coordinate which party should be waiting.

Table 5. Comparison of the efficiency of sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} , for cycle graphs 4, 5, 7, and 8, when parties are allowed to wait and are possibly randomized at the same initial positions ($S = 1$), and $N_{\max} = 1$.

Set	E	Cycle 4	Cycle 5	Cycle 7	Cycle 8
\mathfrak{S}	0	0.625	0.52	0.387 76	0.343 75
\mathfrak{Q}	0	0.645 06	0.529 36	0.406 07	0.357 28
\mathfrak{M}	0	0.65	0.540 07	0.407 19	0.358 98
\mathfrak{N}	0	0.75	0.6	0.428 57	0.375
\mathfrak{S}	1	0.625	0.52	0.387 76	0.343 75
\mathfrak{Q}	1	0.648 72	0.531 29	0.406 31	0.357 45
\mathfrak{M}	1	0.654 91	0.540 16	0.407 74	0.3591
\mathfrak{N}	1	0.75	0.6	0.428 57	0.375

Table 6. Comparison of the efficiency of sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} , for cycle graphs 5, 6, 7, and 8, when parties are not allowed to wait (anti-reflexive graphs) and are possibly randomized at the same initial positions ($S = 1$), and $N_{\max} = 2$.

Set	E	Cycle 5	Cycle 6	Cycle 7	Cycle 8
\mathfrak{S}	0	0.52	0.5	0.387 76	0.3125
\mathfrak{Q}	0	0.522 34	0.5	0.387 76	0.3125
\mathfrak{M}	0	0.550 13	0.5	0.412 73	0.345 06
\mathfrak{N}	0	0.6	0.5	0.428 57	0.375
\mathfrak{S}	1	0.84	0.722 22	0.591 84	0.5
\mathfrak{Q}	1	0.892 71	0.722 22	0.591 84	0.5
\mathfrak{M}	1	0.900 76	0.75	0.624 78	0.531 78
\mathfrak{N}	1	1	0.833 33	0.714 29	0.625

5.3. Cycle graphs with two steps

For the cycle over 4 vertices, when the time limit of the parties is two, $N_{\max} = 2$, the parties can trivially meet simply by pre-agreeing about the focal point (if waiting is allowed). In general, this is true for any graphs having a radius less or equal to N_{\max} . On the other hand, the complexity of calculating the capabilities of boxes from each set \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} grows exponentially with N_{\max} , with base R (the degree of nodes in the graph) as noted in section 4. For these reasons we consider only the case when the parties are not allowed to wait (i.e. anti-reflexive graphs), and cycle graphs of sizes 5, 6, 7, and 8.

The calculations revealed that a gain is obtained only when the parties can be randomized to the same starting location, $S = 1$. The results are shown in table 6.

Interestingly, for cycles over 5, 6, and 7 nodes the value obtained with see-saw for the quantum set \mathfrak{Q} is exactly equal to the LHV value, whereas the ML and NS values are greater than LHV. The result has been confirmed by subsequent reevaluations of the see-saw with different initializations, but still provides only a lower bound, thus the actual quantum capability can be larger.

A quantum gain is possible for the 5 node cycle both in cases with edge meeting allowed, $E = 1$, and not, $E = 0$. We note that the success probability in NS with $E = 1$ is 1, meaning that there exists an NS box that always shows to the parties the proper direction so that they approach each other. Below we analyze this case in more detail.

For Alice starting in node 1 and Bob starting in node 5, an NS box assures that:

- (a) If Bob visits the node 1 and then the node 2, then Alice visits either nodes 2 and then 1, or 5 and then 1 or 5 and then 4; thus they will either meet transposing nodes 1 and 2 in the first case, or 1 and 5 in the second and third case.
- (b) If Bob visits the node 1 and then the node 5, then Alice visits either node 5 and then 1, or 5 and then 4; in both cases, they will meet transposing nodes 1 and 5.
- (c) If Bob visits node 4 and then the node 3, then Alice visits nodes 2 and then 3; they will meet after two steps at node 3.
- (d) If Bob visits the node 4 and then the node 5, then Alice visits nodes 5 and 4; they will meet after two steps transposing nodes 4 and 5.

For LHV boxes, an optimal strategy is to always choose the node with a lower labeling number. For the initial positions 1 and 5 this behaves as the first option for the described NS box, i.e. Bob visits nodes 1 and 2, and Alice visits nodes 2 and 1, and they meet on transposing nodes 1 and 2. For the quantum box, we have two possibilities for Bob, and to both of them Alice can react in two ways:

Table 7. Success probabilities when parties are located at random places with $S = 1$ (allowed the same starting node for both parties) of a reflexive directed cycle graph for a two-step limit $N_{\max} = 2$.

