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ORIGINAL RESEARCH

Billiard in a rotating half-plane

Sergey Kryzhevich[1] · **Alexander Plakhov**[2,3]

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Abstract

The main objective of this research is to study the properties of a billiard system in an unbounded domain with moving boundary. We consider a system consisting of an infinite rod (a straight line) and a ball (a massless point) on the plane. The rod rotates uniformly around one of its points and experiences elastic collisions with the ball. We define a mathematical model for the dynamics of such a system and write down asymptotic formulae for its motions. In particular, we determine existence and uniqueness of solutions. We find all possible grazing impacts of the ball. Besides, we demonstrate that for almost every initial condition, the ball goes to infinity exponentially fast, with the time intervals between neighboring collisions tending to zero. The approach developed in this paper is an original combination of methods of Billiards and Vibro-Impact Dynamics. It could be a base for studying more complicated systems of similar types.

Keywords Billiards · Moving boundary · Elastic impact · Sliding

1 Introduction

Isaac Newton in 1687 [18] considered the problem of least resistance for a body moving in a rarefied medium. He assumed that the medium is rarified, so that the mutual interaction of particles of the medium can be neglected, and that collisions of the particles with the body surface are perfectly elastic. These assumptions greatly simplify the optimization problem.

Alexander Plakhov contributed equally to this work.

✉ Sergey Kryzhevich
serkryzh@pg.edu.pl

Alexander Plakhov
plakhov@ua.pt

1 Institute of Applied Mathematics, Faculty of Applied Physics and Mathematics and BioTechMed Center, Gdańsk University of Technology, ul. Gabriela Narutowicza 11/12, Gdańsk 80-233, Poland

2 Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro 3810-193, Portugal

3 Institute for Information Transmission Problems, Bolshoy Karetny per. 19, build.1, Moscow 127051, Russia

Starting from 1993 [5], many mathematical papers studying various settings and approaches to Newton's problem have appeared (see, e.g., [1, 4, 6, 12, 13, 20–22] among others). It is generally assumed in these papers that the body translates in the medium; see, however, the papers [11, 19, 23, 24] where a combination of translational and rotational motions is considered.

To the best of our knowledge, a regular study concerning the free motion of a body (involving both translation and rotation) in the framework of Newtonian aerodynamics has never been carried out, even in the 2D case. Theorems of existence and uniqueness for the dynamics have not been obtained, and free motion on the plane of special shapes, even the simplest ones, such as an ellipse, a square, or even a line segment, has never been studied.[1]

Here we start with the case which seems to be the simplest one: a line segment. By simplifying further the problem, assume that the mass of the segment is infinite. Initially, it stays at rest in the horizontal position in the plane, and then it starts rotational motion about its center counterclockwise.

Of course, the first hit of each particle is with the right half of the rod. It is assumed that the medium particles do not mutually interact, so it suffices to consider the interaction of each individual particle with the rod. It also makes sense to suppose that the length of the rod is infinite.

Thus, we have the billiard in a moving domain on the plane. The domain is a half-plane rotating uniformly about a fixed point on its boundary. Note that the previous works on billiards in moving boundaries are mainly motivated by studying the mechanism of Fermi acceleration. By contrast, our motivation comes from Newton's least resistance problem.

Apparently, the billiard in a rotating half-plane has never been studied. The system looks very simple, but its study is far from trivial, as will be seen in this paper (Figs. 1 and 2).

Without loss of generality, assume that the angular velocity of the rod equals 1 and the rotation is counterclockwise. It is convenient to consider the dynamics in the rotating coordinate system at the complex plane $\mathbb{C}$, where the rod is represented by the real axis $\mathbb{R}$, with the fixed point being at the origin, and the position of the ball $z(t)$ at the instant of time $t \in \mathbb{R}$ belongs to the closed upper half-plane $\mathbb{C}_0^+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. Between two neighboring impacts, the ball moves uniformly according to the formula

$$z(t) = (z + wt)\exp(-it), \qquad z,\ w \in \mathbb{C}, \tag{1}$$

in the interior of the upper half-plane, $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and reflects elastically when hitting the real line, that is, if $z(t) \in \mathbb{R}$ then

$$\dot{z}(t+0) = \dot{z}(t-0)^* \tag{2}$$

(asterisk means the complex conjugation and dot means the derivative in t).

If $\dot{z}(t-0) \in \mathbb{R}$ in (1), and therefore, the function z is differentiable at t,

$$\dot{z}(t) = \dot{z}(t+0) = \dot{z}(t-0),$$

then we say that a *grazing impact* takes place at the instant t.

The initial position and velocity of the ball are given by

$$(z(0), \dot{z}(0)) = (z_0, \dot{z}_0), \qquad z_0 \in \mathbb{C}^+, \quad \dot{z}_0 \in \mathbb{C}.$$

Taking into account the nature of our problem, we assume that the first hit is from the right half-axis, $\mathbb{R}^+ := (0, +\infty)$, leaving the general case to the future.

[1] The only exception is the disk, whose dynamics is trivial: its motion is rectilinear, and its scalar velocity satisfies a differential equation $\dot{v} = -cv^2$.

