# Unusual dynamics and nonlinear thermal self-focusing of initially focused magnetoacoustic beams in a plasma

Anna Perelomova
Gdansk University of Technology,
Faculty of Applied Physics and Mathematics,
ul. Narutowicza 11/12, 80-233 Gdansk, Poland
anna.perelomova@pg.edu.pl

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**Abstract**

Unusual thermal self-focusing of two-dimensional beams in plasma which axis is parallel to the equilibrium straight magnetic field, is considered. The equilibrium parameters of plasma determine scenario of a beam divergence (usual or unusual) which is stronger as compared to a flow without magnetic field. Nonlinear thermal self-action of a magnetosonic beam behaves differently in the ordinary and unusual cases. Damping of wave perturbations and normal defocusing in gases lead to reduction of the magnitude of initially planar perturbations at the axis of a beam. Additional thermal self-focusing non-specific for gases occurs in plasma under some condition which counteracts this reduction. The theory and numerical examples concern thermal self-action of initially focused (defocused) magnetosonic beam. Dynamics of perturbations in a beam is determined by dimensionless parameters responsible for diffraction, damping of the wave perturbations, initial radius of a beam's front curvature and the ratio of viscous to thermal damping coefficients.

## 1 Introduction

The MHD (magnetohydrodynamic) flow consists in general of variety of different wave and non-wave modes. The number of modes is larger compared with a flow which is not affected by the magnetic field, and their definition depend strongly on the angle between the wave vector and the magnetic field. The linear analysis of a flow considers equilibrium parameters of a medium, geometry of a flow, diffraction and damping factors. Alfvén, fast and slow magnetoacoustic modes specify the wave motion. The entropy mode belongs to the non-wave modes. The correct definition of all modes is the starting point in studies of nonlinear flows. It relies on dispersion relations, or, equivalently, on the links between small-magnitude specific perturbations.

The variety of modes increases in the multi-dimensional flow. A planar wave does not reflect real flow. Dynamics of perturbations in the real beams differs from the rays' behavior. The transversal diffusion of magnitude of perturbations, that is, diffraction, takes place. Since diffraction reduces magnitude of perturbations away from a beam's axis, it has considerable impact on the nonlinear phenomena which enhance in the field of intense perturbations. This concerns nonlinear distortion of a magnetosonic wave form itself and the nonlinear excitation of the entropy mode in its field. The diffraction behavior of beams (including unusual one) is determined by an angle between the magnetic field and direction of a beam propagation and equilibrium parameters of plasma. We concentrate on initially focused beams propagating along axis $z$ which is parallel to the constant straight magnetic field. This study continues and extends the previous results by considering usual and unusual behavior of magnetosonic beams in initially focused (defocused) beams (Sections 3,4)[1]. It considers shear and bulk viscosity as damping mechanisms as well. Unusual thermal self-focusing of a magnetosonic beam increases the magnitude of perturbations and may lead to a pronounced focus in view of joint impact of nonlinearity, diffraction and attenuation and to displacement of a geometrical focus (Sec. 5).

It would be useful to mention contribution of nonlinear wave theory of intense optic beams in concern to self-action in acoustics (e.g., [2, 3]). The possibility of thermal self-action of acoustic beams was suggested by analogy with optic beams by Askaryan [4]. Self-focusing in non-linear optics is induced by the variations in refractive index of materials exposed to intense electromagnetic radiation. A medium whose refractive index enlarges with the electric field intensity acts as a focusing lens. Gain medium may be self-focusing and defocusing [5]. It is well-established that self-focusing in plasma can occur by means of relativistic [6], ponderomotive and thermal (due to growth of the index of refraction induced by collisional heating of a plasma [7]) effects. Defocusing in tunnel ionized plasma is different for Gaussian and super-Gaussian beams and depends on the spot size of the laser [8]. It depends also on the external magnetic field [9]. The mathematical contents in regard to the acoustic and optic wave phenomena are also different. This concerns different patterns of nonlinearity, strong dispersion of optic waves and usually weak dispersion of acoustic waves. In weakly dispersive media, the initial profile becomes distorted in the course of propagation and the spectrum rapidly spreads due to excitation of higher harmonics. This requires special mathematical methods. Response of the optic medium at the fundamental frequency occurs due to cubic nonlinearity and higher order odd nonlinearities of the medium while acoustic nonlinearity in fluids is quadratic. Very special property of acoustic beams in a plasma affected by the magnetic field which has no analogs in optics is possibility of unusual diffraction and corresponding unusual thermal self-action of beams.

## 2 The equations of MHD flow

We start from the system of ideal MHD equations. It makes use of the single-fluid model, a model of an ideal gas consisting of molecules of negligible size, and macroscopic equilibrium quantities of plasma. An internal energy of an ideal gas depends exclusively on its temperature. Ideal magnetohydrodynamics is a reasonably good approximation in most flows of astrophysical plasmas. The full set of ideal MHD equations consists of the continuity equation, momentum equation, energy balance equation and electrodynamic equations (e.g., [10, 11, 12]):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0,$$

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \eta \Delta \vec{v} + \left(\frac{1}{3}\eta + \xi\right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v}), \tag{2.1}$$

$$\frac{Dp}{Dt}-\gamma\frac{p}{\rho}\frac{D\rho}{Dt}=(\gamma-1)\left[\vec{\nabla}\cdot(\chi\vec{\nabla}T)+\xi(\vec{\nabla}\cdot\vec{v})^2+\frac{\eta}{2}\sum_{i,k=1,2,3}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{i,k}\vec{\nabla}\cdot\vec{v}\right)^2\right],$$

$$\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\times(\vec{v}\times\vec{B}),$$

$$\vec{\nabla}\cdot\vec{B}=0,$$

where $p$, $\rho$, $\vec{v}$, $\vec{B}$ are hydrostatic pressure and mass density of plasma, its velocity, the magnetic field, $T$ is plasma's temperature, $\mu_0$ is the permeability of free space, and $\gamma$ denotes the ratio of specific heats under constant pressure and constant density, $\gamma=C_P/C_V$. Thermal conductivity is designated by $\chi$, and $\eta$, $\xi$ mark shear and bulk viscosity. $\delta_{i,k}$ is the Kronecker delta, and $\frac{D}{Dt}$ denotes the material derivative.

