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Equivalence scales for continuous distributions of expenditure

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Abstract

Research background: In the actual sizable populations of households, the standard microeconomic concept of equivalence scales is intractable since its necessary condition of equality of household welfare levels is unlikely to be fulfilled.
Purpose of the article: This paper aims to develop a concept of an equivalence scale, which can be suitable for continuous distributions of expenditures in the population.
Methods: Using household welfare intervals, we get the random equivalence scale (RES) as the ratio of expenditure distributions of the compared populations of households.
Findings & value added: We derive the parametric distribution of RES for the lognormal distributions of expenditures. The truncated distribution of RES is applied to account for possible economies of scale in the household size. A society's inequality aversion can be helpful when selecting a single equivalence scale. We estimate RES for Poland using microdata on expenditures and subjective assessments of household welfare intervals. The estimated equivalence scales turned out to be very flat and dependent on welfare.

Introduction

Economists face severe difficulties when assessing economic inequality and the poverty rate (indeed all welfare issues) in society. Such assessments require data about the distribution of income or expenditure (hereafter treated as exchangeable) among comparable *individuals,* whereas available data from surveys concern *households.* However, households differ in many aspects other than expenditure, e.g., size and demographic composition. To account for household heterogeneity, economists use indices called *equivalence scales*. The microeconomic theory of consumer behaviour offers the concept of the equivalence scale for a pair of households, namely for a household with specific attributes and a reference household, usually a single-person household. The *standard equivalence scale* is the ratio of the households' expenditures if and only *if* the compared households attain the same *welfare level* (Blackorby & Donaldson, 1993).

Besides several advantages of the standard equivalence scales, they suffer from various shortcomings. The next Section offers a broader exposition.

In this paper, we focus on another trouble with the standard equivalence scales: inconsistency between the discrete nature of microeconomic categories and the continuous nature of these categories in the populations of households (*in the population*, for short). Such populations are remarkably sizable. Therefore, a discrete distribution of expenditures approaches a continuous distribution in the population according to the limit theorems.

Dealing with continuous expenditure distributions is not novel for econometric demand models from which equivalence scales are derived. For instance, expenditures will have the (continuous) lognormal distribution when the logarithms of expenditures are a (nonlinear) function of some non-random explanatory variables disturbed by normally distributed shocks.

The abovementioned inconsistency, overlooked in the literature, has significant theoretical consequences. In the population, the microeconomic discrete indirect utility functions become continuous random variables since they are admissible transformations of continuously distributed expenditures for all price vectors. An essential property of a continuous random variable is *zero probability* that it takes on a specific value. However, such an event should not be considered an impossible event but an event that is *unlikely to occur* (Fisz, 1967, p. 36). Thus, the probability that the indirect utility functions of two persons living in different households attain the same specific level is zero. In other words, the abovementioned *'if and only if'* condition of equality of welfare levels is unlikely to be fulfilled in the population. It is hard to rely on a concept which is based on an unlikely assumptions.

Moreover, the ratio of expenditures of a pair of households becomes the ratio of continuous random variables. Therefore, the microeconomic equivalence scale becomes a continuous random variable in the population.

The above circumstances have motivated the aim of this paper: developing the concept of equivalence scales, which could be suitable for continuous distributions of expenditures in the populations. We define *the random equivalence scale* (RES) for households of a specific type, given the group of reference households, as the ratio of expenditure distributions of the households *if and only if* the continuously distributed indirect utility functions of households' members fall into the same welfare *interval*. Thus RES is a continuous random variable, which distribution depends on expenditure distributions of compared households.

In this paper, we develop the parametric distribution of RES for the lognormal distribution of household expenditures. Then, the distribution of RES will also have the lognormal form. This fact makes estimating RES very easy. The truncated distribution of RES can account for possible scale economies in the household size.

To operate with a single equivalence scale in practice, one may use a particular measure of central tendency of RES, e.g. the mean, the median or the mode. In Section 4, we propose ethical recommendations for choosing a particular parameter.

We estimate RES for Poland using micro-data on expenditure from the Polish Household Budget Survey (PHBS) 2015. We use the subjective assessments of households' financial situation for determining welfare intervals. We have found that Polish households enjoyed large economies of scale in 2015. The estimated equivalence scales turned out to be dependent on welfare in that year.

The rest of this paper proceeds as follows. Section 2 offers a review of the literature on equivalence scales. In Section 3, we present the theoretical framework of RES. In Section 4, we develop the distribution of RES when expenditures obey the lognormal distribution. In Section 5, we show a formal relationship between some positional measures of RES and inequality aversion. This relationship can help in selecting a single equivalence scale.

In Section 6, we present the empirical results of the estimation of RES for Poland in 2015. Section 7 offers a summary and concluding remarks.

Literature review

Equivalence scales seek to answer the paradigm question, "how much money does a household need to spend to be as well off as a single person living alone?" (Browning *et al.*, 2013). The problem has a long tradition dating back to Engel (1895). Engel observed that income share devoted to food decreases with income for any family size. Therefore, Engel claimed that households with different compositions are equally well-off if they spend the same income share for food. Chiappori (2016) notices that "despite the numerous problems raised by this approach, it remains by and large the dominant method for devising welfare-related policies and estimating their effects."

Essential innovations have followed Engel's work by, e.g. Sydenstricker and King (1921), Rothbarth (1943), Prais and Houthakker (1955) and Barten (1964).[1] Lewbel and Pendakur (2008) offer a broader presentation of these issues.

For the sake of presentation, we assume the household size h=1,…,m as the attribute accounting for household needs in the population. We shall refer to the household with h≥2 as the 'h-household'. We assume single childless adults (h=1) as the *reference households,* although other specifications are possible.

The standard microeconomic equivalence scales for h-household, given reference household h=1, is defined as follows:

$$z_h = \frac{c(\boldsymbol{p},u_0,h)}{c(\boldsymbol{p},u_0,1)},\ h=2,\dots,m, \tag{1}$$

In Eq. (1), $c(\cdot)$ is the household cost (expenditure) function, i.e., the minimum cost of attaining utility level u_0 for a given demographic attribute h, $\boldsymbol{p}$ is the vector of prices (Deaton & Muellbauer, 1980).

For operational purposes, it is helpful to express the equivalence scale (1) in terms of expenditure x. The inversion of the cost function gives the indirect utility function, $v_h(\boldsymbol{p},x)$, of a person in h-household, h=1,2,…,m,

[1] see Lewbel and Pendakur (2008) for more details.

giving the maximum utility attained when spending x and facing prices $\boldsymbol{p}$. For each h=1,2,…,m, $v(\cdot)$ is continuous and homogeneous of degree zero in x and $\boldsymbol{p}$, increasing in x, quasi-convex, nonincreasing and locally non-satiated in p (Blackorby & Donaldson, 1993). If an analyst estimated equivalence scales using data from a single cross-sectional survey, considering price variability would not be necessary.

Let x and y denote the actual expenditure of h-household and the reference household, respectively. The cost functions and indirect utility functions are related by identities $y=c(\boldsymbol{p},u_0,1)\leftrightarrow v_1(\boldsymbol{p},y)= u_0$ and $x=c(\boldsymbol{p},u_0,h)\leftrightarrow v_h(\boldsymbol{p},x)= u_0$, h=2,…,m (Blackorby & Donaldson, 1993). Then, the equivalence scale (1) will be the ratio of actual expenditures

$$z_h = \begin{cases} \frac{x}{y}, & for\ h = 2,\dots,m \\ 1, & for\ h = 1 \end{cases} . \tag{2}$$

if and only if the households' members attain the same welfare level u_0, i.e., *if and only if* the following identity holds

$$v_h(p,x)=v_1(p,y)=u_0, \tag{3}$$

at all price vectors.

