

Partial hyperbolicity and central shadowing

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Abstract. We study shadowing property for a partially hyperbolic diffeomorphism f . It is proved that if f is dynamically coherent then any pseudotrajectory can be shadowed by a pseudotrajectory with “jumps” along the central foliation. The proof is based on the Tikhonov-Schauder fixed point theorem.

Keywords: partial hyperbolicity, central foliation, Lipschitz shadowing, dynamical coherence.

1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well developed part of the global theory of dynamical systems (see, for example, monographs [12], [13]). This theory is of special importance for numerical simulations and the classical theory of structural stability.

It is well known that a diffeomorphism has the shadowing property in a neighborhood of a hyperbolic set [2], [4] and a structurally stable diffeomorphism has the shadowing property on the whole manifold [11], [17], [19].

There are a lot of examples of non-hyperbolic diffeomorphisms, which have shadowing property (see for instance [14], [21]) at the same time this phenomena is not frequent. More precisely the following statements are correct. Diffeomorphisms with C^1 -robust shadowing property are structurally stable [18]. In [1] Abdenur and Diaz conjectured that C^1 -generically shadowing is equivalent to structural stability, and proved this statement for so-called tame diffeomorphisms. Lipschitz shadowing is equivalent to structural stability [15] (see [21] for some generalizations).

In present article we study shadowing property for partially hyperbolic diffeomorphisms. Note that due to [7] one cannot expect that in general shadowing holds for partially hyperbolic diffeomorphisms. We use notion of central pseudotrajectory and prove that any pseudotrajectory of a partially hyperbolic diffeomorphism can be shadowed by a central pseudotrajectory. This result might be considered as a generalization of a classical shadowing lemma for the case of partially hyperbolic diffeomorphisms.

2 Definitions and the main result

Let M be a compact n – dimensional C^∞ smooth manifold, with a Riemannian metric dist . Let $|\cdot|$ be the Euclidean norm at \mathbb{R}^n and the induced norm on the leaves of the tangent bundle TM . For any $x \in M$, $\varepsilon > 0$ we denote

$$B_\varepsilon(x) = \{y \in M : \text{dist}(x, y) \leq \varepsilon\}.$$

Below in the text we use the following definition of partial hyperbolicity (see for example [6]).

Definition 1. A diffeomorphism $f \in \text{Diff}^1(M)$ is called *partially hyperbolic* if there exists $m \in \mathbb{N}$ such that the mapping f^m satisfies the following property. There exists a continuous invariant bundle

$$T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x), \quad x \in M$$

and continuous positive functions $\nu, \hat{\nu}, \gamma, \hat{\gamma} : M \rightarrow \mathbb{R}$ such that

$$\nu, \hat{\nu} < 1, \quad \nu < \gamma < \hat{\gamma} < \hat{\nu}^{-1}$$

and for all $x \in M$, $v \in \mathbb{R}^n$, $|v| = 1$

$$\begin{aligned} |Df^m(x)v| &\leq \nu(x), & v \in E^s(x); \\ \gamma(x) &\leq |Df^m(x)v| \leq \hat{\gamma}(x), & v \in E^c(x); \\ |Df^m(x)v| &\geq \hat{\nu}^{-1}(x), & v \in E^u(x). \end{aligned} \tag{1}$$

Denote

$$E^{cs}(x) = E^s(x) \oplus E^c(x), \quad E^{cu}(x) = E^c(x) \oplus E^u(x).$$

For further considerations we need the notion of dynamical coherence.

Definition 2. We say that a k – dimensional distribution E over TM is *uniquely integrable* if there exists a k – dimensional continuous foliation W of the manifold M , whose leaves are tangent to E at every point. Also, any C^1 – smooth path tangent to E is embedded to a unique leaf of W .

Definition 3. A partially hyperbolic diffeomorphism f is *dynamically coherent* if both the distributions E^{cs} and E^{cu} are uniquely integrable.