Set	Dir. cycle 4	Dir. cycle 5	Dir. cycle 6	Dir. cycle 7	Dir. cycle 8
\mathfrak{S}	0.625	0.52	0.5	0.387 76	0.343 75
\mathfrak{Q}	0.676 78	0.52	0.5	0.390 44	0.347 17
\mathfrak{M}	0.690 12	0.550 13	0.5	0.412 73	0.361 4
\mathfrak{N}	0.75	0.6	0.5	0.428 57	0.375

- When Bob visits the node 1 and then the node 5, then Alice with *low* probability visits 2 and then 3 or with *high* probability visits 5 and then 1; thus with *high* probability they meet transposing nodes 1 and 5 after one step.
- When Bob visits the node 4 and then the node 3, then Alice with *high* probability visits 2 and then 3 or with *low* probability visits 5 and then 1; thus with *high* probability they meet at the node 3 after two steps.

Whereas for the pair of settings 1 and 5 both LHV and NS strategies win with certainty, and the quantum strategy wins with high probability, the LHV fails e.g. for the pair of settings 1 and 4. On contrary, for these settings, the NS box satisfies:

- If Bob visits the node 3 and then the node 2, then Alice visits nodes 2 and then 3; the parties will meet transposing nodes 2 and 3.
- If Bob visits node 3 and then the node 4, then Alice visits nodes 5 and then 4; the parties will meet at node 4.
- If Bob visits the node 5 and then the node 1, then Alice visits either nodes 2 and then 1, or 5 and then 1, or 5 and then 4; they will meet either after two steps at the node 1 in the first case or after one step at the node 5 in the second and third case.
- If Bob visits the node 5 and then the node 4, then Alice visits either node 5 and then 1, or 5 and then 4; in both cases, they meet in the first step at the node 5.

This ensures that the NS box wins with a probability of 1. For the quantum box, similarly, as in the previous case we have the following possibilities:

- When Bob visits the node 3 and then the node 2, then Alice with *high* probability visits 2 and then 3 or with *low* probability visits 5 and then 1; thus with *high* probability they meet transposing nodes 2 and 3 after two steps.
- When Bob visits the node 5 and then the node 4, then Alice with *low* probability visits 2 and then 3 or with *high* probability visits 5 and then 1; thus with *high* probability, they meet at the node 5 after one step.

The quantum box that we consider provides success for any pair of settings with probabilities between 0.83 and 1. We observe that the probability 1 is obtained exactly for these settings where $y = x + 1$, with $6 \equiv 1$, i.e. for $(x, y) \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$.

5.4. Directed cycles with two steps

As the last example, we consider reflexive directed cycle graphs and a two-step limit, $N_{\max} = 2$. The reflexive directed cycle graphs over N nodes has edges of the form (i, i) and $(i, i + 1)$, where $i \in \{1, \dots, N\}$, and $N + 1 \equiv 1$. This means that from each node a party can either wait or move in the node-increasing number direction.

By using these graphs we reduce the complexity of the problems to be analyzed, *viz.* for $N_{\max} = 2$ a box has to output one of only four possible outcomes, that are further interpreted as instructions for two subsequent moves of the party. The formalism remains the same as in the previous sections.

We note that in this case there is no possibility for edge meeting, so cases with $E = 0$ and $E = 1$ are trivially equivalent. Recall that in the ordinary, bidirectional, cycles, in most of the cases with $S = 0$ we observed no quantum advantage. In the directed cycles we observed that all cases with gain have $S = 1$, see table 7. For directed cycle with 9 nodes, there was no quantum advantage, and the success probability was $1/4$ for $S = 0$, and $1/3$ for $S = 1$ for all sets \mathfrak{S} , \mathfrak{Q} , \mathfrak{M} , and \mathfrak{N} .

Let us first analyze the peculiar case of a directed 6-node cycle. An optimal LHV strategy is for instance:

- If the party is in the node 1 then waits for two steps.
- If the party is in the node 2, then move to the node 3 and then 4.

- (c) If the party is in the node 3, then wait, and in the second step move to the node 4.
- (d) If the party is in the node 4 then waits for two steps.
- (e) If the party is in the node 5, then move to the node 6 and then 1.
- (f) If the party is in the node 6, then move to the node 1 and wait.

This strategy is the same for both parties, Alice and Bob. Direct calculations show that this gives a success probability of 0.5. This result cannot be improved by using quantum, ML, or NS boxes. The successful pairs of settings are $\{(1, 5), (1, 6), (2, 3), (2, 4), (3, 4), (5, 6)\}$, and their reverses, and all pairs (a, a) .