The quantity y is called the *household equivalent income,* namely the income necessary to provide a single childless adult with the same utility level that each member of the h-household enjoys. To measure inequality and the poverty rate, an analyst assigns the household equivalence income to each household member. In this way, an analyst transforms household income distribution into one in which each household is an identical single adult and is equivalent in terms of welfare to the actual one (Donaldson & Pendakur, 2004).

The standard model of equivalence scales assumes that a household is a unique consumption unit whose behaviour can be adequately described by a single interpersonally comparable utility function. In this so-called unitary approach, all household members enjoy the same household resource and the same level of wellbeing (ordinally) measured by that function.

Substituting y=x/z_h in Eq. (3) gives an alternative definition of an equivalence scale, namely

$$v_h(p,x)=v_1(p, x/z_h), \tag{4}$$

Then the equivalence scale z_h will be the solution to Eq. (4) provided such a solution exists.

The standard equivalence scale cannot be identified from demand data alone.[2] Pendakur (2018) clarifies this problem as follows. Let $\phi_h(\cdot)$ be a household-specific monotonic function of utility. Applying this function to the right-hand side of Eq, (4) gives[3]

$$\phi_h\,[v_h(p,x)]=v_1(p, x/z_h), \tag{5}$$

Any choice of $\phi_h(\cdot)$ provides a new equivalence scale while leaving demand curves unaffected. Therefore, there is an infinite number of equivalence scales, which are consistent with demand behaviour. Thus, one cannot identify and estimate a single equivalence scale from demand data alone. Equivalence scales depend on the exact choice of ϕ_h (cardinalisation of utility functions) and cannot be derived from ordinal preferences alone (Pendakur, 2018).

In the literature, there are various attempts to overcome the nonidentification of microeconomic equivalence scales. One can identify a single equivalence scale assuming the independence of utility u_0. This assumption is known as the *independence of base* (*IB*) (Lewbel, 1989) or *equivalence-scale exactness* (ESE) (Blackorby & Donaldson, 1993).[4] Blundel and Lewbel (1991) estimate IB and statistically reject it. Nevertheless, the authors observe that imposing the IB restriction has almost no effect on the estimated equivalence scales.

Blundell and Lewbel (1991) showed that one could identify *changes* in equivalence scales (1) having observations of the demands of the compared households in two separate price regimes. However, not *changes* but *the level* of an equivalence scale is necessary for most applications.

Some researchers use additional sources of identifying information. In specially designed surveys, respondents evaluate their welfare subjectively. It means the cardinalisation of indirect utility functions, i.e. the choice of the function ϕ_h. Then an equivalence scale is derived from Eq. (5). In the

[2] A quantity is said to be identified, if it cn be calculated from data.

[3] Pendakur (2018) do not apply this monotonic transformation to $v_1(\cdot)$; although it is possible, it makes the point harder to see.

[4] See Donaldson and Pendakur (2004, 2006) for more general assumptions.

Leyden school approach, a respondent evaluates an income that would be *very good*, *good*,…, *rather bad*, *bad* (see, e.g., van Praag, 1968, 1991; Goedhart *et al*., 1977; Kapteyn & van Praag, 1978). In this approach, the authors use cardinal, fully comparable utilities. Van Praag and van der Sar (1988) provide a version using ordinally fully comparable utilities. Zaidi and Burchardt (2005) use this methodology to consider equivalence scales for disability. The main drawback of this approach is that a respondent has to assess hypothetical incomes that he would never experience.

Some authors use the information on how satisfied a respondent is with his *actual income* (see, e.g., Melenberg & van Soest, 1996; Bellemare *et al*., 2002; Schwarze, 2003; Bollinger *et al*., 2012; Biewen & Juhasz, 2017). The answers in the form *good, rather good, neither good nor bad, rather bad, and bad* are the ordinal measurement of household welfare. Data on such assessments are available in typical household budget surveys. The general restriction implied is that such qualifications have the same meaning to every respondent. Tinbergen (1991) argued that: "This restriction can be accepted since in discussion on the policy resulting from the use of welfare measurements the *same words we **also** used* either to accept or to reject the policy."

Koulovatianos *et al.* (2005, 2019) performed vignette surveys where respondents *directly* evaluate equivalent incomes for various household compositions. The authors maintained that respondents could perform such evaluations.

Pendakur (2018) notices that asking people to direct evaluating utilities and equivalence scales is understandable and straightforward. However, the estimation of the scales may be debatable if the utility measures are noisy. Moreover, reported utility functions and equivalence scales may not be consistent with demand behaviour.

Some authors search for nonsubjective proxies of welfare. For instance, Jackson (1968) uses the budget share for food, the concept coming back to Engel (see Lewbel, 1999, pp. 190–192, for a more exhaustive review of Engel's scales). Recently, Szulc (2009) used several 'wellbeing covariates' for appraising household welfare and estimated equivalence scales for Poland with the help of the matching estimator.

The standard model of equivalence scales has raised various theoretical and normative difficulties. Chiappori (2016) summarises the difficulties as follows:

> *"Equivalence scales are at best severely biased, at worst totally misleading. Approaches based on income effects - which constitute in practice the main reference of policy-related empirical works - are structurally incompatible with the very notions (economies of scale) they try to capture. Moreover, the empirical estimation of both price- and income-based equivalence scales is largely arbitrary, not only because of the counterfactual assumptions on which they rely, but also and more deeply due to their reliance on interpersonal comparison of utility levels. Lastly, their normative implications are deceptive, if not plain nefarious, in particular, because they totally disregard issues linked with intrafamily allocation and inequality."*

Recently, the collective household models offered an alternative to the unitary approach. Chiappori (1992) introduced the *efficient collective household model*, treating households as collections of individuals. Browning *et al.* (2013) improve the model by introducing the concept of the *indifference scale*. The model replaces the abovementioned paradigm question of equivalence scales with the question: "How much income would an individual living alone need to attain the same indifference curve over goods that this individual attains as a member of a family of given composition? (Chiappori, 2016). The *indifferent income* is the household income divided by the indifference scale.

Indifference scales and indifferent incomes need not use interpersonal comparisons of welfare. They compare only for the same type of person, e.g. a person living alone and the same person living in a composite household, e.g. when being married. Thus, they can be identified entirely from indifference curves. However, Pendakur (2018) notices that the persons living alone are still heterogeneous.

Pendakur (2018) advocates for the use of indifference scales to deal with the fact that people live in heterogeneous economic environments and the use of equivalence scales to deal with the fact that people are themselves different from each other. He notices that equivalence scales and indifference scales are *complements*, not *substitutes*. The author uses both equivalence scales and indifference scales for developing the concept of *individual equivalent income.* A person's individual equivalent income is the amount of money that a reference individual needs to be as well-of as that person. Thus this concept can account for heterogeneity across people and the types of households in which people live.

Browning's *et al*. (2013) address the problem of estimating personal budget shares, which reflect the distribution of household incomes among its members. The budget shares account for income inequality within households. Dunbar *et al.* (2013, 2021) solve the problem for the case when demand functions of single persons are not observable. Pendakur (2018) notices that this case is essential for two reasons. First, household members may not have the same utility functions as reference individuals. Second, some people cannot be observed when living alone. For instance, children cannot live alone and so cannot reveal their preferences.

The collective household model accounts for economies of scale in household expenditures explicitly. The economies of scale reflect the mechanism that explains why the cost of living of a composite household is less than the sum of costs of living of its members taken independently (Chiappori, 2016). Economies of scale may arise from various sources, e.g. sharing household resources like food or shelter.

A general conclusion from the above considerations is that the collective household models do not entirely dismiss the equivalence scales. Pendakur's (2018) claim that equivalence scales and indifference scales are complementary leaves room for the former scales.