The analysis is readily reduced to two dimensions, if one considers all perturbations as functions of two Cartesian coordinates $x$ and $z$ and time [13]. All equilibrium quantities are constant and subscripted by 0, and all perturbations are superscripted by apostrophes. Since the bulk flow is absent, $\vec{v}_0=\vec{0}$, apostrophes by components of velocity are dropped. The equilibrium magnetic field is aligned along axis $z$, $\vec{B}_0=(0,0,B_0)$. The MHD equations may be rearranged in the following leading-order form correct to the second order of the perturbations:

$$\frac{\partial\rho'}{\partial t}+\rho_0\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}\right)=N_1,$$

$$\frac{\partial v_x}{\partial t}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x}-\left(\frac{4\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\partial^2 v_x}{\partial x^2}-\left(\frac{\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\partial^2 v_z}{\partial x\partial z}-\frac{\eta}{\rho_0}\frac{\partial^2 v_x}{\partial x^2}-\frac{\eta}{\rho_0}\frac{\partial^2 v_x}{\partial z^2}-\frac{B_0}{\rho_0\mu_0}\left(\frac{\partial B_x'}{\partial z}-\frac{\partial B_z'}{\partial x}\right)=N_2,$$

$$\frac{\partial v_y}{\partial t}-\frac{\eta}{\rho_0}\frac{\partial^2 v_y}{\partial x^2}-\frac{\eta}{\rho_0}\frac{\partial^2 v_y}{\partial z^2}-\frac{B_0}{\rho_0\mu_0}\frac{\partial B_y'}{\partial z}=N_3, \tag{2.2}$$

$$\frac{\partial v_z}{\partial t}-\frac{\eta}{\rho_0}\frac{\partial^2 v_z}{\partial x^2}-\left(\frac{\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\partial^2 v_x}{\partial x\partial z}-\left(\frac{4\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{\rho_0}\frac{\partial p'}{\partial z}=N_4,$$

$$\frac{\partial p'}{\partial t}+\gamma p_0\frac{\partial v_z}{\partial z}-\frac{\chi}{\rho_0 C_P}\frac{\partial^2\gamma p'}{\partial x^2}+\frac{\gamma\chi p_0}{\rho_0^2 C_P}\frac{\partial^2\rho'}{\partial x^2}-\frac{\chi}{\rho_0 C_P}\frac{\partial^2\gamma p'}{\partial z^2}+\frac{\gamma\chi p_0}{\rho_0^2 C_P}\frac{\partial^2\rho'}{\partial z^2}=N_5,$$

$$\frac{\partial B_x'}{\partial t}-B_0\frac{\partial v_x}{\partial z}=N_6,$$

$$\frac{\partial B_y'}{\partial t}-B_0\frac{\partial v_y}{\partial z}=N_7,$$

$$\frac{\partial B_z'}{\partial t}+B_0\frac{\partial v_x}{\partial x}=N_8,$$

where $N_1\dots N_8$ consist of quadratically nonlinear terms including these ones proportional to damping coefficients.

$$N_1=-\rho'\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}\right)-v_x\frac{\partial\rho'}{\partial x}-v_z\frac{\partial\rho'}{\partial z}$$

$$N_2=-v_x\frac{\partial v_x}{\partial x}-v_z\frac{\partial v_x}{\partial z}+\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x}-\frac{B_0}{\rho_0^2\mu_0}\rho'\left(\frac{\partial B_x'}{\partial z}-\frac{\partial B_z'}{\partial x}\right)+$$

$$\frac{B_z'}{\rho_0\mu_0}\left(\frac{\partial B_x'}{\partial z}-\frac{\partial B_z'}{\partial x}\right)-\frac{B_y'}{\rho_0\mu_0}\frac{\partial B_y'}{\partial x}- \tag{2.3}$$

$$\left(\frac{4\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\rho'}{\rho_0}\frac{\partial^2 v_x}{\partial x^2}-\left(\frac{\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\rho'}{\rho_0}\frac{\partial^2 v_z}{\partial x\partial z}-\frac{\eta}{\rho_0}\frac{\rho'}{\rho_0}\frac{\partial^2 v_x}{\partial x^2}-\frac{\eta}{\rho_0}\frac{\rho'}{\rho_0}\frac{\partial^2 v_x}{\partial z^2}$$

$$N_3=-v_x\frac{\partial v_y}{\partial x}-v_z\frac{\partial v_y}{\partial z}-\frac{B_0}{\rho_0^2\mu_0}\rho'\frac{\partial B_y'}{\partial z}+\frac{B_z'}{\rho_0\mu_0}\frac{\partial B_y'}{\partial z}+\frac{B_x'}{\rho_0\mu_0}\frac{\partial B_y'}{\partial x}-\frac{\eta}{\rho_0^2}\rho'\frac{\partial^2 v_y}{\partial x^2}-\frac{\eta}{\rho_0^2}\rho'\frac{\partial^2 v_y}{\partial z^2}$$

$$N_4=-\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial z}+\frac{B_x'}{\rho_0\mu_0}\frac{\partial B_z'}{\partial x}-\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{B_x^2+B_y^2}{2\mu_0}\right)-v_z\frac{\partial v_z}{\partial z}-$$

$$\left(\frac{\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\rho'}{\rho_0}\frac{\partial^2 v_x}{\partial x\partial z}-\left(\frac{4\eta}{3\rho_0}+\frac{\xi}{\rho_0}\right)\frac{\rho'}{\rho_0}\frac{\partial^2 v_z}{\partial z^2},$$

$$N_5=-\gamma p'\frac{\partial v_z}{\partial z}-v_z\frac{\partial p'}{\partial z}+(\gamma-1)\xi\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}\right)^2+$$

$$(\gamma-1)\eta\left(\frac{4}{3}\left(\frac{\partial v_x}{\partial x}\right)^2+\frac{4}{3}\left(\frac{\partial v_z}{\partial z}\right)^2+\frac{2}{3}\frac{\partial v_x}{\partial z}\frac{\partial v_z}{\partial x}+\left(\frac{\partial v_x}{\partial z}\right)^2+\left(\frac{\partial v_z}{\partial x}\right)^2+\left(\frac{\partial v_y}{\partial x}\right)^2+\left(\frac{\partial v_y}{\partial z}\right)^2\right)+$$

$$\frac{\gamma\chi}{\rho_0^3C_P}\frac{\partial^2}{\partial x^2}\left(p_0\rho'^2-\rho_0\rho'p'\right)+\frac{\gamma\chi}{\rho_0^3C_P}\frac{\partial^2}{\partial z^2}\left(p_0\rho'^2-\rho_0\rho'p'\right),$$