An optimal LHV strategy for 4 nodes is the following. For any setting, Alice waits at her initial node, as in the well know approach called *wait-for-mummy* [34]. Bob moves in the following way:

- (a) If Bob is in node 1, then wait for one step and then move to node 2.
- (b) If Bob is in node 2, then wait for one step and then move to node 3.
- (c) If Bob is in the node 3, then move to the node 4 and then 1.
- (d) If Bob is in the node 4, then move to the node 1 and then 2.

This deterministic asymmetric strategy provides a success probability of 0.625. If we fix Alice's part of the box to the wait-for-mummy strategy, then quantum boxes are not able to provide any advantage, whereas without this constraint there exists a quantum box that gives a success probability of at least 0.67678.

5.5. Robustness to noise

Any future experimental realization of the proposed rendezvous method will need to be robust against noise and quantum state imperfections. To estimate what is the critical noise tolerance that still allows the proposed protocols to show quantum advantage we model the noise as an effective replacement of the maximally entangled state $|\Phi\rangle$ with the noised one given by the formula

$$\nu|\Phi\rangle\langle\Phi| + (1 - \nu)\rho_{\text{mixed}}, \quad (11)$$

where ρ_{mixed} is the maximally mixed state modeling uniform classical probability distribution of all possible states, i.e. the white noise. The critical value of ν denoted ν_{crit} is defined as the level of noise when their quantum strategy has the same efficiency as the LHV strategy.

To illustrate the robustness we considered the case of cubic graphs from table 2, viz. cubic graphs number 4 to 9, anti-reflexive. For conciseness, we set $E = 1$ and $S = 0$. The values of ν_{crit} are 0.75, 0.80252, 0.83777, 0.75, 0.75, 0.86695, respectively. We obtain similar values for other cases with quantum advantage. This result shows that the proposed protocols are feasible for experimental and practical realizations using current technologies.

6. Conclusions

In this paper, we have proposed a novel quantum protocol that allowed us to complete the rendezvous task on a network for asymmetric, location-aware agents, and time limits with higher probability than using only classical resources. Our study covered cases of cubic graphs and cycles with 1 and 2 step limits. Due to the high complexity of the problem, we were not able to analyze more complicated instances and leave more optimized numerical or analytical approaches for future research.

We imagine a couple of scenarios, either current or future, where the problem can arise. They all refer to cases when for some reason the starting locations are not known in advance, and communication in the course of the game is forbidden.

One such scenario involves agents (e.g. a fleet of drones) performing some secret mission in a hostile territory, where they do not know, where they will be located at a certain moment, after which they are assumed to group in one of the pre-agreed points; the agents also cannot communicate, as any signal could be detected by enemies and reveal their presence.

Another scenario involves a group of miners, speleologists, or rescuers, that are exploring separately some undergrounds or caves, knowing their map. In this scenario, the walls may shield any communication signals, and if they need to group after some specified time period, then entanglement would possibly provide an advantage to their task. The same situation may apply to medics or scientists working surrounded by devices that are fragile to any sort of wireless communication.

A futuristic application would be a case when parties are starting at some far distant planets or solar systems when immediate communication is prohibited by the speed of light. In this case, even without sending any signals, they can increase the chances that they will direct their spaceships toward the same focal point.

In some cases, especially when large distances are considered, it may be the case that communication in principle would be possible, but would require large amounts of energy. This would be particularly the case when the direction where a signal should be sent is unknown, and thus there is a need to widespread any message in all space. This can be contrasted with quantum entanglement which, after a technology to keep and transport entangled sub-systems is enhanced, will work equally efficiently and, in principle, with no energy cost, no matter how large the distance is.

The last example refers to the case when parties possibly *could* communicate, but they are *not willing* to do so. This would for example be the case when two antagonists want to arrange a point to meet (e.g. to duel or pass a ransom), but in such a way, that their location prior to the meeting is not revealed in any sense, as they do not trust each other and are afraid that the other party knowing the traversed path may prepare a trap. The NS condition prevents from providing any piece of information about the location of one party to the other party, still allowing them to coordinate their meeting point. This scenario provides some analogies with zero-knowledge proofs [97].

It is interesting to investigate whether a similar approach with quantum non-locality can be applied to closely related problems, like other search games [98, 99] or different graph problems with agents, possibly with the accompanying of another quantum phenomena of the so-called Elitzur-Vaidman bomb testing or interaction free measurement [100, 101].

The aim of this work was to bridge the two, till now independent, research areas of rendezvous in classical computer science, and Bell non-locality in quantum information science.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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