Bali (2012) argues that any model based on the unitary approach may be considered a particular case of a collective model. Therefore, the standard model of equivalence scales is not in contrast with the collective household models. The author concludes that there are still good reasons to estimate equivalence scales if enough empirical evidence supports their use.

The concept of the random equivalence scale

We assume that household expenditure in the population is a continuous random variable. As countries' household populations are sizable, discrete distributions of household expenditures can be approximated by continuous distributions according to the limit theorems. In the Introduction, we noticed that econometric demand models, the basis of deriving equivalence scales, implicitly assume continuous expenditures distributions for estimating purposes.

We assume the household size, $h=1,\ldots,m$, as an attribute accounting for differences in household needs. The term 'h-household' will denote the household of the size $h\geq 2$. The household of a single childless person ($h=1$) will denote the reference household.

We assume that positive valued continuous random variable Y with the density function $f_Y(y)$ ($Y \sim f_Y(y)$, for short) describes the distribution of expenditure in the population of reference households. Similarly, positive valued continuous random variable $X \sim f_x(x)$ describes the distribution of expenditure in the population of *h*-households.

Proposition 1.

Let indirect utility functions $v_1(\boldsymbol{p},y)$ and $v_h(\boldsymbol{p},x)$, quoted in (3), be continuous and monotonic transformations of expenditures (Blackorby & Donaldson, 1993). Then, under the above specifications of X and Y, the indirect utility functions will also be continuous random variables for all price levels, namely $V_1=v_1(\boldsymbol{p},Y)$ and $V_h=v_h(\boldsymbol{p},X)$.

Proof. For proof, it is enough to notice that any monotonic and continuous transformation of continuous random variables gives other continuous random variables (Fisz, 1963, p. 40).

It follows from Proposition 1 that the probability $\mathrm{P}(V_1=u_0)=\mathrm{P}(V_h=u_0)=0$. In other words, condition (3) is unlikely to occur in the population. To make this condition feasible, we substitute utility level u_0 in (3) by an interval, say (u_a,u_b) having non-zero probability. We assume a finite number k of disjoint intervals, which cover the whole domain of V_1 and V_h.

Definition 1.

We say that the members of h-households and the reference households are equally well-off if and only if their indirect utility functions V_1 and V_h fall into the same welfare interval with nonzero probability.

Allowing for welfare intervals having non-zero probability implies a continuum of x/y quotients of actual expenditure in Eq. (2). We treat these ratios as the realisations of the positive-valued continuous random variable Z

$$Z = \frac{X}{Y}, \tag{6}$$

with the density function $f_z(z)$ for all $z>0$.

Definition 2.

Z (6) *is said to be the random equivalence scale (RES) for h-households if and only if the members of h-households and reference households are equally well-off in the sense of definition* 1.

There is a sequence $Z_1,\dots,Z_k$ of RESs for k welfare intervals. The case when Z_i, i=1,…,k are independent (a testable condition) is the stochastic counterpart of the IB/ESE condition.

We could get a parametric form of $f_z(z)$ if the parametric forms of $f_x(x)$ and $f_y(y)$ were known. We may assume that X and Y are independent. Then, we can derive the density function $f_z(z)$ of RES (6) from the following formula

$$f_z(z) = \int_0^{\infty} y f_x(y \cdot z) f_y(y) dy, \tag{7}$$

(Fisz, 1967, p. 62).

RES (7) can reveal potential economies of scale in the household size (*economies of size,* for short) under some restriction on its domain. As equivalence scales measure the extent to which households share goods internally, higher values mean lower economies of scale (Bali, 2012). RES (6) for h household may take on all positive values. However, only values from [1,h], h=2,3,…,m, interval are admissible for Z to account for size economies. The upper bound h, yielding *expenditure per capita,* determines the lack of economies of scale. The lower bound 1, yielding *expenditure per household*, determines the maximum economies of scale.

For h-household, RES values greater than h would reflect *diseconomies of* size. For instance, the cost of a disabled household member could be greater than the cost of an able-bodied member. Also, the cost of a child could be greater than the cost of an adult person. Diseconomies of size could also arise if there are overcrowding effects as household size increases (Coulter *et al*., 1992). When diseconomies of size are suspected, one can apply the non-truncated distribution of RES.

It is worth adding that this way of accounting for size economies by RES is still valid for other specifications of household attributes than the household size. For instance, if an h-household comprised a certain number of adults and children, then h would still be the upper limit of RES.

Restriction RES's domain to [1,h] interval implies the *truncated distribution* of Z (6), with the following density function $f_t(z)$

$$f_t(z) = \begin{cases} \frac{f_Z(z)}{F_Z(h)-F_Z(1)}, for\ z \in [1,h] \\ 0, for\ z \notin [1,h] \end{cases} \tag{8}$$

where $f_z(\cdot)$ and $F_z(\cdot)$ denote the density and distribution functions of the non-truncated Z, respectively.

To get a single equivalence scale, one may use a parameter of the truncated distribution of $Z_t \sim f_t(z)$, e.g. the mean, the median, the mode, etc. In Section 5, we shall see that a society's inequality aversion can help choose a particular equivalence scale.

RES is a formalisation of actual circumstances which practitioners have inevitably experienced when estimating standard equivalence scales based on sizable sample data. Although one can discern the continuous character of expenditure distributions and welfare functions, he either ignores them or makes *ad hoc* assumptions not embodied in the standard equivalence scale model.

For instance, Jackson (1968) calculated equivalence scales for the USA measuring household welfare by the percentage of income spent on food. More specifically, she applied small income intervals that give welfare intervals. She presented the following result: "A *typical* adult living alone requires 36% of the income of a *typical* family of four to attain the same standard of living or *welfare level* as the family [our emphases]." Note that the expression 'equal welfare level' is misleading since Jackson used welfare intervals. Moreover, Jackson obtained *many* expenditures (realisations of X and Y, in our terms) for each welfare interval. In other words, she received *expenditure distributions* for each interval. Jackson's equivalence scale of 2.78 is not the definitional *number x/y* (2) of expenditure of *two individual households*, but the ratio E[X]/E[Y] of average expenditures designating a 'typical' family.

The random equivalence scale in the lognormal distribution of expenditure

Let expenditure in the general population of households obey the lognormal distribution with the density function

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}, x > 0, \sigma > 0 \tag{9}$$

where μ and σ^2 are the mean and the variance of logarithms of x, respectively (Aitchison & Brown, 1957). We shall use the common shorthand $X \sim \Lambda(\mu,\sigma^2)$ for a positive-valued random variable X with the lognormal distribution (9).

The lognormal distribution has been widely used in many scientific branches. In 1970, Johnson and Kotz predicted that "It is quite likely that the lognormal distribution will be one of the most widely applied distributions in practical statistical work in the near future." Johnson and Kotz, 1970, p. 128). Crow and Shimizu (1988) present the most recent developments in the theory of this distribution and its applications in various fields.

The use of the lognormal distribution as a theoretical model of income distributions has a long tradition, dated back to Kapteyn (1903), Gibrat (1931) and Kalecki (1945). Aitchison and Brown' (1957) monograph offered a unified theory of the distribution in question. Kleiber and Kotz (2003) devoted an extensive chapter for a concise presentation of the theory and application of the lognormal distribution to modelling income distributions.

In application, the lognormal distribution seems to fit well incomes in the middle range but fails in the upper tail (Hill, 1959; Cowell, 1977), specifically the top 3-4 percentiles (Airth, 1985). Nevertheless, some empirical findings support good fitting in a whole income range (see, e.g. Lopez & Servén, 2006).