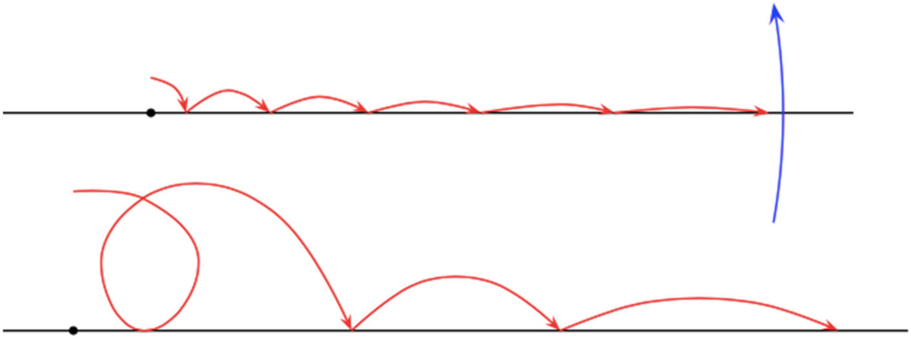

Fig. 1 Various trajectories of the ball in the rotating coordinate system. The rod rotates counterclockwise

Definition 1 A function $z(t)$, $0 \le t < T \le +\infty$ is called a billiard trajectory, if there exists a finite or countable sequence of values $0 < t_1 \ldots < t_m < \ldots \le T$ such that

(a) for $0 \le t < t_1$, $t_1 < t < t_2, \ldots, t_m < t < T$, the function $z(t)$ lies in $\mathbb{C}^+$ and satisfies (1), for certain complex $z = z_n$ and $w = w_n$;
(b) $z(t_n) \in \mathbb{R}$ and the left and right derivatives $\dot{z}(t_n - 0)$ and $\dot{z}(t_n + 0)$ satisfy (2) (except of course for the case when $n = m$ and $t_m = T$).

Remark 1 Note that there exist billiard trajectories that cannot be extended to the future beyond a certain time instant. Namely, consider the function

$$z(t) = r[1 + i(t - t_1)]\exp(-i(t - t_1)), \quad t \in [0,\ t_1], \qquad r > 0,\ \ 0 < t_1 < t_*, \tag{3}$$

Here t_* is the smallest positive solution of the equation $t = \tan t$, $t_* \approx 4.49341$. This choice of t_* guarantees that the function z in (3) takes values in $\mathbb{C}^+$.

We have $z(t_1) = r$, $\dot{z}(t_1 - 0) = 0$. It is impossible to extend the function z to a right half-neighborhood of t_1 to a function $z(t) = (z_1 + w_1 t)\exp(-it)$ satisfying $z(t_1) = r$, $\dot{z}(t_1 + 0) = 0$ and taking values in $\mathbb{C}^+$. Indeed, the only function of this kind satisfying these conditions coincides with the function in (3); however, it takes values outside $\mathbb{C}^+$.

Remark 2 Observe that the condition $\dot{z}(t_1 - 0) = 0$ is coarser than grazing (in the latter case only the imaginary part is zero). The 'regular' grazing corresponds to a quadratic tangency while the 'degenerate' case corresponds to the cubic tangency provided it occurs out of the origin.

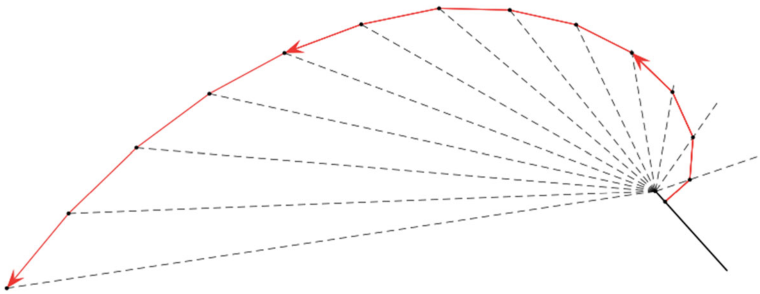

Fig. 2 A trajectory of the ball in a resting coordinate system

Note that the initial conditions of the function in (3) are:

$$z(0) = r(1 - it_1)\exp(it_1), \qquad \dot{z}(0) = -rt_1 \exp(it_1),$$

and designate the set of all such initial conditions by

$$\mathcal{M} := \{r(1 - i\tau, -\tau)\exp(i\tau) : \ r > 0, \ 0 < \tau < t^*\} \subset \mathbb{C}^2.$$

Sliding motion. The natural extension of the function (3) beyond t_1 should be the following:

$$z(t) = \begin{cases} r[1 + i(t - t_1)]\exp(-i(t - t_1)), & \text{if} \quad 0 \le t \le t_1, \\ r\cosh(t - t_1), & \text{if} \quad t \ge t_1. \end{cases}$$

The second line in this equation means that for $t \ge t_1$ the ball moves along the rod, being always subject to an inertial force from the rod in the orthogonal direction. Since the angular velocity of the rod equals 1, the motions along the rod satisfy the equation $\ddot{x} = x$ where x on the right-hand side stands for the centrifugal force. Here we assume the absence of friction (the presence of friction would make the impact inelastic). We call such a regime of motion *sliding*. We observe that transitions from billiard to sliding and vice-versa can only happen if $\dot{x}(t - 0) = y(t) = \dot{y}(t - 0) = 0$.

Model of the system. The free-flight billiard motion described by (1) corresponds to the complex second-order o.d.e

$$\ddot{z} + 2i\dot{z} - z = 0$$

which can be rewritten as follows (recall that $z = x + iy$)

$$\begin{cases} \ddot{x} - 2\dot{y} - x = 0 \\ \ddot{y} + 2\dot{x} - y = 0. \end{cases}$$

The sliding regime corresponds to the equation $\ddot{x} - x = 0$, $y \equiv 0$.

Taking into account impacts, we follow the approach, formulated by Paoli and Schatzmann [17] and, also, in the earlier paper by Moreau [14]:

$$\begin{cases} \ddot{x} - 2\dot{y} - x = 0 \\ \ddot{y} + 2\dot{x} - y = \mu \end{cases} \tag{4}$$

Here μ is a locally finite measure supported on the set $I = I_1 \bigcup I_2$ where

$$I_1 = \{\tau \in \mathbb{R} : y(\tau) = 0, \dot{y}(\tau - 0) < 0\};$$
$$I_2 = \{\tau \in \mathbb{R} : y(\tau) = \dot{y}(\tau - 0) = 0 \quad \text{and}$$
$$(\dot{x}(\tau) > 0 \quad \text{or} \quad \dot{x}(\tau) = 0, \quad x(\tau) > 0)\}.$$

Observe that the set I_1 is countable. The measure is defined by the formula

$$d\mu = \left(-2\sum_{\tau \in I_1} \delta(t - \tau)\dot{y}(\tau - 0) + 2\chi_{I_2}(t)\dot{x}(t) \right) dt.$$

Here δ stands for the Dirac function and χ_{I_2} for the indicator function of the set I_2.