$$N_6=\frac{\partial}{\partial z}(v_xB_z'-v_zB_x'),$$

$$N_7=\frac{\partial}{\partial z}(v_yB_z'-v_zB_y')-\frac{\partial}{\partial x}(v_xB_y'-v_yB_x'),$$

$$N_8=\frac{\partial}{\partial x}(v_zB_x'-v_xB_z').$$

The system(2.2),(2.3) makes use of the equation of state for an ideal gas,

$$T=\frac{p}{\rho(C_P-C_V)}. \tag{2.4}$$

Equations without account for damping were derived in references [13, 14, 15]. A set of equations (2.2),(2.3) describes dynamics of perturbations correct to the second order in the magnetoacoustic Mach number $M$. The Mach number measures the nonlinear effects of a flow and equals the ratio of amplitude of velocity to the speed of magnetosonic perturbations. The linear limit of Eqs (2.2) (the case $N_1=...=N_8=0$ determines the dynamic equations for small magnitude perturbations).

# 3 Quasi-planar dynamics of small-amplitude magnetosound perturbations

For definiteness, a beam propagating along the positive direction of axis $z$, is considered. The linearized equations Eqs(2.2) ($N_1=...=N_8=0$) in the case of zero $\xi$, $\eta$, $\chi$ determine speed of a wave along axis $z$ if one assumes a planar wave with all perturbations proportional to $\exp(i\omega t-ik_zz)$, where $\omega$ denotes frequency of wave oscillations, and $k_z$ is the wave vector of the planar wave. Apart from Alfvén waves and the entropy non-wave mode, there are four roots of the dispersion relation $\omega=Ck_z$ [16, 17]

$$C^4+c_0^2C_A^2-C^2(c_0^2+C_A^2)=0. \tag{3.5}$$

There are two absolute values of propagation speed of the planar wave, $C = c_0$ and $C = C_A$, where

$$c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad C_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \tag{3.6}$$

denote the acoustic speed in non-magnetized gas in equilibrium and the Alfvén speed. The roots of Eq.(3.5) $C$ coincide with speeds of propagation of the quasi-planar wave along axis $z$. The plasma beta symbolized by $\beta = \frac{2}{\gamma}\frac{c_0^2}{C_A^2}$ is the ratio of the plasma pressure to the magnetic pressure. Most tokamaks deal with $\beta$ of order 0.01, and spherical tokamaks typically operate at values about 0.1. The Sun's corona has $\beta$ around 0.01. Active regions have much higher $\beta$, over 1 in some cases. A large value of plasma beta is a characteristic of stellar terrestrial environment, for example, the solar atmosphere. There are reports of laboratory production of an extremely large plasma beta, termed ELB plasma [18]. Hence, in physically meaningful cases, the plasma beta can be varied from $10^{-3}$ to values as large as $10^3$. The case $c_0 = C_A$ is especial and is not paid attention in this study.

## 3.1 Case $C = c_0$ and $c_0 \neq C_A$

This case describes $p$-modes, that it, magnetosonic modes with non-zero perturbations of pressure. Substituting all perturbations in the form of the plane waves proportional to $\exp(i\omega t - ik_x x - ik_z z)$, considering weak damping and treating $\frac{k_x}{k_z}$ as a small parameter, we arrive at the leading-order dispersion relation:

$$\omega = c_0 k_z \left(1 + \frac{c_0^2 k_x^2}{2(c_0^2 - C_A^2)k_z^2}\right) + i\left(\frac{(\gamma-1)\chi}{2C_P\rho_0} + \frac{2\eta}{3\rho_0} + \frac{\xi}{2\rho_0}\right) k_z^2. \tag{3.7}$$

The effects relating to diffraction and damping due to thermal conduction and viscosity are supposed to be of the same order, so Eq.(3.7) is valid to the second order of the generic small parameter $m$ which is responsible for diffraction and damping effects. Going to derivation of dynamic equation, one makes use of correspondence of operators $\partial/\partial t$ and $\partial/\partial x$, $\partial/\partial z$ to $i\omega$, $-ik_x$ and $-ik_z$, respectively. Any field perturbation $\varphi$ inherent to this mode may be considered as a function of the retarded time $\tau = t - z/c_0$ and "slow" coordinates $mz$, $\sqrt{m}x$. This choice of variables suggests that the spatial variations occur more slowly along axis $z$ of a beam than along axis $x$ to an observer which moves at speed $c_0$ along axis $z$ [2, 19, 20] ($\varphi$ may represent any perturbation from the set $\rho'$, $v_x$, $v_z$, $p'$, $B'_x$, $B'_z$). Discarding terms $O(m^2)$, collecting terms of order $m$ and rearranging equation from the slow scale back to coordinates $x$, $z$, one arrives at the dynamic equation

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \varphi}{\partial z} - \frac{1}{c_0^3}\left(\frac{(\gamma-1)\chi}{2C_P\rho_0} + \frac{2\eta}{3\rho_0} + \frac{\xi}{2\rho_0}\right)\frac{\partial^2 \varphi}{\partial \tau^2}\right) = \pm\frac{D^2 c_0}{2}\frac{\partial^2 \varphi}{\partial x^2}, \tag{3.8}$$

where

$$D = \frac{c_0}{\sqrt{|c_0^2 - C_A^2|}}.$$

The upper sign in the right-hand side of Eq.(3.8) corresponds to the case $c_0 > C_A$, and the lower one to the case $c_0 < C_A$. Eq.(3.8) resembles the famous equation for the small magnitude perturbations in weakly diverging beams in a gas in the absence of magnetic field ($\varphi$ may represent any perturbation from the set $\rho'$, $v_x$, $v_z$, $p'$) [19, 20]:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \varphi}{\partial z} - \frac{1}{c_0^3}\left(\frac{(\gamma-1)\chi}{2C_P\rho_0} + \frac{2\eta}{3\rho_0} + \frac{\xi}{2\rho_0}\right)\frac{\partial^2 \varphi}{\partial \tau^2}\right) = \frac{c_0}{2}\frac{\partial^2 \varphi}{\partial x^2}. \tag{3.9}$$