Fitting the lognormal distribution may also give ambiguous results concerning income data and expenditure data. Lopez and Servén (2006) reported a better fit to income data than to expenditure data. Battistin *et al*. (2009) provided a reverse result: the distribution of consumption is much closer to the lognormal than income.

Application of three or four-parameter distributions could be an alternative to the lognormal distribution since the former distributions usually fit income or expenditure data better than the lognormal one (see, e.g. Bresson, 2009). This circumstance poses a tradeoff between goodness-of-fit and analytical tractability. In this paper, we apply the lognormal distribution because of analytical purposes.

There is a strict connection between the lognormal distribution $\Lambda(\mu,\sigma^2)$ and the normal distribution $N(\mu,\sigma)$. If $Y = \ln X \sim N(\mu,\sigma)$, then $X \sim \Lambda(\mu,\sigma^2)$. Thus, the distribution function $F(x)$ of the lognormal distribution (10) is equal to $\Phi((x-\mu)/\sigma)$, where $\Phi(\cdot)$ is the distribution function of the standard normal distribution N(0.1).

The measures of central tendency in the lognormal distribution (7) are of the following form:

The mean

$$M = exp\{ \mu + \sigma^2/2\}, \tag{10}$$

The median

$$Me = exp\{ \mu\}, \tag{11}$$

The harmonic mean

$$H = \exp\{\mu - \sigma^2/2\} \tag{12}$$

The mode

$$Mo = exp\{ \mu - \sigma^2\} \tag{13}$$

(Aitchison & Brown, 1957).[5] The geometric mean equals the median in the lognormal distribution.

Lemma 1 (Aitchison & Brown, 1957, p.11). If $X \sim \Lambda(\mu_x,\sigma_x^2)$ and $Y \sim \Lambda(\mu_y,\sigma_y^2)$ are independent, then $Z=X/Y \sim \Lambda(\mu_z,\sigma_z^2)$, where $\mu_z=\mu_x-\mu_y$ and $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$.

In other words, if X and Y obey the lognormal distribution, then Z will also follow it.

It is worth noticing that using three or four-parameter theoretical distributions does not provide compact forms of the density functions of the ratio X/Y. The final Section offers a broader discussion of this issue.

The truncated distribution of RES in [1,h] interval can account for possible economies of scale in the household size. Using (8), we can get the density function $f_t(z)$ and the distribution function $F_t(z)$ of the two-sided truncated lognormal distribution of Z_t. Note that

$$F_z(h) = \Phi\left(\frac{lnh - \mu_z}{\sigma_z}\right) = \Phi(\beta)$$

[5] Aitchison and Brown (1957) do not present the harmonic mean. However, H can be calculated easily as H=1/E[X^{-1}], using the fact that if $X \sim \Lambda(\mu,\sigma)$, then $X^{-1} \sim \Lambda(-\mu,\sigma)$ (Aitchison & Brown, 1957, p. 10). Then E[X^{-1}]=exp{$-\mu+\sigma^2/2$} and H=exp($\mu - \sigma^2/2$}.

and

$$F_z(1) = \Phi\left(-\frac{\mu_z}{\sigma_z}\right) = \Phi(\alpha)$$

where $\alpha = -\frac{\mu_z}{\sigma_z}$ and $\beta = \frac{\ln h - \mu_z}{\sigma_z}$.

Substituting these quantities and the density function $f_z(z)$ of $Z \sim \Lambda(\mu_z, \sigma_z^2)$ in (8), we get, after some algebra, the density function of the truncated lognormal distribution, namely

$$f_t(z) = \begin{cases} \frac{1}{z\sigma_z\sqrt{2\pi}(\Phi(\beta)-\Phi(\alpha))} \exp\left\{-\frac{(\ln z-\mu_z)^2}{2\sigma_z^2}\right\}, for\ z \in [1,h] \\ 0, for\ z \notin [1,h] \end{cases} \tag{14}$$

The distribution function of the truncated lognormal distribution has the form

$$F_t(z) = \begin{cases} 0, for\ z < 1 \\ \frac{\Phi\left(\frac{\ln z-\mu_z}{\sigma_z}\right)-\Phi(\alpha)}{\Phi(\beta)-\Phi(\alpha)},\ for\ 1 \le z < h\ , \\ 1, for\ z \ge h \end{cases} \tag{15}$$

(see Johnson *et al.*, 1994, p. 10). The symbols α, β, $\Phi(\cdot)$, and $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\}$ will be used throughout this paper.

Fig. 1 illustrates the density functions of the truncated and the non-truncated distributions of RES for three-person households reporting the ‘Average’ level of welfare.

In Fig. 1, the non-truncated distribution of Z is distributed as $\Lambda(0.439,0.429)$ (see Table 1 in Section 6). Thus, the probability of the appearance of the nonadmissible values of Z outside the interval [1,3], namely $P(Z<1 \vee Z>3)=0.405$, is remarkably high.

The *r*th moment of the truncated lognormal distribution is

$$E[Z_t^r|1 \le z \le h] = E[Z^r]\frac{\Phi(r\sigma_z-\alpha)-\Phi(r\sigma_z-\beta)}{\Phi(\beta)-\Phi(\alpha)}, \tag{16}$$

where $E[Z^r]=\exp\{r\mu_z+r^2\sigma_z^2\}$ is the *r*th moment in the non-truncated lognormal distribution $Z \sim \Lambda(\mu_z,\sigma_z^2)$ (Wang *et al.*, 2012). It is worth noting that formula (14) is valid for all real *r*.

Let M_z, Me_z, H_z, and Mo_z denote the mean, the median, the harmonic mean and the mode of the non-truncated distribution of $Z\sim\Lambda(\mu_z,\sigma_z^2)$ (formulae (10)–(13)), respectively. Then, these measures of central tendency for the truncated lognormal distribution are as follows:

The mean

$$M_t = corr_1 \cdot M_z, \tag{17}$$

where $corr_1 = \frac{\Phi(\sigma_z-\alpha)-\Phi(\sigma_z-\beta)}{\Phi(\beta)-\Phi(\alpha)}$.

The median

$$Me_t = corr_2 \cdot Me_z, \tag{18}$$

where $corr_2 = \exp\left\{\sigma_z \Phi^{-1}\left(\frac{\Phi(\beta)+\Phi(\alpha)}{2}\right)\right\}$.

The geometric mean

$$G_t = corr_3 Me_z, \tag{19}$$

where $corr_3 = \exp\left\{\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\sigma_z\right\}$.

The harmonic mean

$$H_t = corr_4 \cdot H_z, \tag{20}$$

where $corr_4 = \frac{\Phi(\beta)-\Phi(\alpha)}{\Phi(\sigma_z+\beta)-\Phi(\sigma_z+\alpha)}$.

The mode

$$Mo_t = \begin{cases} 1, for\ \mu_z \le \sigma_z^2 \\ Mo_z, for\ \sigma_z^2 < \mu_z < \ln h + \sigma_z^2 \\ h,\ \ for\ \mu_z \ge \ln h + \sigma_z^2 \end{cases} \tag{21}$$

Note that the median is not equal to the geometric mean in the truncated distribution of Z_t.

One can derive (17) from (16) putting r=1. Formula (18) can be derived by solving the equation $F_t(Me_t)$=1/2, where $F_t(\cdot)$ is the distribution function (15) of the truncated lognormal distribution of Z_t. Calculating $\exp\{E[lnZ_t]\}$ will give formula (19). Note that $Y = lnZ_t$ has the truncated normal distribution, with the density function $g(y) = \frac{1}{\sigma_z}\phi(\frac{y-\mu_z}{\sigma_z})/(\Phi(\beta)-\Phi(\alpha))$. Then,

(19) is obtained using $E_g[Y] = \mu_z + \frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\sigma_z$ (Johnson *et al.*, 1994, p. 10). Formula (20) can be derived as $(E[Z^{-1}|1<z<h])^{-1}$, using (16) with r=-1. The mode Mo_t (21) is the maximum of density function $f_t(z)$ (14). Three variants of Mo_t appear dependently on whether mode Mo_z of the non-truncated distribution of Z falls into the interval $[1,h]$ or not (see Fig. 1).