If f is dynamically coherent then distribution E^c is also uniquely integrable and corresponding foliation W^c is a subfoliation of both W^{cs} and W^{cu} . For a discussion how often partially hyperbolic diffeomorphisms are dynamically coherent see [5], [9].

In the text below we always assume that f is dynamically coherent.

For $\tau \in \{s, c, u, cs, cu\}$ and $y \in W^\tau(x)$ let $\text{dist}_\tau(x, y)$ be the inner distance on $W^\tau(x)$ from x to y . Note that

$$\text{dist}(x, y) \leq \text{dist}_\tau(x, y), \quad y \in W^\tau(x). \quad (2)$$

Denote

$$W_\varepsilon^\tau(x) = \{y \in W^\tau(x), \text{dist}_\tau(x, y) < \varepsilon\}.$$

Let us recall the definition of the shadowing property.

Definition 4. A sequence $\{x_k : k \in \mathbb{Z}\}$ is called d -pseudotrajectory ($d > 0$) if $\text{dist}(f(x_k), x_{k+1}) \leq d$ for all $k \in \mathbb{Z}$.

Definition 5. Diffeomorphism f satisfies the *shadowing property* if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d -pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists a trajectory $\{y_k\}$ of the diffeomorphism f such that

$$\text{dist}(x_k, y_k) \leq \varepsilon, \quad k \in \mathbb{Z}. \quad (3)$$

Definition 6. Diffeomorphism f satisfies the *Lipschitz shadowing property* if there exist $\mathcal{L}, d_0 > 0$ such that for any $d \in (0, d_0)$, and any d -pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists a trajectory $\{y_k\}$ of the diffeomorphism f , satisfying (3) with $\varepsilon = \mathcal{L}d$.

As was mentioned before in a neighborhood of a hyperbolic set diffeomorphism satisfies the Lipschitz shadowing property [2], [4], [13].

We suggest the following generalization of the shadowing property for partially hyperbolic dynamically coherent diffeomorphisms.

Definition 7 (see for example [10]). An ε -pseudotrajectory $\{y_k\}$ is called *central* if for any $k \in \mathbb{Z}$ the inclusion $f(y_k) \in W_\varepsilon^c(y_{k+1})$ holds (see Fig. [1]).

Definition 8. A partially hyperbolic dynamically coherent diffeomorphism f satisfies the *central shadowing property* if for any $\varepsilon > 0$ there exists $d > 0$ such that for any d -pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists an ε -central pseudotrajectory $\{y_k\}$ of the diffeomorphism f , satisfying (3).



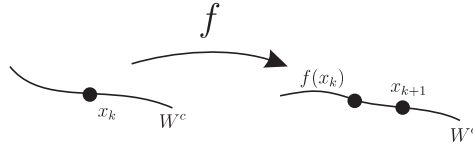


Figure 1: Central pseudotrajectory

Definition 9. A partially hyperbolic dynamically coherent diffeomorphism f satisfies the *Lipschitz central shadowing property* if there exist $d_0, \mathcal{L} > 0$ such that for any $d \in (0, d_0)$ and any d -pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists an ε -central pseudotrajectory $\{y_k\}$, satisfying (3) with $\varepsilon = \mathcal{L}d$.

Note that the Lipschitz central shadowing property implies the central shadowing property.

We prove the following analogue of the shadowing lemma for partially hyperbolic diffeomorphisms.

Theorem 1. *Let diffeomorphism $f \in C^1$ be partially hyperbolic and dynamically coherent. Then f satisfies the Lipschitz central shadowing property.*

Note that for Anosov diffeomorphisms any central pseudotrajectory is a true trajectory.

Let us also mention the following related notion [10].

Definition 10. Partially hyperbolic, dynamically coherent diffeomorphism f is called *plaque expansive* if there exists $\varepsilon > 0$ such that for any ε -central pseudotrajectories $\{y_k\}, \{z_k\}$, satisfying

$$\text{dist}(y_k, z_k) < \varepsilon, \quad k \in \mathbb{Z}$$

hold inclusions

$$z_0 \in W_\varepsilon^c(y_0), \quad k \in \mathbb{Z}.$$

In the theory of partially hyperbolic diffeomorphisms the following conjecture plays important role [3], [10].