The divergence is more manifested in the presence of a magnetic field since $D > 1$. The diffraction length for the harmonic oscillations with frequency $\omega$,

$$z_d = \omega a^2/(2Dc_0) \tag{3.10}$$

($a$ denotes the characteristic width of a beam at a transducer) is shorter in this case compared to that in the non-magnetized gas ($D = 1$),

$$z_{d,0} = \omega a^2/(2c_0). \tag{3.11}$$

### 3.2 Case $C = C_A$ and $c_0 \neq C_A$

This case is not acoustic. Similar to subcec.3.1 manipulations yield the leading-order dispersion relation corresponding to a beam propagating in the positive direction of axis $z$,

$$\omega = C_A k_z \left(1 + \frac{C_A^2 k_x^2}{2(C_A^2 - c_0^2)k_z^2}\right). \tag{3.12}$$

An equation which governs perturbations in a beam may be extracted from Eq.(3.12) by assuming perturbations as $\varphi(t - z/C_A, mz, \sqrt{m}x)$. We arrive at

$$\frac{\partial^2 \varphi}{\partial \tau \partial z} = \begin{cases} \frac{D^2 C_A^3}{2c_0^2}\frac{\partial^2 \varphi}{\partial x^2}, & C_A > c_0 \\ -\frac{D^2 C_A^3}{2c_0^2}\frac{\partial^2 \varphi}{\partial x^2}, & C_A < c_0 \end{cases}. \tag{3.13}$$

where $\varphi$ may take value of any non-zero wave perturbation which specifies the wave modes. Propagation of non-zero perturbations is not affected by damping mechanisms. The cases $C_A > c_0$ and $C_A < c_0$ impose normal and unusual diffraction, respectively, and resemble the Alvfén mode which propagates in the positive direction of axis $z$ with the speed $C_A$. This is important difference between these two modes. Namely, the Alvfén mode propagates as a planar wave without diffraction due to exact dispersion relation $\omega = C_A k_z$.

## 4 Nonlinear dynamics of weakly divergent focused magnetosonic beams

Both cases $C = c_0$ and $C = C_A$ impose unusual diffraction in dependence on the ratio $\frac{C_A}{c_0}$. Eq.(3.8) concerns perturbation in magnetoacoustic beam. It contains term responsible for the thermal conduction and viscosity and includes Eq.(3.13) as a particular mathematical case (with different coefficient by diffraction term). We concentrate on the studies of dynamics of perturbations in initially focused acoustic beams. Only beams propagating with speed $c_0$ ($p$-modes) may excite entropy perturbations and experience thermal self-action.

Going to the weakly nonlinear dynamics, we consider effects due to nonlinearity, diffraction and attenuation of the same order in smallness (hence, the Mach number $M$ and the generic parameter $m$ responsible for diffraction and damping due to thermal conduction and viscosity are of comparative smallness). These effects are considered as independent and having only small impact on the wave process. The dynamic equations correct to the second order of perturbations follow from the system (2.2),(2.3). All perturbations are treated as functions of retarded time and $\tau = t - z/c_0$ and "slow" coordinates $mz$, $\sqrt{m}x$. The links of perturbations in the beam are required for the proper evaluation of the nonlinear term. The leading-order links in the beam are as follows:

$$v_x = v_y = 0, \quad v_z = \frac{c_0}{\rho_0}\rho', \quad p' = c_0^2\rho', \quad B'_x = B'_y = B'_z = 0. \tag{4.14}$$

for any $C$ except for the case $C = c_0 = C_A$. In particular, equation which describes perturbation of density in the magnetosonic beam propagating in the positive direction of axis $z$ with the speed $c_0$ takes the form:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \rho'}{\partial z} - \frac{\gamma+1}{2\rho_0 c_0}\rho'\frac{\partial \rho'}{\partial \tau} - \frac{1}{c_0^3}\left(\frac{(\gamma-1)\chi}{2C_P\rho_0} + \frac{2\eta}{3\rho_0} + \frac{\xi}{2\rho_0}\right)\frac{\partial^2\rho'}{\partial \tau^2}\right) = \begin{cases} \frac{D^2 c_0}{2}\frac{\partial^2\rho'}{\partial x^2}, & c_0 > C_A \\ -\frac{D^2 c_0}{2}\frac{\partial^2\rho'}{\partial x^2}, & c_0 < C_A \end{cases}. \tag{4.15}$$

An equation which describes perturbations in a planar wave affected by thermal conduction in a weakly nonlinear flow at arbitrary angle between the equilibrium magnetic field and wave wector, has been derived in Ref. [14]. Eq.(4.15) incorporates weak nonlinearity, damping due to thermal conduction and viscosity, and diffraction (including unusual one). It complements the dynamic equation derived in Ref.[1] by terms associated with viscosity. The case $c_0 > C_A$ corresponds to the usual diffraction but with the shorter diffraction length $z_d$, and the case $c_0 < C_A$ is especial. The famous Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation for a Newtonian beam is in fact (4.15) with $D = 1$. There are no analytical solutions to Eq.(4.15) in general. In the case of negligible nonlinearity, (4.15) transforms into (3.8) with $\varphi$ replaced by $\rho'$:

$$\frac{\partial \rho'}{\partial z\partial \tau} - \frac{1}{c_0^3}\left(\frac{(\gamma-1)\chi}{2C_P\rho_0} + \frac{2\eta}{3\rho_0} + \frac{\xi}{2\rho_0}\right)\frac{\partial^3\rho'}{\partial \tau^3} = \begin{cases} \frac{D^2 c_0}{2}\frac{\partial^2\rho'}{\partial x^2}, & c_0 > C_A \\ -\frac{D^2 c_0}{2}\frac{\partial^2\rho'}{\partial x^2}, & c_0 < C_A \end{cases}. \tag{4.16}$$