As mentioned earlier, a particular measure of central tendency of the truncated distribution of RES may serve as a single equivalence scale. The numerical calculation of M_t, Me_t, G_t, and H_t is straightforward, as it requires only calculations of the measures of central tendency of the non-truncated distribution of $Z \sim \Lambda(\mu_z,\sigma_z^2)$ and their adjustment by the corresponding correction $corr_i$ (i=1,2,3,4). The selection of the Mo_t depends on the position of Mo_z of the non-truncated distribution of Z toward interval $[1,h]$.

Ethical recommendations for the choice of a single equivalence scale

RES offers many equivalence scale values as a continuous random variable, whereas practice demands a single equivalence scale. This Section seeks an answer to the question of whether and how this demand could be fulfilled.

Coulter *et al.* (1992a) argue that equivalence scales are part of a social evaluation process. Adjusting household incomes by distinct equivalence scales yields different income distributions; hence different assessments of inequality and the poverty rate. Therefore, any ethical evaluation of income distributions *implicitly* embodies an indirect assessment of underlying equivalence scales.

Equivalence scales are also used for other purposes, notably for building up indexing schemes for social benefits, payments or exemptions, setting alimony and child support allowances, payments for life insurance and legal compensation for wrongful death (Lewbel & Weckstein, 1995). Thus, choosing a particular equivalence scale may have significant social consequences. Thus, the choice of an equivalence scale is not *ethically neutral*.

The primary type of social judgements are those summarised by the concepts of inequality and poverty aversion. Distributional assessments should also consider social judgements about differences in needs (Coulter *et al.*, 1992b).

Let a society or a social decisionmaker appraise a socially important random variable *D~g(d)*, e.g. income or expenditure, with the following utility function

$$u(d) = \begin{cases} \frac{d^{1-\varepsilon}}{1-\varepsilon} & for 0 \leq \varepsilon \neq 1 \\ log\, d & for \varepsilon = 1 \end{cases} , \quad (22)$$

(Atkinson, 1970). Parameter $\varepsilon \geq 0$ measures D's marginal utility elasticity and characterises the social decisionmaker's *inequality aversion.*

The *equally distributed equivalent income* (EDEI) is a convenient summary characteristic of appraisals of income. EDEI is the income in a hypothetical one-point (egalitarian) distribution that gives the same average utility (social welfare) as the initial distribution of incomes (Atkinson, 1970). Formally, EDEI, is the solution d_ε, say, to the following equation

$$u(d_\varepsilon) = E_g[u(D)], \quad (23)$$

where E_g is the operator of the mathematical expectation with respect to *g(d)*. We may formally solve Eq. (23) for *any* random variable *D~g(d)* provided E_g exists.

If a social decisionmaker evaluates income or expenditure with the help of utility function (22), the RES will indirectly become an object of the evaluation. This observation means the admission of a formal appraisal of the distribution of RES by utility function (22).

The distribution of RES exhibits *inequality.* RES would be a single number if and only if its distribution were 'egalitarian', i.e., the one-point distribution. Although actual distributions of RES are not 'egalitarian', we may formally get the hypothetical egalitarian distribution with the help of Eq. (23). Note that now a social decisionmaker having inequality aversion ε is interested in eradicating inequality in the distribution of RES.

If we apply the utility function (22) with $\varepsilon \neq 1$ for the evaluation of the truncated lognormal distribution of $Z_t \sim f_t(z)$, the solution z_ε to Eq. (23) will have the form $z_\varepsilon = \{E_{f_t}[Z_t^{1-\varepsilon}]\}^{1/(1-\varepsilon)}$. We may apply the formula (16) with r=1-ε to determine the expectation E_{f_t}. For $\varepsilon \neq 1$, we get

$$z_\varepsilon = \exp\{\mu_z + (1-\varepsilon)\sigma_z^2/2\} \left[\frac{\Phi((1-\varepsilon)\sigma_z - \alpha) - \Phi((1-\varepsilon)\sigma_z - \beta)}{\Phi(\beta) - \Phi(\alpha)}\right]^{1/(1-\varepsilon)} \quad (24)$$

Formula (24) specifies a single equivalence scale z_ε that would be preferable by a social decisionmaker with inequality aversion ε≠1.

When ε=1, Eq. (23) takes on the form: $ln\, z_\varepsilon = E_g[lnZ_t]$, where $g(y)$ is the density function of the random variable $Y = lnZ_t$. It is easy to see that z_ε will then be the geometric mean G_t (19), i.e.,

$$z_\varepsilon = \exp\left\{\mu_z + \frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\sigma_z\right\} \tag{25)}$$

Formula (25) specifies a single equivalence scale z_ε preferable by a social decisionmaker with ε=1.

Setting some particular values of inequality aversion ε in (24) and respecting (25), we get

$$z_\varepsilon = \begin{cases} M_t, & for\ \varepsilon = 0 \\ G_t, & for\ \varepsilon = 1 \\ H_t, & for\ \varepsilon = 2 \\ cMo_z, & for\ \varepsilon = 3 \end{cases} \tag{26}$$

where M_t, G_t, and H_t are the mean, the geometric mean, and the harmonic mean, respectively, of the truncated lognormal distribution of Z_t. Mo_z is the mode of the non-truncated distribution of Z and $c = \left[\frac{\Phi(\beta)-\Phi(\alpha)}{\Phi(\beta+2\sigma_z)-\Phi(\alpha+2\sigma_z)}\right]^{1/2}$.

One can interpret the result (25) as follows. If the social decisionmaker were inequality neutral (ε=0), he/she would recommend the mean M_t. as the equivalence scale. If the social decisionmaker exhibited a 'moderate' aversion to inequality (ε=1, say), he/she would recommend the geometric mean G_t as a single equivalence scale. If the social decisionmaker had inequality aversion ε=2, he/she would recommend an equivalence scale based on the harmonic mean H_t. Finally, if the social decisionmaker exhibited a 'strong' aversion to inequality (say, ε=3), he/she would recommend an equivalence scale proportional to mode Mo_z.

In general, one may use formulae (24) and (25) for calculating a single equivalence scale for any level of inequality aversion. Several methods of estimating country inequality aversion have been proposed in the literature (see, among others, Kot, 2020 for a broader explanation).

Eq. (26) shows how equivalence scales coincide formally with the measures of central tendency of the truncated distribution of RES for some particular values of ε. On the other hand, if an analyst used a particular measure of central tendency as a single equivalence scale, he would implicitly assume a particular level of society's inequality aversion. For instance, Jackson's (1968) equivalence scale of the form E[X]/E[Y] seems to be proper for an inequality-neutral society, provided the lognormal distribution of expenditures.

Eq. (24) also reveals the relationship between inequality aversion and economies of size. As EDEI is generally a declining function of ε (Lambert, 2001, p. 101), the greater the inequality aversion, the smaller the equivalence of scale, and, therefore, the greater economies of size.

Empirical illustration

We estimate RESs using micro-data on monthly household expenditure from the PHBS 2015. For the sake of presentation, we select 35,627 households inhabited by, at most, five persons. The five groups of households comprise 96 per cent of all Polish households in that year. We chose households of childless single individuals as the reference group. Fig.2 shows the density functions of expenditure distributions of the selected groups of households.