Conjecture 1 (Plague Expansivity Conjecture). *Any partially hyperbolic, dynamically coherent diffeomorphism is plaque expansive.*

Let us note that if the diffeomorphism f in Theorem 1 is additionally plaque expansive then leaves $W^c(y_k)$ are uniquely defined (see Remark 1 below).

Among results related to Theorem 1 we would like to mention that partially hyperbolic dynamically coherent diffeomorphisms, satisfying plaque expansivity property are leaf stable (see [10, Chapter 7], [16] for details).

3 Proof of Theorem 1

In what follows below we will use the following statement, which is consequence of transversality and continuity of foliations W^s, W^{cu} .

Statement 1. *There exists $\delta_0 > 0, L_0 > 1$ such that for any $\delta \in (0, \delta_0]$ such that for any $x, y \in M$ satisfying $\text{dist}(x, y) < \delta$ there exists unique point $z = W_\varepsilon^s(x) \cap W_\varepsilon^{cu}(y)$ for $\varepsilon = L_0\delta$.*

Note that for a fixed diffeomorphism f , satisfying the assumptions of the theorem, it suffices to prove that its fixed power f^m satisfies the Lipschitz central shadowing property. Since foliations $W^\tau, \tau \in \{s, u, c, cs, cu\}$ of f^m coincide with the corresponding foliations of the initial diffeomorphism f we can assume without loss of generality that conditions (1) hold for $m = 1$. Note that a similar claim can be done using adapted metric, see [8].

Denote

$$\lambda = \min_{x \in M} (\min(\hat{\nu}^{-1}(x), \nu^{-1}(x))) > 1.$$

Let us choose l so big that

$$\lambda^l > 2L_0.$$

Arguing similarly to previous paragraph it is sufficient to prove that f^l has the Lipschitz central shadowing property and hence, we can assume without loss of generality that $l = 1$.

Decreasing δ_0 if necessarily we conclude from inequalities (1) that

$$\text{dist}_s(f(x), f(y)) \leq \frac{1}{\lambda} \text{dist}_s(x, y), \quad y \in W_{\delta_0}^s(x) \quad (4)$$

and

$$\text{dist}_u(f(x), f(y)) \geq \lambda \text{dist}_u(x, y), \quad y \in W_{\delta_0}^u(x).$$

Denote

$$I_r^\tau(x) = \{z^\tau \in E^\tau(x), |z^\tau| \leq r\}, \quad \tau \in \{s, u, c, cs, cu\}, \quad r > 0,$$

$$I_r(x) = \{z \in T_x M, |z| \leq r\}, \quad r > 0.$$

Consider standard exponential mappings $\exp_x : T_x M \rightarrow M$ and $\exp_x^\tau : T_x W^\tau(x) \rightarrow W^\tau(x)$, for $\tau \in \{s, c, u, cs, cu\}$. Standard properties of exponential mappings imply that there exists $\varepsilon_0 > 0$, such that for all $x \in M$ maps \exp_x, \exp_x^τ are well defined on $I_{\varepsilon_0}(x)$ and $I_{\varepsilon_0}^\tau(x)$ respectively and $D \exp_x(0) = \text{Id}$, $D \exp_x^\tau(0) = \text{Id}$. Those equalities imply the following.

Statement 2. For $\mu > 0$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that for any point $x \in M$, the following holds.

A1 For any $y, z \in B_\varepsilon(x)$ and $v_1, v_2 \in I_\varepsilon(x)$ the following inequalities hold

$$\frac{1}{1 + \mu} \text{dist}(y, z) \leq |\exp_x^{-1}(y) - \exp_x^{-1}(z)| \leq (1 + \mu) \text{dist}(y, z),$$

$$\frac{1}{1 + \mu} |v_1 - v_2| \leq \text{dist}(\exp_x(v_1), \exp_x(v_2)) \leq (1 + \mu) |v_1 - v_2|.$$

A2 Conditions similar to **A1** hold for \exp_x^τ and dist_τ , $\tau \in \{s, c, u, cs, cu\}$.