The analytical solution to it may be obtained seeking $\rho'$ in the form [19]

$$\rho' = A(x, z)\exp(i\omega\tau), \tag{4.17}$$

which transforms (4.16) to

$$\frac{\partial A}{\partial Z} + \frac{z_d}{z_{tv}}A = \mp\frac{i}{4}\frac{\partial^2 A}{\partial X^2}, \tag{4.18}$$

where $X = x/a$, $Z = z/z_d$ are dimensionless coordinates, and

$$z_{tv} = \frac{2c_0^3}{\omega^2}\left(\frac{(\gamma-1)\chi}{C_P\rho_0} + \frac{4\eta}{3\rho_0} + \frac{\xi}{\rho_0}\right)^{-1} \tag{4.19}$$

designates the characteristic scale of damping due to viscosity and thermal diffusion. Eq.(4.18) differs from the parabolic equation for the complex amplitude by the term $\frac{z_d}{z_{tv}}A$ [2, 19]. We will consider sound beams with an initially curved wave front and Gaussian transversal distribution of amplitude. The boundary condition for a focused beam at $Z = 0$ is

$$\frac{A(X,0)}{A_0} = \exp\left(-X^2(1 \mp i\delta)\right), \quad \delta = \frac{\omega a^2}{2Rc_0} = \frac{z_d}{R}, \tag{4.20}$$

where $R$ is an initial radius of a beam's front curvature, and $A_0$ denoted the magnitude of perturbation at $X = 0$ $Z = 0$. Positive $\delta$ correspond to the initially focused beams, and negative $\delta$ correspond to defocused beams with negative $R$. The solution to Eq.(4.18) with the boundary condition (4.20) takes the form

$$\frac{A(X,Z)}{A_0}=\frac{\exp\left(-\frac{X^2(1\mp i\delta)^2}{1\mp iZ-\delta Z}-\frac{z_d}{z_{tv}}Z\right)}{\sqrt{1\mp iZ-\delta Z}}. \tag{4.21}$$

The module of $A$,

$$|A(X,Z)|=\frac{A_0}{\sqrt[4]{Z^2+(1-\delta Z)^2}}\exp\left(-\frac{X^2}{Z^2+(1-\delta Z)^2}-\frac{z_d}{z_{tv}}Z\right) \tag{4.22}$$

gets smaller away from the axis of a beam in the both cases of positive and negative signs. The magnitude of the wave perturbation at the axis of a beam $|A(0,Z)|$ is proportional to $1/\sqrt[4]{Z^2+(1-\delta Z)^2}$, and the width of a beam increases as $\sqrt{Z^2+(1-\delta Z)^2}$. The perturbation of density which amplitude satisfies Eq.(4.18) with the boundary condition (4.20), takes the form

$$\rho'=A_0\frac{\exp\left(-\frac{X^2}{Z^2+(1-\delta Z)^2}-\frac{z_d}{z_{tv}}Z\right)\sin\left(\omega\tau-\frac{X^2((1+\delta^2)Z-\delta)}{Z^2+(1-\delta Z)^2}\pm\frac{1}{2}\arctan\left(\frac{Z}{1-\delta Z}\right)\right)}{\sqrt[4]{Z^2+(1-\delta Z)^2}}. \tag{4.23}$$

The phase of perturbations at the axis of a beam ($X=0$) equals $\omega\tau\pm\frac{1}{2}\arctan\left(\frac{Z}{1-\delta Z}\right)$. An excess dimensionless speed of perturbations $\frac{\Delta c}{c_0}$ at the axis of a beam in the vicinity of transducer (that is, for $Z\ll 1$) equals approximately $\pm\frac{c_0}{2z_d\omega}$. Usually, perturbations at the axis of a beam propagate faster than that in the planar wave in the same equilibrium parameters of plasma (sign plus, $c_0>C_A$), but they propagate slower in the unusual case (sign minus, $c_0<C_A$). The magnitude of perturbation in density in the focal point $z_f$ achieves maximum (if focal point exists). The geometrical focus at the axis of a focused beam without damping is suited at (exists if $\delta>0$)

$$Z_{f,0}=\frac{z_f}{z_d}=\frac{\delta}{1+\delta^2}. \tag{4.24}$$

Maximum dimensionless perturbation of density in the focal point equals $\sqrt[4]{1+\delta^2}A_0$. The dimensionless focal distance at the axis of damped beam equals ($\delta>0$, if $1+\delta^2>4z_d/z_{tv}$)

$$Z_f=\frac{\delta}{1+\delta^2}+\frac{\sqrt{1-+2\delta^2+\delta^4}-\delta^2-1}{4(1+\delta^2)z_d/z_{tv}}. \tag{4.25}$$

Perturbation of density achieves maximum $\rho'_{max}$

$$\rho'_{max}=A_0\frac{2^{3/4}\sqrt{\frac{z_d}{z_{tv}}}\exp\left(\frac{\delta^2+1-4z_d\delta/z_{tv}-\sqrt{1-16z_d^2/z_{tv}^2+2\delta^2+\delta^4}}{4(1+\delta^2)}\right)}{\sqrt[4]{1+\delta^2-\sqrt{1+2\delta^2+\delta^4-16z_d^2/z_{tv}^2}}} \tag{4.26}$$

at the axis of a beam in a focal point, in the case of weak damping, the focal point shifts towards a transducer approximately at the dimensionless distance $\frac{2z_d/z_{tv}}{(1+\delta^2)^2}$ (compared to the case with zero thermal conduction and viscosity). The magnitude of density perturbation achieves in the focal point value $A_0(\sqrt[4]{1+\delta^2}-\frac{\delta}{\sqrt[4]{(1+\delta^2)^3}}\frac{z_d}{z_{tv}})$. If total damping is strong enough,

$$\frac{z_d}{z_{tv}}\geq\frac{1+\delta^2}{4}, \tag{4.27}$$

it prevents formation of the pronounced focal point. The dimensionless focal distance and maximum dimensionless perturbation of density $\frac{\rho'_{max}}{A_0}$ at the axis of a beam (for the parameters $\frac{z_d}{z_{tv}}$ and $\delta$ where the local maximum exists) are shown at Fig.1