We specify five categories of household welfare using the answers to the following PHBS' question: 'Please, evaluate the financial situation of their households: 1 very good, 2 rather good, 3 average (neither good nor bad), 4 rather bad, 5 bad [original wording of HBS]. We treat these disjoint categories as the indicators of unobserved and disjoint household welfare intervals. Although the categories exhibit an order, the ordinal values of household welfare are not necessary for the construction of RES.

The estimation of single equivalence scales based on RES is straightforward. For this, it is enough to estimate the parameters μ and σ^2 of the lognormal distribution of expenditure for selected household sizes within each welfare category separately.

We can also estimate the lognormal distribution parameters for selected household sizes, *independently* of welfare categories. This 'Overall' version is a stochastic counterpart of the IB/ESE assumption.

To check whether the expenditures obey the lognormal distribution, we apply *the spread ratio test* Θ, which verifies the normality of the logarithms of expenditure. The test has the following form:

$$\Theta = \frac{\mu - u_{0.01}}{u_{0.99} - \mu} \tag{24}$$

where μ is the mean and the quantities $u_{0.01}$, $u_{0.99}$ are the quantiles of order 0.01 and 0.99, respectively. An approximate range for acceptance of the normal distribution is $0.7<\Theta<1.3$ (Thomopoulos, 2017).[6]

In the literature, there are almost forty tests of normality having miscellaneous and often unknown power. As income data samples are usually huge, some of the most powerful tests seem to be intractable. For instance, the chi-square test prescribes rejecting almost all theoretical income distributions for large samples (McDonald & Xu, 1995). Some of the other popular tests of goodness of fit perform poorly when statistical hypotheses are composite.

We use the spread ratio test because of its simplicity and tractability for sizable data samples.[7] We have not found in the literature arguments that the test performs worse than other tests. The analyse of the power of this test is outside of the scope of the paper.

Table 1 presents the calculated values of the spread ratio test. Examining Table 1 shows that we cannot reject the lognormal form of expenditure distributions for all cases, except the case of 2-person households in the 'Very good' category of welfare. This case makes up only 3.33% of all households.

Table 1 also reveals that the welfare distribution among Polish households in 2015 was not symmetric toward the 'Average' category. Approximately 15 % of households declared welfare categories below 'Average', whereas about 28% claimed welfare categories above 'Average'.

Table 2 presents the estimates of the lognormal distribution parameters of expenditure. These estimates, i.e. (μ_x, σ_x^2) for h-households and (μ_y, σ_y^2) for reference households are the basis for the calculation of the parameters μ_z and σ_z^2 of the non-truncated lognormal distribution of RES, according to Lemma 1, with $\mu_z=\mu_x-\mu_y$ and $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$. Table 3 presents the results.

We apply formulae (24) and (25) for calculating single equivalence scales, assuming four levels of inequality aversion ε=0, 1, 2, and 3, which correspond to the measures of central tendency in the distribution of RES, namely M_t, G_t, H_t and cMo, respectively in eq. (26). The results are presented in Table 4. For comparison, we also show the LIS equivalence scale (the square root of household size).

[6] Unfortunately, Thomopoulos (2017) does not report the power of this test against alternative distributions. .

[7] An anonymous referee supposes that the Q test checks mainly symmetry of hypothesized distribution.

Examining Table 4 shows that the equivalence scales for Polish households are very flat compared to the per-capita equivalence scale. One can also see that the larger the inequality aversion, the smaller the equivalence scales and, therefore, the larger economies of size.

Figure 3 illustrates the welfare independent (Overall) equivalence scales and the LIS scale. One can see that the Overall scale is very flat. One can also notice that the greater the household size, the greater the diversification of equivalence scales concerning inequality aversion. The LIS equivalence scale seems to be closest to the Overall version of RES for ε=1. Thus, LIS seems to be a proper equivalence scale if the Polish society in 2015 showed a moderate aversion to inequality.

Formal testing, whether equivalence scales depend on welfare or not, is cumbersome. We cannot apply the analysis of variance because our data violate its assumptions.[8] However, we can examine this problem graphically.

Figures 4–7 illustrate the relationship between welfare and equivalence scales for each household size separately. The graphs in the figures display this relationship for equivalence scales in the form of measures of central tendency of RES, namely the mean (ε=0), the median (ε=1), the harmonic mean (ε=2) and the mode (ε=3).

Visual examining Figures 4–7 shows two interesting features. First, the level of an equivalence scale depends on inequality aversion, namely the greater ε, the smaller an equivalence scale, therefore, the more significant economies of size. Second, equivalence scales seem to depend on welfare (expenditure). One may hypothesise that equivalence scales in question declines with welfare: the more sizable a household, the more transparent this relationship.[9]

We qualify the declining equivalence scales with welfare (expenditure) pattern as hypothetical since it relies only on visual inspection. Several researchers have reported such a pattern (see, e.g. Donaldson & Pendakur, 2004, 2006; Koulovatianos *et al.*, 2005; Majumder & Chakrabarty, 2008; Balli & Tiezzi, 2011).

[8] I am grateful to an anonymous referee for this remark.

[9] We have observed the same pattern when we use average expenditures of reference household within each welfare category (Donaldson & Pendakur, 2004).

Conclusions

This paper shows what would happen with the microeconomic equivalence scales if continuous distributions approximated the discrete distributions of incomes or expenditure. Such an approximation seems to be entirely accurate for vast populations of households.

The continuous distribution of income or expenditure renders the microeconomic concept of equivalence scales impractical, as its necessary condition of equality of household welfare levels is unlikely to be fulfilled. The use of welfare intervals overcomes this unlikeliness and leads to the concept of the random equivalence scale as a continuous random variable.

The calculation of single equivalence scales based on the estimated distribution of RES is straightforward. Only expenditure distributions of the h-households and the reference households are necessary for deriving the RES distribution. There is no need to consider the underlying microeconomic mechanisms that generate these distributions. In particular, it is not necessary to estimate a demand system.

The concept of RES is heuristically prolific. The truncated distribution of RES can account for economies of size. RES can also predict new economic phenomena, namely the relationship between a society's attitude toward inequality and equivalence scales and the relationship between inequality aversion and economies of size. These relationships have not been analysed yet.

One can apply RES for any specification of household attributes. For instance, if various compositions of adults and children are determined, the h-index could identify these compositions. Nevertheless, the household size will still be valid for truncating the distribution of RES when accounting for economies of size.

The empirical results of this research suggest that equivalence scales depend on household welfare (expenditure). Thus, RES can provide results consistent with results obtained from estimated demand systems. Nevertheless, these conclusions should be treated with some caution since we have drawn them from limited empirical data (one country and one year).

The recent version of RES has two apparent limitations: the subjective measurement of welfare intervals and the lognormal distribution of expenditure. Further research is needed to check the robustness of RES to other specifications of household welfare. When a set of nonsubjective household welfare indicators were convincingly specified, either Szulc's (2009) or a cluster analysis could be applied to determine utility intervals.

Although the lognormal form of household expenditures turned out to be suitable for our data, other theoretical forms recommended in the literature might be considered (see Kleiber & Kotz, 2003, for a concise review). For some distributions, the density functions of the ratio of two random variables are known. Malik (1967) and Ahuja (1969) showed that the ratio of two independent gamma distributions has the GB2 distribution. Mielke and Flueck (1976) found the ratio of two Weibull random variables. The gamma and Weibull distributions seem to be plausible theoretical models of income distributions (e.g., Salem & Mount, 1974; Bartels & van Metelen, 1975).

Pollastri and Zambruno (2010) derived an equivalence scale as the ratio of two Dagum (1977) distributions. [10] Unfortunately, the density function and the distribution function of the ratio do not have compact forms. The authors approximated the distribution function of the ratio by numerical integration. They used the median as an equivalence scale. The authors did not restrict the domain of the ratio to [1,h] interval. Thus the scale does not account for possible economies of size.