The second equation of (4) can be treated as follows:

$$y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}(s + 0)\, ds,$$

$$\dot{y}(t + 0) = \dot{y}(t_0) + 2x(t) - 2x(t_0) - \int_{t_0}^{t} y(s)\, ds + \mu(I \bigcap [t_0, t]).$$

for any $t, t_0 \in \mathbb{R}, t > t_0$.

Observe that in the instant t_0 of billiard-to-sliding transition, we have $\ddot{x}(t) > 0$, so $x(t)$ increases for $t > t_0$ and no further transition to the billiard motion is possible. Moreover, in the sliding regime, one must have $x(t) > 0$, otherwise, the solution switches to the billiard mode immediately.

Notice that our model of interaction is frictionless both in impact and sliding modes. If friction is there, the pattern becomes much more sophisticated. For instance, if the dry friction is there, in order to exclude undesirable effects, one has to add stochastic terms (see [2, 7–10, 14–17, 25] for various particular cases of this approach). We postpone these studies to the future.

The main result of this paper is the following theorem.

Theorem 1 *Consider a function* $z(t) = (z_0 + w_0 t)\exp(-it)$, $t \in [0, t_1]$, $t_1 > 0$, *with the initial conditions* $(z(0), \dot{z}(0)) \notin \mathcal{M}$ *and such that* $z(t) \in \mathbb{C}^+$ *for* $0 \le t < t_1$ *and* $z(t_1) \in \mathbb{R}^+$. *Then this function can be uniquely extended to a billiard trajectory* $z(t)$, $t \ge 0$. *Additionally, the instants of subsequent hits* $t_1 < t_2 < t_3, \ldots$ *are correctly defined and the following is true.*

(a) The impact velocities $\dot{z}(t_k - 0)$ *are not real values for any* $k > 1$. *In other words, grazing of* $z(t)$ *cannot take place for* $t = t_k$ *if* $k = 2, 3, 4, \ldots$.

(b) The sequence $\{r_n = z(t_n)\} \subset \mathbb{R}$ *is strictly monotone increasing, and tends to infinity as* $n \to \infty$. *Moreover*

$$r_n = o(\exp(\alpha n)) \tag{5}$$

for any $\alpha > 0$.

(c) Denote $\delta_n := t_{n+1} - t_n$. *The sequence* $\{\delta_n\}$ *is monotone decreasing, besides*

$$\sum_{n=1}^{\infty} \delta_n = \infty. \tag{6}$$

Remark 3 In particular, we prove that the assumption $(z(0), \dot{z}(0)) \notin \mathcal{M}$ makes it impossible to fall into sliding regime in the future.

Remark 4 We study motion with the first collision with the positive half-line postponing the case of the first impact with the negative half-line to the future.

Remark 5 Observe that the unboundedness of the rod is the reason for the ball to speed up infinitely. This is the principal difference between the current result and, for instance, that of the paper [3] where the velocities of particles are bounded. However, in some cases (see, for example, [26]), the billiard in a bounded domain with a moving boundary may cause motions with exponentially increasing velocities.

2 Proof of Theorem 1

Lemma 1 *Let* $z(t)$ *be a billiard motion on* $[0, t_1]$. *Either* $Im\, \dot{z}(t_1 - 0) < 0$, *or* $Im\, \dot{z}(t_1 - 0) = 0$ *and* $Re\, \dot{z}(t_1 - 0) < 0$.

Recall that the equality $\mathrm{Im}\, \dot{z}(t_1 - 0) = 0$ implies grazing.

Proof Change the time variable, $s = t - t_1$, and denote

$$f(s) := z(t_1 + s) = r(1 + ws)\exp(-is), \qquad s \in [-t_1,\ 0],$$

where $w = a + ib$ is a complex value. Clearly, $f(0) = r > 0$.

One has $\dot{f}(s) = r(w - i - iws)\exp(-is)$ and $\ddot{f}(s) = -r(2iw + 1 + ws)\exp(-is)$, hence $\dot{f}(0^-) = r(w - i)$ and $\ddot{f}(0^-) = -r(2iw + 1)$. Using that $\dot{f}(0^-) = \mathrm{Re}\,\dot{z}(t_1 - 0) + i\,\mathrm{Im}\,\dot{z}(t_1 - 0)$, we obtain

$$\mathrm{Re}\,\dot{z}(t_1 - 0) = ra \quad \text{and} \quad \mathrm{Im}\,\dot{z}(t_1 - 0) = r(b - 1). \tag{7}$$

Taking into account that $\mathrm{Im}\, f(s) > 0$ for $s < 0$ and using the Taylor decomposition $f(s) = f(0) + s\dot{f}(0) + \frac{1}{2}s^2\ddot{f}(0) + \ldots$, we conclude that either $b < 1$, or $b = 1$ and $a \le 0$. The case $b = 1$, $a = 0$ should be excluded, since in this case $(z(0), \dot{z}(0)) = (f(-t_1), \dot{f}(-t_1)) = r\exp(it_1)(1 - it_1, -t_1) \in \mathcal{M}$. Using (7), one obtains the statement of Lemma 1. □

Lemma 2 *There is an infinite sequence $t_1 < t_2 < \ldots$ of hits such that $z(t)$ can be uniquely extended to a billiard trajectory on each interval $[0,\ t_n)$, $n = 1, 2, \ldots$, so that the values $r_n = z(t_n)$ are positive and form a strictly monotone increasing sequence. Additionally, for $n \ge 2$, $\mathrm{Im}\,\dot{z}(t_{n+1} - 0) < 0$ and $\mathrm{Re}\,\dot{z}(t_{n+1} - 0) > 0$; hence grazing may take place only at t_1.*

In particular, this lemma claims that the motion, once switching from sliding to billiard mode cannot switch to sliding again.