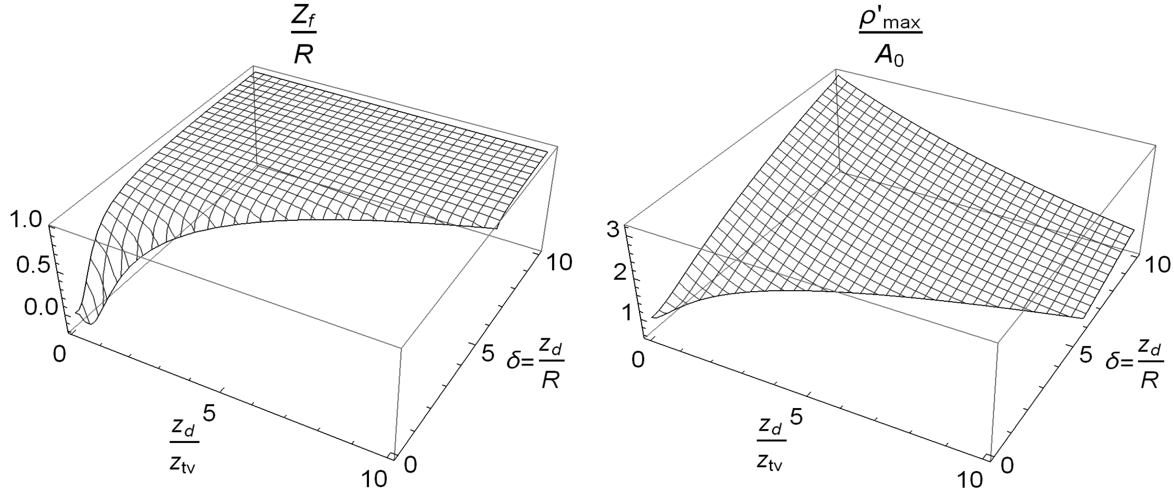



Figure 1. *The dimensionless focal distance $\frac{Z_f}{R}$ and maximum dimensionless perturbation of density $\frac{\rho'_{max}}{A_0}$ at the axis of a beam (for the parameters $\frac{z_d}{z_{tv}}$ and $\delta$ where the local maximum exists).*

# 5 Thermal self-action of magnetosonic beams

Nonlinearity alters waveforms of the finite-magnitude perturbations in the course of propagation. The nonlinear interaction of modes which is responsible for an irreversible transfer of momentum and energy from the macroscopic to microscopic motion, go into play. The reason for interaction is damping and nonlinearity [19, 20]. Excitation of the entropy perturbations in the field of intense magnetosonic wave leads to the heterogeneous enlargement of the background temperature in a plane perpendicular to the axis of a beam and to formation of thermal lenses. Diffraction plays a key role in propagation of a wave beam and hence in the magnetosonic heating. Since the sound speed depends on temperature, the distortions of a wave front are stronger in the vicinity of a beam axis. The acoustic heating leads to the defocusing of a beam, if $(\partial c_0/\partial T)_p > 0$, and to the focusing of a beam otherwise. This phenomenon is known as thermal self-action and was firstly studied with regard to the Newtonian flows [21, 22]. Thermal self-action of quasi-harmonic sound waves in the case of weak nonlinearity has much in common with thermal self-action of optic waves [23, 24]. Acoustic nonlinearity in a weakly dispersive medium leads to the higher harmonic generation in the course of wave propagation and hence wave perturbations can not be longer considered as quasi-harmonic. The coupling system which describes magnetoacoustic heating due to loss of beam's energy, consists of Eq.(4.15) supplemented by the term taking into account variation of the local sound speed due to variation of the background temperature $T'$, and the diffusion equation which is responsible for this variation,

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\rho'}{\partial z}-\frac{T'}{2c_0T_0}\frac{\partial\rho'}{\partial\tau}-\frac{\gamma+1}{2\rho_0c_0}\rho'\frac{\partial p'}{\partial\tau}-\frac{1}{c_0^3}\left(\frac{(\gamma-1)\chi}{2C_P\rho_0}+\frac{2\eta}{3\rho_0}+\frac{\xi}{2\rho_0}\right)\frac{\partial^2\rho'}{\partial\tau^2}\right)=\pm\frac{D^2c_0}{2}\frac{\partial^2\rho'}{\partial x^2}, \tag{5.28}$$

$$\frac{\partial T'}{\partial t} - \frac{\chi}{\rho_0 C_P}\frac{\partial^2 T'}{\partial x^2} = \frac{c_0}{C_P}F, \tag{5.29}$$

where $F$ designates the magnetosonic force of heating. Eq.(5.28) takes into account the equality valid for an ideal gas, $(\partial c_0/\partial T)_p = c_0/2T_0$. $T'$ is not an acoustic quantity but a non-wave perturbation which specifies the entropy mode governed by Eq.(5.29). In the case of the periodic or nearly periodic magnetosonic perturbations, $F$ is the quantity averaged over a period [2, 25].

$$F = \frac{1}{c_0\rho_0^2}\left(\frac{(\gamma-1)\chi}{C_P\rho_0} + \frac{4\eta}{3\rho_0} + \frac{\xi}{\rho_0}\right)\left\langle\left(\frac{\partial\rho'}{\partial\tau}\right)^2\right\rangle. \tag{5.30}$$

The "fast" time may be eliminated from Eq.(5.28) by substitution (4.17) [25]. This yields the dynamic equation for the complex amplitude $A$,

$$\frac{\partial A}{\partial Z} - i\frac{\omega T' C_P(\gamma-1)z_d}{2c_0^3}(1+r_{tv})\,A + \frac{z_d}{z_{tv}}A = \mp\frac{i}{4}\frac{\partial^2 A}{\partial X^2}, \tag{5.31}$$

where $r_{tv}$ denotes ratio of viscous to thermal damping,

$$r_{tv} = \frac{(4\eta/3+\xi)C_P}{(\gamma-1)\chi}. \tag{5.32}$$

Eq.(5.31) couples to Eq.(5.29) by means of the force $F$ (5.30). The stationary heating imposes $\partial T'/\partial t = 0$. For the preliminary evaluations of $T'$, we assume that $A$ is a solution to (4.18) with the boundary condition (4.20), that is, (4.21). The magnetosonic force takes the form

$$F = A_0^2\frac{\omega^2(\gamma-1)\chi}{2c_0C_P\rho_0^3}\frac{\exp\left(-\frac{2X^2}{Z^2+(1-\delta Z)^2} - 2\frac{z_d}{z_{tv}}Z\right)}{\sqrt{(1-\delta Z)^2+Z^2}}(1+r_{tv}). \tag{5.33}$$