Deriving truncated forms of the abovementioned distributions might be analytically and computationally much more cumbersome. The recent mathematical literature does not offer very much on this issue.

Thus, even if the lognormal distribution does not fit income data perfectly, it can provide preliminary estimates of the random equivalence scales. It would be interesting to determine how robust such estimates are to the choice of theoretical income distributions.

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[10] I was not aware of Pollastri and Zambruno's (2010) paper when I elaborated the concept of RES.

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Annex

Table 1. The spread-ratio test Θ

Size	*1 Very good*	*2 Rather good*	*3 Average*	*4 Rather bad*	*5 Bad*	*N*	*[%]*
1	0.830	0.865	0.856	1.021	1.072	7455	20.93
2	0.670	0.884	0.905	0.920	1.031	12040	33.79
3	0.928	0.878	0.856	1.058	0.856	7446	20.90
4	0.840	0.871	0.864	1.012	1.054	6199	17.40
5	1.165	0.718	1.016	0.980	1.020	2487	6.98
N	3503	6563	20078	3983	1500	35627	
[%]	9.83	18.42	56.36	11.18	4.21		100.00

Note: *N* is the number of households

Source: own elaboration using data from Polish HBS 2015.

Table 2. Parameters of the lognormal distribution of expenditure

Size	*1 Very good*		*2 Rather good*		*3 Average*		*4 Rather bad*		*5 Bad*		*6 Overall*	
	μ	σ^2	μ	σ^2	μ	σ^2	μ	σ^2	μ	σ^2	μ	σ^2
1	7.80	0.35	7.61	0.27	7.30	0.22	7.08	0.21	6.85	0.25	7.31	0.30
2	8.21	0.33	8.03	0.23	7.74	0.21	7.44	0.21	7.33	0.22	7.79	0.27
3	8.39	0.31	8.21	0.22	7.95	0.20	7.64	0.18	7.53	0.21	8.01	0.26
4	8.46	0.31	8.29	0.21	8.03	0.20	7.78	0.16	7.69	0.23	8.11	0.25
5	8.45	0.31	8.32	0.21	8.10	0.19	7.89	0.17	7.87	0.17	8.14	0.22

Source: Own elaboration using data from Polish HBS 2015.

Table 3. Parameters of the non-truncated lognormal distribution of $Z \sim \Lambda(\mu_z, \sigma_z^2)$

Size	*1 Very good*		*2 Rather good*		*3 Average*		*4 Rather bad*		*5 Bad*		*6 Overall*	
	μ_z	σ_z^2	μ_z	σ_z^2	μ_z	σ_z^2	μ_z	σ_z^2	μ_z	σ_z^2	μ_z	σ_z^2
2	0.41	0.68	0.42	0.50	0.44	0.43	0.36	0.43	0.48	0.47	0.41	0.68
3	0.59	0.66	0.60	0.49	0.65	0.43	0.56	0.40	0.68	0.46	0.59	0.66
4	0.66	0.66	0.68	0.48	0.73	0.42	0.70	0.38	0.85	0.48	0.66	0.66
5	0.65	0.67	0.71	0.48	0.80	0.41	0.81	0.38	1.02	0.42	0.65	0.67

Source: own calculations using data from Table 2.

Table 4. Random equivalence scales for Poland 2015

RES	*Size*	*Equivalence scales within welfare categories* Very good	Rather good	Average	Rather bad	Bad	*Overall*	*LIS*
M_t ε=0	2	1.4481	1.4505	1.4546	1.4451	1.4581	1.4561	1.4142
	3	1.8298	1.8349	1.8557	1.8211	1.8613	1.8615	1.7321
	4	2.1431	2.1343	2.1702	2.1445	2.2528	2.2047	2.0000
	5	2.3564	2.3616	2.4284	2.4390	2.6586	2.4624	2.2361
G_t ε=1	2	1.4202	1.4229	1.4270	1.4176	1.4297	1.4283	1.4142
	3	1.7464	1.7531	1.7746	1.7413	1.7782	1.7783	1.7321
	4	1.9949	1.9920	2.0287	2.0064	2.0911	2.0570	2.0000
	5	2.1449	2.1606	2.2293	2.2422	2.4470	2.2519	2.2361
H_t ε=2	2	1.3928	1.3956	1.3998	1.3906	1.4032	1.4008	1.4142
	3	1.6663	1.6743	1.6957	1.6647	1.6995	1.6973	1.7321
	4	1.8572	1.8595	1.8953	1.8770	1.9667	1.9166	2.0000
	5	1.9575	1.9806	2.0468	2.0609	2.2484	2.0585	2.2361
cMo ε=3	2	1.3664	1.3694	1.3736	1.3647	1.3767	1.3743	1.4142
	3	1.5932	1.6019	1.6225	1.5945	1.6254	1.6224	1.7321
	4	1.7381	1.7441	1.7773	1.7630	1.8397	1.7924	2.0000
	5	1.8046	1.8310	1.8915	1.9058	2.0668	1.8950	2.2361

Note: M_t, G_t, H_t and cMo_z are the arithmetic mean, the geometric mean, the harmonic mean in the truncated distribution of *Z*, respectively whereas Mo_z is the mode in the non-truncated distribution of *Z* (see eq. 24).

Source: own calculations using data from Table 3.

Figure 1. Truncated and non-truncated distributions of RES for three-person households

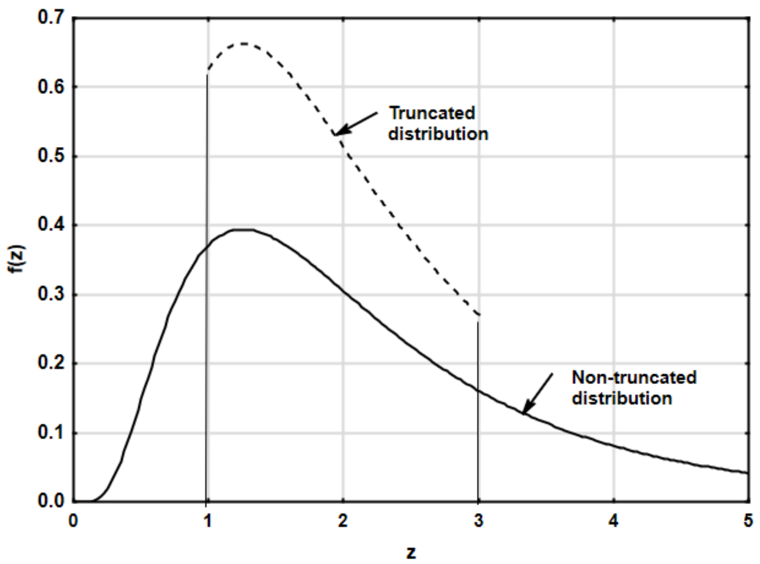


Figure 2. Expenditure distributions for selected household sizes (the Gaussian kernel estimates)