Proof Let us prove by induction that for any natural n there are $t_1 < t_2 < \ldots < t_n$ such that $z(t)$ can be extended to a billiard trajectory on $[0,\ t_n]$, with $r_k = z(t_k)$, $k = 1, \ldots, n$ being real positive values and $z(t) \in \mathbb{C}^+$ for the resting values of t, and $\mathrm{Im}\,\dot{z}(t_k - 0) < 0$ for $k \ne 1$. It will be clear from the proof that the extension is unique.

The claim is obviously true for $n = 1$. Assume that it is true for a certain $n \ge 1$, that is, $z(t)$ is extended to $[0,\ t_n]$, $r_n > 0$, and either $\mathrm{Im}\,\dot{z}(t_n - 0) < 0$, or $\mathrm{Im}\,\dot{z}(t_n - 0) = 0$ and $\mathrm{Re}\,\dot{z}(t_n - 0) < 0$. Let us show that $z(t)$ can be uniquely extended to $[t_n,\ t_{n+1}]$, with $r_{n+1} > 0$ and $\mathrm{Im}\,\dot{z}(t_{n+1} - 0) < 0$.

Denote $f(s) := z(t_n + s)$. As yet, the function f is defined for $[-t_n,\ 0]$, with either $\mathrm{Im}\,\dot{f}(0^-) < 0$, or $\mathrm{Im}\,\dot{f}(0^-) = 0$ and $\mathrm{Re}\,\dot{f}(0^-) < 0$. We are going to extend it to an interval $[0,\ \delta_n]$, with $\delta_n = t_{n+1} - t_n$ to be defined, and look for the function in the form

$$f(s) = f_n(s) = r_n(1 + w_n s)\exp(-is), \quad s \in [0,\ \delta_n].$$

We have $\dot{f}(0^+) = r_n(w_n - i)$, and by (2), $\dot{f}(0^+) = \dot{f}(0^-)^*$. Thus, the value $w_n = a_n + ib_n$ is defined by the initial conditions

$$a_n = \frac{1}{r_n}\mathrm{Re}\,\dot{z}(t_n - 0) \quad \text{and} \quad b_n = 1 - \frac{1}{r_n}\mathrm{Im}\,\dot{z}(t_n - 0). \tag{8}$$

It follows that either $b_n > 1$, or $b_n = 1$ and $a_n < 0$.

We have

$$f(s) = r_n(1 + a_n s + ib_n s)(\cos s - i\sin s), \tag{9}$$

and so,

$$f(s) \in \mathbb{R} \iff b_n s\cos s - (1 + a_n s)\sin s = 0.$$

We define δ_n as the smallest positive value satisfying

$$\frac{\delta_n}{\tan\delta_n} = \frac{1 + a_n\delta_n}{b_n}. \tag{10}$$

In Fig. 3, there are shown the functions $s/\tan s$ and $(1 + a_n s)/b_n$. The function $\phi(s) := s/\tan s$ is concave on $[0, \pi)$, $\phi(0) = 1$, $\phi'(0) = 0$, and $\phi(s) \to -\infty$ as $s \to \pi$. The case $b_n > 1$ is shown in the left figure, and the case $b_n = 1$, $a_n < 0$, in the right figure. In both cases, there is a unique solution $s = \delta_n$ of equation (10) in the interval $(0, \pi)$. Besides, for $0 < s < \delta_n$,

$$\frac{s}{\tan s} > \frac{1 + a_n s}{b_n},$$

and so, $\mathrm{Im}\, f(s) > 0$.

Let us check that $r_{n+1} > r_n$. Indeed,

$$\begin{aligned} r_{n+1} = f(\delta_n) &= r_n\big[(1 + a_n\delta_n)\cos\delta_n + b_n\delta_n \sin\delta_n\big] \qquad (11)\\ &= r_n b_n \delta_n \sin\delta_n \left[\frac{1 + a_n\delta_n}{b_n}\,\frac{\cos\delta_n}{\delta_n \sin\delta_n} + 1\right]\\ &= r_n b_n \delta_n \sin\delta_n \left[\frac{\delta_n}{\tan\delta_n}\,\frac{\cos\delta_n}{\delta_n \sin\delta_n} + 1\right]\\ &= r_n b_n \delta_n \sin\delta_n \left(\cot^2\delta_n + 1\right) = r_n b_n \frac{\delta_n}{\sin\delta_n} > r_n. \end{aligned}$$

The velocity of the ball equals

$$\dot f(s) = r_n[w_n - i(1 + w_n s)]\exp(-is) = r_n[a_n + ib_n - i(1 + (a_n + ib_n)s)]\exp(-is).$$

We have

$$\begin{aligned} \dot z(t_{n+1} - 0) = \dot f(\delta_n) = r_n\big[&(a_n + b_n\delta_n)\cos\delta_n + (b_n - 1 - a_n\delta_n)\sin\delta_n\big]\\ &+ir_n\big[(b_n - 1 - a_n\delta_n)\cos\delta_n - (a_n + b_n\delta_n)\sin\delta_n\big]. \end{aligned}$$