Making use of (5.29), (5.33), we rearrange (5.31) as

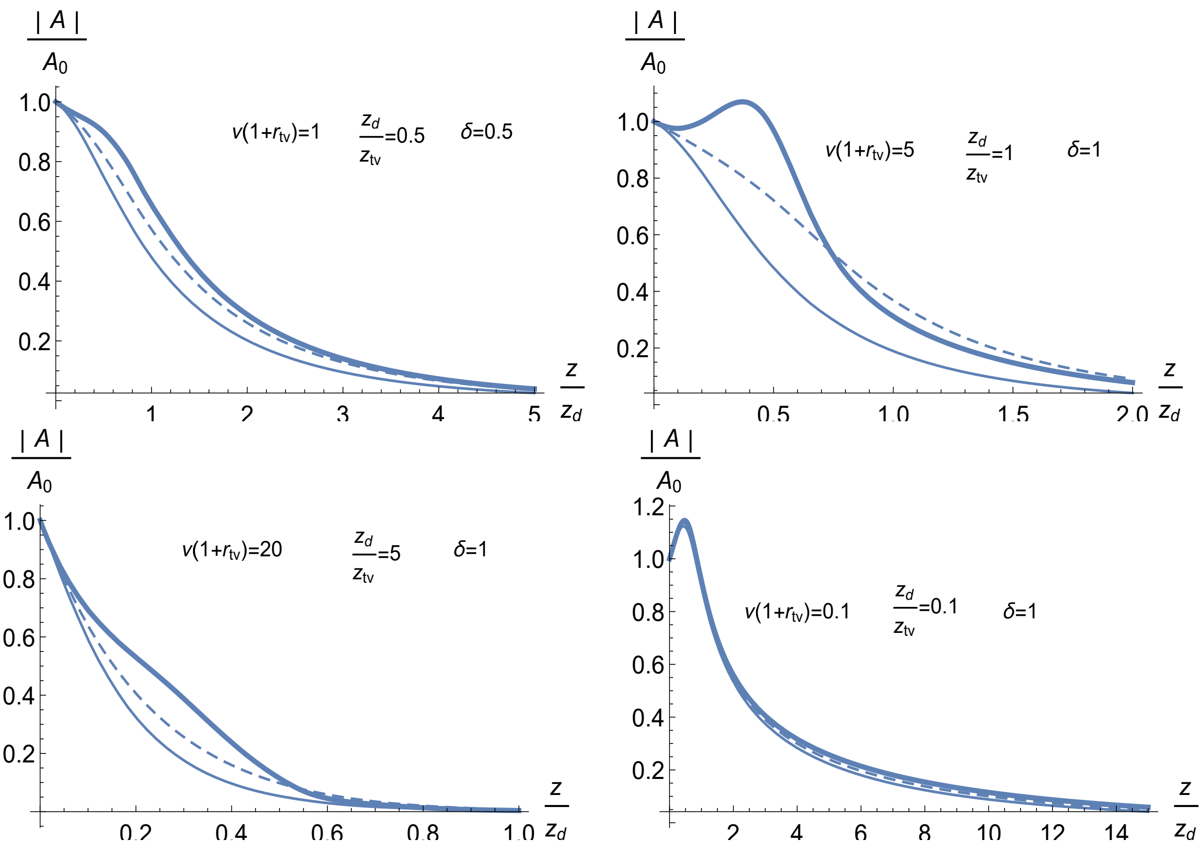
$$\frac{\partial A}{\partial Z} + i\nu(1+r_{tv})\exp\left(-2\frac{z_d}{z_{tv}}Z\right)\left(\exp\left(-\frac{2X^2}{Z^2+(1-\delta Z)^2}\right)\sqrt{Z^2+(1-\delta Z)^2} + \right. \tag{5.34}$$

$$\left.\sqrt{2\pi}X\mathrm{erf}\left(\frac{\sqrt{2}X}{\sqrt{Z^2+(1-\delta Z)^2}}\right)\right)A + \frac{z_d}{z_{tv}}A = \mp\frac{i}{4}\frac{\partial^2 A}{\partial X^2},$$

where

$$\nu = \frac{(\gamma-1)^2D^2}{2(\gamma+1)^2}\frac{z_d^2}{z_{nl}^2}, \tag{5.35}$$

and $\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp(-s^2)ds$ is the error function. So, dynamics is determined by four dimensionless parameters: the ratio of thermoviscous to diffraction effects, $\frac{z_d}{z_{tv}}$, squared ratio of nonlinear to diffraction effects $\nu$, the ratio of viscous to thermal effects, $r_{tv}$, and $\delta$. Eq.(5.34) is solved numerically in *Mathematica* with the boundary condition (4.20). Zero boundary conditions are set at $X = \pm 100$. $Z$ varies from 0 till 100. The dimensionless magnitude at the beam's axis $|A|/A_0$ for some sets of $\nu(1+r_{tv})$, $z_d/z_{tv}$ and $\delta$ in the unusual (bold lines) and ordinary cases (thin lines) of diffraction are shown in Fig.2. The broken curves represent $|A|/A_0$ at the axis of a beam in accordance to (4.22), that is, they represent a case without thermal self-action. Neither nonlinear distortion of a beam nor its nonlinear self-action are considered by the broken curves but only the initial curvature of a wave profile and linear impact of diffraction, viscosity and thermal conduction. Positive $\delta$ correspond to focused at a transducer beams and negative $\delta$ correspond to initially defocused beams. Usually, thermal self-defocusing of a beam in a gaseous medium takes place. The unusual case results in the larger magnitudes at the axis of a beam's propagation than that in the course of usual diffraction. This resembles thermal self-focusing when $(\partial c_0/\partial T)_p < 0$ which does not happen to gases but to the most liquids apart from water.