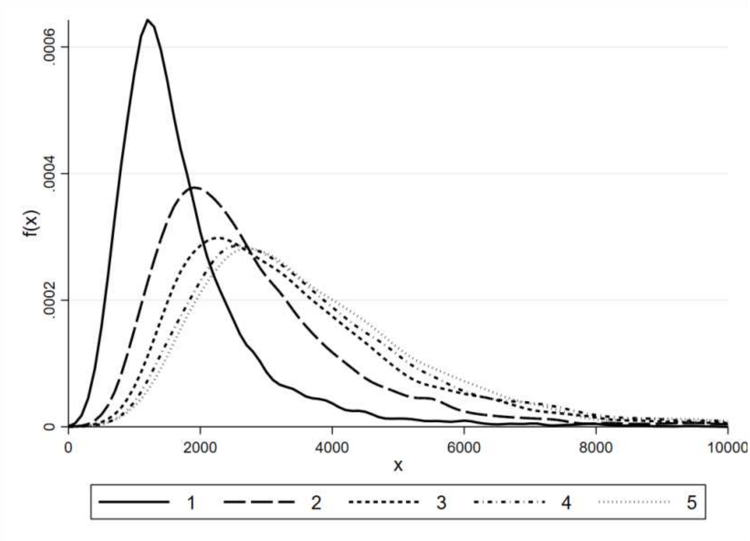
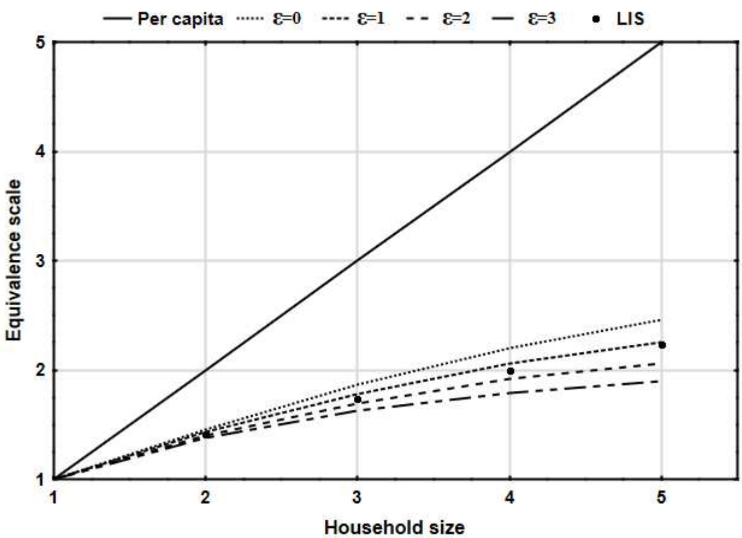
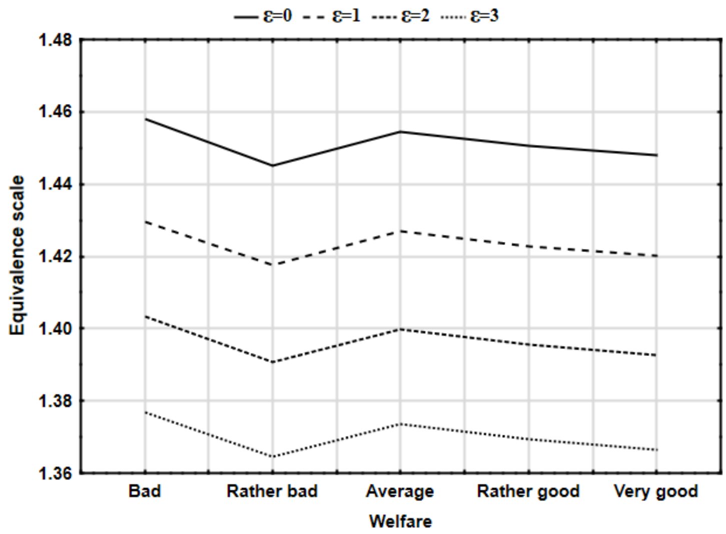


Figure 3. The overall (welfare independent) equivalence scales for various levels of inequality aversion ε. For comparison, the Per capita and LIS scales are added



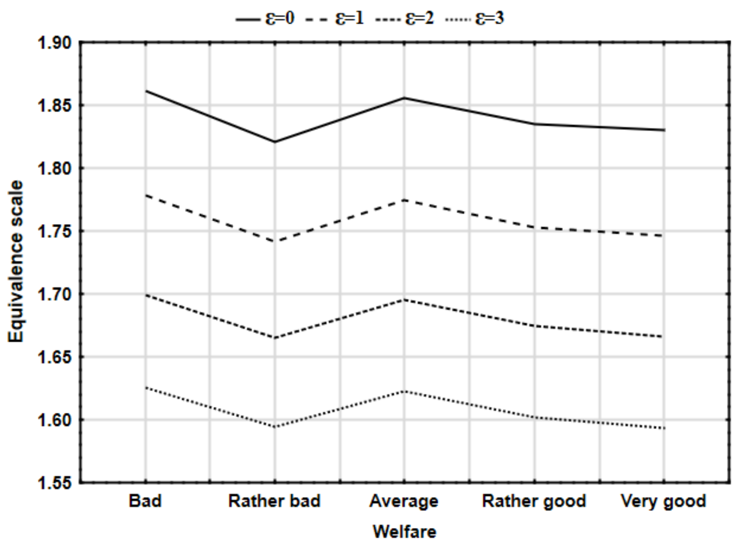
Source: own elaboration using data from Table 4.

Figure 4. Equivalence scales for two-person households



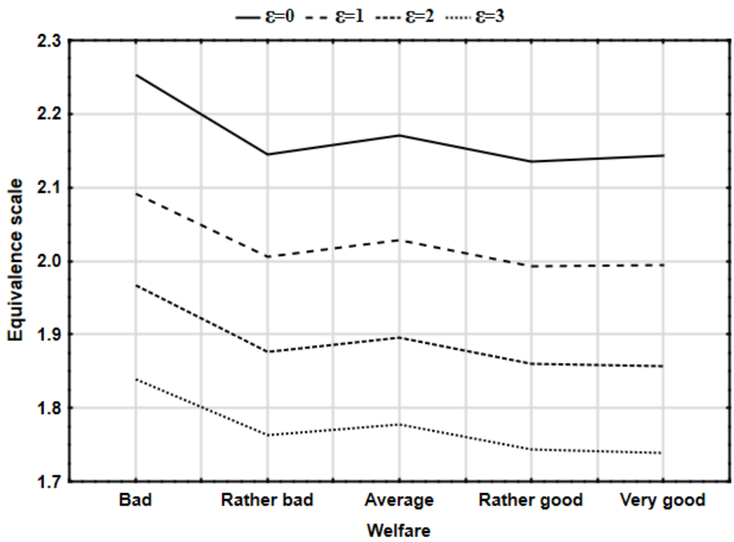
Source: own elaboration using data from Table 4.

Figure 5. Equivalence scales for three-person households



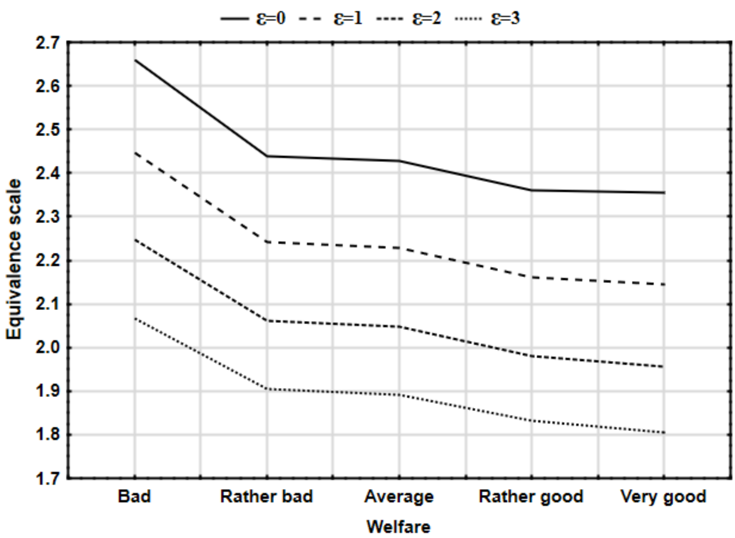
Source: own elaboration using data from Table 4.

Figure 6. Equivalence scales for four-person households



Source: own elaboration using data from Table 4.

Figure 7. Equivalence scales for five-person households



Source: own elaboration using data from Table 4.