Let us show that $\mathrm{Im}\,\dot z(t_{n+1} - 0) < 0$ and $\mathrm{Re}\,\dot z(t_{n+1} - 0) > 0$. By (10) we have

$$a_n = b_n \cot\delta_n - \frac{1}{\delta_n}, \qquad (12)$$

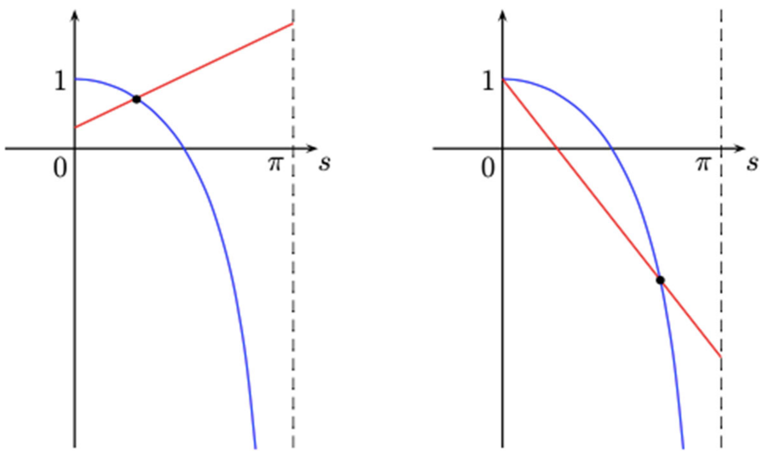

Fig. 3 The functions $\phi(s) = s/\tan s$ and $(1 + a_n s)/b_n$. In the left figure, $b_n > 1$. In the right figure, $b_n = 1$ and $a_n < 0$

and using that $b_n \geq 1$ we get

$$\frac{1}{r_n}\operatorname{Im}\dot{z}(t_{n+1}-0) = (b_n - 1 - a_n\delta_n)\cos\delta_n - (a_n + b_n\delta_n)\sin\delta_n \tag{13}$$

$$= -b_n\frac{\delta_n}{\sin\delta_n} + \frac{\sin\delta_n}{\delta_n} \leq -\frac{\delta_n}{\sin\delta_n} + \frac{\sin\delta_n}{\delta_n} < 0.$$

Further, utilizing (10) and (12) we have

$$\frac{1}{r_n}\operatorname{Re}\dot{z}(t_{n+1}-0) = (a_n + b_n\delta_n)\cos\delta_n + (b_n - 1 - a_n\delta_n)\sin\delta_n \tag{14}$$

$$= \frac{b_n}{\sin\delta_n} - \frac{\cos\delta_n}{\delta_n} \geq \frac{1}{\sin\delta_n} - \frac{\cos\delta_n}{\delta_n} = \frac{2\delta_n - \sin(2\delta_n)}{2\delta_n\sin\delta_n} > 0.$$

□

We maintain the notation adopted in the proof of Lemma 2; namely, the part of the trajectory in $[t_n, t_{n+1}]$ has the form

$$f(s) = z(t_n + s) = r_n(1 + w_n s)\exp(-is), \quad s \in [0, \delta_n], \quad \text{with } w_n = a_n + ib_n.$$

The following Corollary follows immediately from Lemma 2 and (8).

Corollary 1 *For $n \geq 2$, $a_n > 0$ and $b_n > 1$.*

Further, using (8), (11), (13), and (14), one comes to the iterative formulas

$$a_{n+1} = \frac{(a_n + b_n\delta_n)\cos\delta_n + (b_n - 1 - a_n\delta_n)\sin\delta_n}{(1 + a_n\delta_n)\cos\delta_n + b_n\delta_n\sin\delta_n}$$

$$= \frac{a_n\cos\delta_n + b_n\sin\delta_n}{(1 + a_n\delta_n)\cos\delta_n + b_n\delta_n\sin\delta_n} \tag{15}$$

$$= \frac{1}{\delta_n} - \frac{\cos\delta_n\sin\delta_n}{b_n\delta_n^2}, \tag{16}$$

$$b_{n+1} = 2 - \frac{b_n\cos\delta_n - a_n\sin\delta_n}{(1 + a_n\delta_n)\cos\delta_n + b_n\delta_n\sin\delta_n}$$

$$= 2 - \frac{1}{b_n}\left(\frac{\sin\delta_n}{\delta_n}\right)^2. \tag{17}$$

Lemma 3 *The time intervals δ_n strictly monotonically converge to 0: $\delta_n \downarrow 0$ as $n \to \infty$.*

Proof By (10), the function $g(s) = g_n(s) := \frac{1+a_{n+1}s}{b_{n+1}} - \frac{s}{\tan s}$ vanishes at $s = \delta_{n+1}$. Besides, $g(s) < 0$ for $0 < s < \delta_{n+1}$ and $g(s) > 0$ for $\delta_{n+1} < s < \pi$. Using (16) and (17), one easily checks that

$$g(\delta_n) = \frac{1 + a_{n+1}\delta_n}{b_{n+1}} - \frac{\delta_n}{\tan\delta_n} = \frac{2 - \frac{\cos\delta_n\sin\delta_n}{b_n\delta_n}}{2 - \frac{1}{b_n}\left(\frac{\sin\delta_n}{\delta_n}\right)^2} - \frac{\delta_n\cos\delta_n}{\sin\delta_n} > 0, \tag{18}$$

hence $\delta_{n+1} < \delta_n$.

According to (17), $b_{n+1} < 2$. Using (16), one has $1+a_{n+1}\delta_{n+1} < 1+a_{n+1}\delta_n < 2$. Thus, we conclude that

$$\text{for } n \geq 2, \quad 1 < b_n < 2, \quad 0 < a_n < \frac{1}{\delta_n}, \quad \text{and} \quad \frac{1+a_n\delta_n}{b_n} < 1.$$

The sequence $\{\delta_n\}$ is decreasing, and, therefore, converges to a value $c \geq 0$. It remains to prove that $c = 0$.