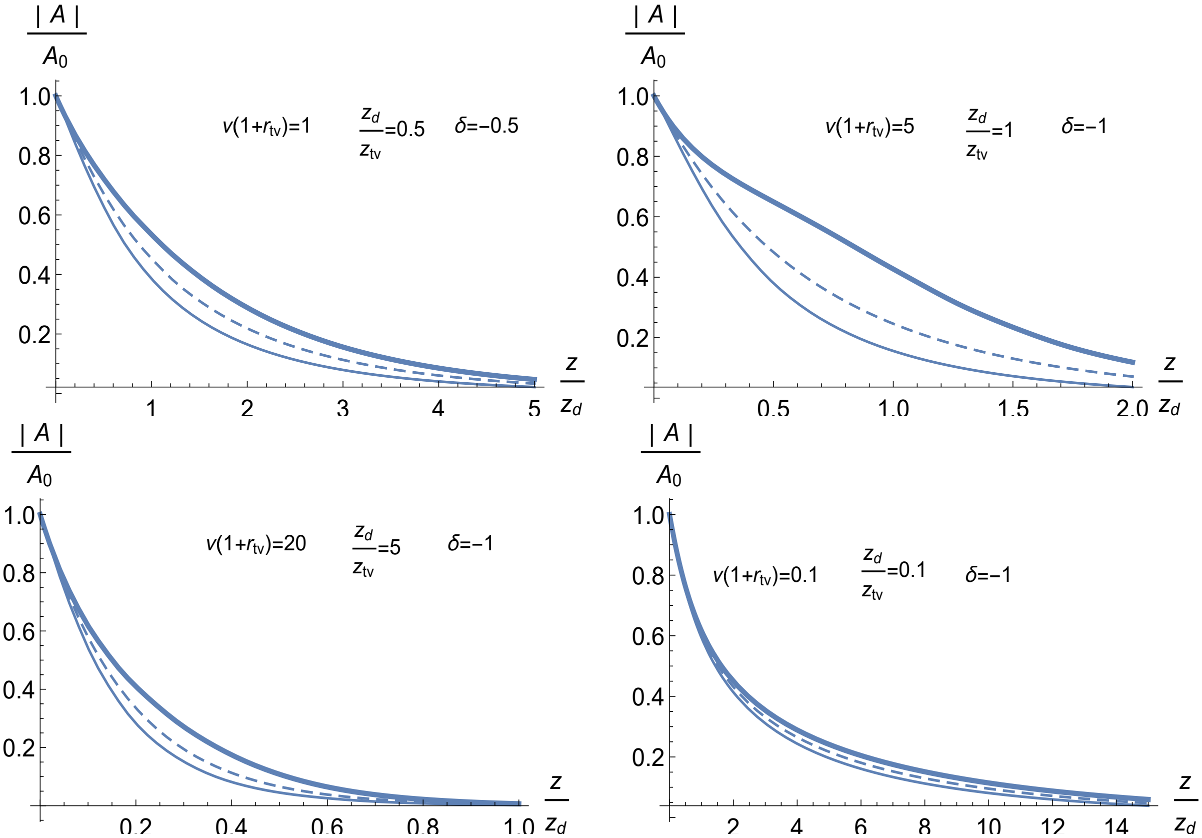



Figure 2. *The dimensionless amplitude at the axis of a beam $|A|/A_0$ for various $\nu(1+r_{tv})$, $z_d/z_{tv}$, $\delta$ ($\delta > 0$ for initially focused beams, and $\delta < 0$ for initially defocused beams). Bold lines correspond to the unusual case (sign plus on the right of* (5.34)*), and thin lines correspond to the ordinary case (sign minus). The boundary condition at the transducer is given by* (4.20)*. The dotted lines represent a linear case without thermal self-action, that is, $|A|/A_0$ determined by* (4.22)*.*

# 6 Concluding Remarks

The linear dynamic equations which describe perturbations in the weakly diverging beams which propagate parallel to the magnetic field ($p$-modes,(3.8); non-acoustic modes,(3.13)), take the form different from that in a Newtonian flow. The case $C = c_0 \neq C_A$ yields an unusual sign of the diffraction term in an equation describing perturbations if $c_0 < C_A$. This has been reported in Ref.[1]. The divergence is more pronounced for magnetosonic beams affected by the magnetic field compared to that in non-magnetized gas. The scale of transversal divergence is smaller $\frac{c_0}{\sqrt{|c_0^2-C_A^2|}}$ times. Eq.(4.15) incorporates nonlinear, diffraction and damping effects due to viscosity and thermal conduction on dynamics of perturbation of density in a magnetosound wave. It resembles the KZK equation with still unavailable analytical solutions [2, 19, 26] but with shorter diffraction length and possible unusual scenario of diffraction.

Sections 4,5 represent the main results of this study. They consider dynamics and non-linear thermal self-focusing of initially focused (or defocused) weakly divergent magnetosonic beams in thermoviscous plasma. Eq.(4.21) describes magnitude of perturbation of density in the focused beam with account for unusual diffraction if nonlinear distortion of the magnetosonic perturbations is weak. Sec.5 considers stationary thermal self-action of initially focused or defocused Gaussian periodic beams. The thermal self-action is modeled by Eq.(5.31) which couples to Eq.(5.29) by means of a force $F$ (Eq.(5.30)). Magnitude of perturbation of density in the magnetosonic Gaussian beam satisfying the boundary condition (4.20) is determined by (5.34) which considers unusual diffraction as well. The quasi-linear approach does not lead to the broadening of frequency spectrum and allows to consider the quasi-harmonic wave propagation, hence the nonlinearity goes into play only in coupling of the entropy and magnetosonic modes. The dynamics is determined by four parameters, $\nu$ (Eq.(5.35)), $z_d/z_{tv}$ (Eqs(3.10),(4.19)), $r_{tv}$ (Eq.(5.32)) and $\delta$ (Eq.(4.20)). In the course of self-focusing, thermal and viscous damping coefficients contribute individually by means of $r_{tv}$. The unusual thermal self-action leads to growth of a beam's magnitude at the axis and resembles an additional focusing of a beam which takes place in a medium with negative temperature coefficient $(\partial c_0/\partial T)_p$ (majority of liquids). The nonlinear effects have impact on a position of a focal point which moves towards a transducer. This occurs because the thermal lens becomes stronger in the course of the heating of a medium. Damping of a medium also shifts the focal point towards transducer and makes it less pronounced or prevents its formation. As for nonlinear thermal self-action caused by the Newtonian beams with discontinuities, the peak pressure at the axis decreases due to the nonlinear absorption [25]. The nonlinear distortion of a wave front is of a key importance in this case.

The numerical results of this study confirm theoretical conclusions. They are conducted in *Mathematica*. Numerical evaluations are founded on Eq.(5.31) and thus rely on weak nonlinear effects and do not concern waves with discontinuities. In an unusual case of diffraction, a beam is additionally focused which is not specific in gases. The upper series in Fig.2 reveal possibility for a pronounced focus to appear in the case of unusual propagation while it is absent in the usual cases due to damping despite the waveform is focused initially (positive $\delta$ is responsible for the initial focusing of a beam). The last four illustrations in Fig.2 show the case of a beam defocused initially ($\delta < 0$). An additional focusing in unusual cases is evident. Summarizing, a wide variety of linear and nonlinear behavior of perturbations in a beam rely on the type of divergence, usual or unusual. The results can have practical application and provide information about profile of perturbations for any set of parameters. The character of perturbations propagation in a beam may indicate the equilibrium parameters of plasma, damping coefficients, initial radius of a beam's front curvature and the characteristic frequency of perturbations in observations, also remote.

## DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study, which is a purely theoretical one.

## CONFLICT OF INTEREST

The author has no conflicts to disclose.

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