Assume the contrary, that is, $0 < c < \pi$; then $\frac{1+a_n\delta_n}{b_n} \to \frac{c}{\tan c} < 1$ as $n \to \infty$. Since $0 < a_n < \frac{1}{\delta_n}$ and $1 < b_n < 2$, there exist partial limits $\lim_{k\to\infty} a_{n_k}$ and $\lim_{k\to\infty} b_{n_k} =: \beta$. Using (16) and (17), one obtains

$$\frac{1+a_{n_k}\delta_{n_k-1}}{b_{n_k}} = \frac{2 - \frac{\cos\delta_{n_k-1}\sin\delta_{n_k-1}}{b_{n_k-1}\delta_{n_k-1}}}{2 - \frac{1}{b_{n_k-1}}\left(\frac{\sin\delta_{n_k-1}}{\delta_{n_k-1}}\right)^2} > \frac{\delta_{n_k-1}}{\tan\delta_{n_k-1}}. \tag{19}$$

Using that $\lim_{k\to\infty}\delta_{n_k} = c > 0$ and

$$\lim_{k\to\infty}\frac{1+a_{n_k}\delta_{n_k-1}}{b_{n_k}} = \lim_{k\to\infty}\frac{1+a_{n_k}\delta_{n_k}}{b_{n_k}} = \lim_{k\to\infty}\frac{\delta_{n_k}}{\tan\delta_{n_k}} = \frac{c}{\tan c}$$

and passing to the limit $k \to \infty$ in (19), we get

$$\frac{c}{\tan c} = \frac{2 - \frac{\cos c\sin c}{\beta c}}{2 - \frac{1}{\beta}\left(\frac{\sin c}{c}\right)^2} \geq \frac{c}{\tan c}.$$

Thus,

$$\frac{2 - \frac{\cos c\sin c}{\beta c}}{2 - \frac{1}{\beta}\left(\frac{\sin c}{c}\right)^2} = \frac{c\cos c}{\sin c},$$

and therefore, $\sin c = c\cos c$. This equation does not have solutions for $c \in (0, \pi)$. We come to a contradiction. □

The following statement excludes the possibility of the so-called chatter (infinitely many impacts over a finite time interval).

Lemma 4 *Let $\{t_n : n \in \mathbb{N}\}$ be a sequence of successive impacts of a billiard trajectory $z(t)$. Then Eq.* (6) *takes place.*

Remark 6 From Lemmas 3 and 4 it follows that the billiard trajectory is defined for all $t \geq 0$ unless degenerate grazing occurs. The ball keeps moving infinite time, making infinitely many reflections from the rod, with the time intervals between impacts decreasing to zero.

Proof Recall that the function $g = g_n$ is defined by $g(t) = \frac{1+a_{n+1}t}{b_{n+1}} - \frac{t}{\tan t}$. Recall that $1 < b_n < 2$, hence $2 - \frac{1}{b_n}\left(\frac{\sin\delta_n}{\delta_n}\right)^2 > 1$. We have

$$g(\delta_{n+1}) = \frac{1+a_{n+1}\delta_{n+1}}{b_{n+1}} - \frac{\delta_{n+1}}{\tan\delta_{n+1}} = 0,$$

and by (18) and using that $\frac{1+a_n\delta_n}{b_n} = \frac{\delta_n}{\tan\delta_n} \to 1$ as $n \to \infty$, we get

$$g(\delta_n) = \frac{2 - \frac{\cos\delta_n\sin\delta_n}{b_n\delta_n}}{2 - \frac{1}{b_n}\left(\frac{\sin\delta_n}{\delta_n}\right)^2} - \frac{\delta_n}{\tan\delta_n} = \frac{2}{2 - \frac{1}{b_n}\left(\frac{\sin\delta_n}{\delta_n}\right)^2}\left(1 - \frac{\delta_n}{\tan\delta_n}\right)$$

$$< 2\left(1 - \frac{\delta_n}{\tan \delta_n}\right) = \frac{2\delta_n^2}{3} + \alpha(\delta_n^2), \quad \text{where } \frac{\alpha(\xi)}{\xi} \to 0 \text{ as } \xi \to 0.$$

On the other hand, $g'(t) \geq \frac{a_{n+1}}{b_{n+1}} > \frac{a_{n+1}}{2}$ for all t. Further, using (15), one finds

$$\frac{1}{a_{n+1}} = \delta_n + \frac{1}{a_n + b_n \tan \delta_n},$$

and so, taking n_0 sufficiently large, so that $\delta_{n_0} < \pi/2$, for $n \geq n_0$ we have $\frac{1}{a_{n+1}} < \delta_n + \frac{1}{a_n}$.

Assume that

$$\sum_{n=1}^{\infty} \delta_n = c < \infty.$$

Then for $n \geq n_0 + 1$,

$$\frac{1}{a_n} < \frac{1}{a_{n_0}} + \sum_{n_0}^{n-1} \delta_i \leq \frac{1}{a_{n_0}} + c,$$

hence $a_n/2 \geq c_1 > 0$ for all n, where $c_1 = \frac{1}{2}\left(\frac{1}{a_{n_0}} + c\right)^{-1}$ is a positive value. Thus, $g'(t) = g_n'(t) > c_1$, and

$$\delta_n - \delta_{n+1} \leq \frac{g(\delta_n) - g(\delta_{n+1})}{\inf_t g'(t)} \leq \frac{2}{3c_1}\delta_n^2 + \frac{1}{c_1}\alpha(\delta_n^2).$$

It follows that for $c_2 > 2/(3c_1)$ and for n sufficiently large,

$$\delta_{n+1} \geq \delta_n - c_2\delta_n^2 \implies \frac{1}{\delta_{n+1}} \leq \frac{1}{\delta_n} + c_2(1 + o(1)), \quad n \to \infty,$$

and so, $\delta_n \geq \frac{1}{c_2 n}(1 + o(1))$, and $\sum\limits_{n=1}^{\infty} \delta_n = \infty$. We come to a contradiction. □

Lemma 5 *We have $a_n \to 1$ and $b_n \to 1$ as $n \to \infty$.*

Proof Using that $\delta_n \to 0$ as $n \to \infty$, $b_n > 1$, and taking account of (17), one obtains

$$b_{n+1} = 2 - \frac{1}{b_n} + \xi_n,$$

where $\xi_n \to 0$ as $n \to \infty$. Hence we have

$$b_n \geq 2 - \frac{1}{b_n} = b_{n+1} - \xi_n. \tag{20}$$

Let $\beta = \lim_{k\to\infty} b_{n_k}$ be the limit superior of b_n. Taking $n = n_k - 1$ in (20), we see that $\lim_{k\to\infty} b_{n_k-1}$ exists and coincides with β. Passing to the limit $k \to \infty$ in the equality

$$\beta_{n_k} = 2 - \frac{1}{\beta_{n_k-1}} + \xi_{n_k-1},$$

one finds $\beta = 2 - 1/\beta$, whence $\beta = 1$. It follows that $\lim_{n\to\infty} b_n = 1$.

Since by (10), $b_n = \frac{(1+a_n\delta_n)\sin \delta_n}{\delta_n \cos \delta_n}$, making this substitution in (16) and using that by Corollary 1, $a_n > 0$ for $n \geq 2$, after some algebra one obtains

$$a_{n+1} = a_n \frac{\cos^2 \delta_n}{1 + a_n\delta_n} + \delta_n\left(\frac{\sin \delta_n}{\delta_n}\right)^2 < \frac{a_n}{1 + a_n\delta_n} + \delta_n. \tag{21}$$

Hence we have

$$\Delta a_n := a_{n+1} - a_n < \delta_n \Big(1 - \frac{a_n^2}{1 + a_n \delta_n}\Big) < \delta_n.$$

Since $\delta_n \to 0$, for any $0 < \varepsilon < 1$ there exists $n_0 = n_0(\varepsilon)$ such that for all $n \ge n_0$, $\delta_n < \varepsilon$, and therefore, for $a > 1 + \varepsilon$ holds $1 - \frac{a^2}{1+a\delta_n} < -\frac{\varepsilon}{3}$. It follows that

$$\text{if } n \ge n_0 \text{ and } a_n > 1 + \varepsilon, \quad \text{then } \Delta a_n < 0. \tag{22}$$

Let us prove that $a_n < 1 + 2\varepsilon$ for n sufficiently large. First, for some $n_1 \ge n_0$ holds $a_{n_1} \le 1 + \varepsilon$; otherwise the sequence a_n, $n \ge n_0$ is monotone decreasing with the increments $\Delta a_n < -\frac{\varepsilon}{3}\,\delta_n$, and therefore, tends to $-\infty$.

Second, for $n > n_1$ the inequality holds $a_n < 1 + 2\varepsilon$. Otherwise let $n_2 > n_1$ be the smallest value that does not satisfy this inequality; then we have $\Delta a_{n_2-1} > 0$, and therefore, by (22), $a_{n_2-1} \le 1 + \varepsilon$. On the other hand, $\Delta a_{n_2-1} < \delta_{n_2-1} < \varepsilon$, hence $a_{n_2} < 1 + 2\varepsilon$, in contradiction with our assumption.

It follows that $\limsup a_n \le 1$.

Further, from (21) one derives

$$\begin{aligned}\Delta a_n &= -a_n \frac{\sin^2 \delta_n}{1 + a_n \delta_n} + \delta_n \left[\Big(\frac{\sin \delta_n}{\delta_n}\Big)^2 - \frac{a_n^2}{1 + a_n \delta_n}\right] \\ &> \delta_n \left[-a_n\,\delta_n + \Big(\frac{\sin \delta_n}{\delta_n}\Big)^2 - a_n^2\right].\end{aligned} \tag{23}$$

Let us show that for all $0 < \varepsilon < 1$ there exist infinitely many values of n for which $a_n > 1-\varepsilon$. Indeed, otherwise all a_n for n sufficiently large lie in $[0,\ 1-\varepsilon]$, and the sum over n of the right hand sides in (23) is greater than $\varepsilon \sum_n \delta_n(1 + o(1))$, and therefore, diverges to $+\infty$. It follows that $a_n \to +\infty$, which is impossible.

Fix $0 < \varepsilon < 1$. Using (23), we see that there exists m_0 such that for all $n \ge m_0$, the inequality $0 \le a_n \le 1 - \varepsilon$ implies $\Delta a_n \ge \frac{\varepsilon}{2}\,\delta_n > 0$. Additionally, since both sequences a_n and δ_n are bounded, there exists a constant $c > 0$ such that $\Delta a_n \ge -c\delta_n$. Choose a subsequence $m_0 < n_1 < n_2 < \dots n_k < \dots$ such that $a_{n_k} > 1 - \varepsilon$ for all k.

Let a_{s_k} be the smallest value among $\{a_{n_k+1}, a_{n_k+2}, \dots, a_{n_{k+1}}\}$. If $a_{s_k} \le 1 - \varepsilon$ then $a_{s_k-1} > 1 - \varepsilon$. Indeed, if $s_k = n_k + 1$, this is obvious, and if $s_k \ge n_k + 2$ then $\Delta a_{s_k-1} \le 0$, and hence $a_{s_k-1} > 1 - \varepsilon$. We have

$$a_{s_k} = a_{s_k-1} + \Delta a_{s_k-1} > 1 - \varepsilon - c\delta_{s_k-1} \ge 1 - \varepsilon - c\delta_{n_k}.$$

Since δ_{n_k} converges to zero, we conclude that the limit inferior of a_n is $\ge 1 - \varepsilon$, and taking into account that ε is arbitrarily small, $\liminf a_n \ge 1$. Lemma 5 is proven. □

Corollary 2 *The sequence r_n tends to infinity and $r_{n+1}/r_n \to 1$.*

Proof Using formula (9) one has

$$\frac{r_{n+1}}{r_n} = (1 + a_n\delta_n + i b_n \delta_n)(\cos \delta_n - i \sin \delta_n) = 1 + \delta_n + o(\delta_n).$$

Here we used the statement of Lemma 5. The sum of δ_n diverges due to Lemma 4, so $r_n \to \infty$. Besides, the last formula implies Eq.(5). □

Overall, Claim (a) of Theorem 1 follows from Lemma 2. Claim (b) follows from Lemma 2 and Corollary 2. Claim (c) follows from Lemmas 3 and 4. □

3 Conclusion

In a nutshell, the dynamics of the considered system can be regarded as follows: there are the billiard mode and the sliding one. If the initial conditions for the solution are such that the billiard motion is possible (a neighborhood of the corresponding positive semi-trajectory lies in the upper half-plane), the ball moves this way, even if the sliding motion is possible.

The solutions are defined for any initial conditions and are unique as time increases. The so-called chattering (infinitely many impacts on a finite time interval) is impossible for the considered system.

There might be the following scenarios of forward-in-time motion with the first collision taking place with a positive half-line.

1. A billiard motion, extendable to $[0, \infty)$ going to infinity as time increases.
2. A billiard motion that switches to a sliding regime with no further switches to billiard mode.
3. A sliding motion that switches to a billiard mode.
4. A sliding motion which never switches to the billiard regime.

For billiard motions, extendable to infinity, the solutions and their velocities tend to infinity whilst time intervals between neighbor impacts tend to zero.

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References

1. Brock F, Ferone V, Kawohl B. A Symmetry Problem in the Calculus Of Variations. Calc Var. 1996;4:593–9.
2. Broomhead DS, Gutkin E. The Dynamics of Billiards With No-Slip Collisions. Phys D. 1993;67:188–97.
3. Burdzy K, Duarte M, Gauthier C-E, Graham CR, Martin JS. Fermi Acceleration in Rotating Drums. J Math Phys. 2022;63:062706.
4. Buttazzo G, Ferone V, Kawohl B. Minimum Problems Over Sets of Concave Functions and Related Questions. Math Nachr. 1995;173:71–89.
5. Buttazzo G, Kawohl B. On Newton's Problem of Minimal Resistance. Math Intell. 1993;15:7–12.
6. Buttazzo G, Guasoni P. Shape Optimization Problems Over Classes of Convex Domains. J Convex Anal. 1997;4(2):343–51.
7. Champneys A, Várkonyi P. The Painlevé Paradox in Contact Mechanics. IMA J Appl Math. 2016;81:538–88.
8. Cox C, Feres R. No-Slip Billiards in Dimension Two. Contemp Math. 2017;698:91–110.
9. Cox C, Feres R, Ward W. Differential Geometry of Rigid Body Collisions and Non-Standard Billiards. Discrete Contin Dyn Syst. 2016;36(11):6065–99.

10. Feres R. Random Walks Derived from Billiards. Dynamics, Ergodic Theory, and Geometry. Math Sci Res Inst Publ. 2007;54:179–222. Cambridge University Press, Cambridge.
11. Kryzhevich S. Motion of a Rough Disc in Newtonian Aerodynamics. In: Optimization in the Natural Sciences. EmC-ONS 2014. Communications in Computer and Information Science, vol 499. Springer; 2015. p. 3–19.
12. Lachand-Robert T, Oudet E. Minimizing Within Convex Bodies Using a Convex Hull Method. SIAM J Optim. 2006;16:368–79.
13. Lachand-Robert T, Peletier MA. Newton's Problem of the Body of Minimal Resistance in the Class of Convex Developable Functions. Math Nachr. 2001;226:153–76.
14. Moreau J-J. Unilateral Contact and Dry Friction in Finite Freedom Dynamics. In: Moreau J-J, Panagiotopoulos PG, editors. Nonsmooth Mechanics and Applications. CISM Courses and Lectures 302. Vienna: Springer-Verlag; 1988.
15. Palffy-Muhoray P, Virga EG, Wilkinson M, Zheng X. On a Paradox in the Impact Dynamics of Smooth Rigid Bodies. Math Mech Solids. 2019;24(3):573–97.
16. Paoli L, Schatzman M. Schéema Numérique Pour un Modèle de Vibrations Avec Constraints Unilatérale et Perte D'énergie aux Impacts, en Dimension Finie. CR Acad Sci Paris Sér I Math. 1993;317(2):211–5.
17. Paoli L, Schatzman M. Mouvement à un Nombre Fini de Degrées de Liberté avec Contraintes Unilatérales: Cas avec Perte Dénergie. RAIRO Modél Math Anal Numér. 1993;27(6):673–717.
18. Newton I. Philosophiae Naturalis Principia Mathematica. London: Streater; 1687.
19. Plakhov A. Billiards and Two-Dimensional Problems of Optimal Resistance. Arch Ration Mech Anal. 2009;194:349–82.
20. Plakhov A. Optimal Roughening of Convex Bodies. Canad J Math. 2012;64:1058–74.
21. Plakhov A, Torres D. Newton's Aerodynamic Problem in Media of Chaotically Moving Particles. Sbornik Math. 2005;196:885–933.
22. Plakhov A. Problems of Minimal Resistance and the Kakeya Problem. SIAM Rev. 2015;57:421–34.
23. Plakhov A, Tchemisova T, Gouveia P. Spinning Rough Disk Moving in a Rarefied Medium. Proc R Soc Lond A. 2010;466:2033–55.
24. Plakhov A, Tchemisova T. Problems of Optimal Transportation on the Circle and Their Mechanical Applications. J Diff Eqs. 2017;262:2449–92.
25. Rudzis P. Rough Collisions. arXiv:2203.09102.
26. Zhou J. A rectangular Billiard With Moving Slits. Nonlinearity. 2020;33:1542–71.