A3 For $y \in W_\varepsilon^\tau(x)$, $\tau \in \{s, c, u, cs, cu\}$ the following holds

$$\text{dist}_\tau(x, y) \leq (1 + \mu) \text{dist}(x, y).$$

A4 If $\xi < \varepsilon$ and $y \in W_\xi^{cs}(x) \cap W_\xi^{cu}(x)$ then

$$\text{dist}_c(x, y) \leq (1 + \mu)\xi.$$

Consider small enough $\mu \in (0, 1)$ satisfying the following inequality

$$(1 + \mu)^2 L_0 / \lambda < 1. \tag{5}$$

Choose corresponding $\varepsilon > 0$ from Statement [2](#). Let $\delta = \min(\delta_0, \varepsilon / L_0)$.

For a pseudotrajectory $\{x_k\}$ consider maps $h_k^s : U_k \subset E^s(x_k) \rightarrow E^s(x_{k+1})$ defined as the following:

$$h_k^s(z) = (\exp_{x_{k+1}}^s)^{-1}(p)$$

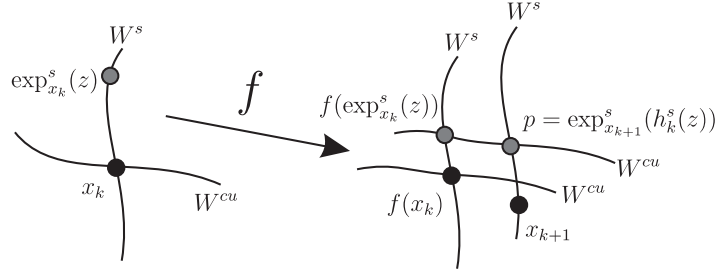


Figure 2: Definition of map h_k^s

where

$$p = W_{L_0\delta_0}^{cu}(f(\exp_{x_k}^s(z))) \cap W_{L_0\delta_0}^s(x_{k+1}) \quad (6)$$

and U_k is the set of points for which map h_k^s is well-defined (see Fig. 2). Note that maps $h_k^s(z)$ are continuous. The following lemma plays a central role in the proof of Theorem 1

Lemma 1. *There exists $d_0 > 0$, $L > 1$ such that for any $d < d_0$ and d -pseudotrajectory $\{x_k\}$ maps h_k^s are well-defined for $z \in I_{Ld}^s(x_k)$ and the following inequalities hold*

$$|h_k^s(z)| \leq Ld, \quad k \in \mathbb{Z}. \quad (7)$$

Proof. Inequality (5) implies that there exists $L > 0$ such that

$$L_0(1 + L(1 + \mu)/\lambda)(1 + \mu) < L. \quad (8)$$

Let us choose $d_0 < \delta_0/2L$. Fix $d < d_0$, d -pseudotrajectory $\{x_k\}$, $k \in \mathbb{Z}$ and $z \in I_{Ld}^s(x_k)$.

Condition A2 of Statement 2 implies that

$$\text{dist}_s(x_k, \exp_{x_k}^s(z)) \leq Ld(1 + \mu).$$

Inequality (4) implies the following

$$\text{dist}_s(f(x_k), f(\exp_{x_k}^s(z))) \leq \frac{1}{\lambda}Ld(1 + \mu).$$

Inequalities (2) and $\text{dist}(f(x_k), x_{k+1}) < d$ imply (see Fig. 3 for illustration)

$$\begin{aligned} \text{dist}(x_{k+1}, f(\exp_{x_k}^s(z))) &\leq \text{dist}(x_{k+1}, f(x_k)) + \text{dist}(f(x_k), f(\exp_{x_k}^s(z))) \leq \\ &d \left(1 + \frac{1}{\lambda}L(1 + \mu) \right) < Ld < \delta_0. \end{aligned}$$

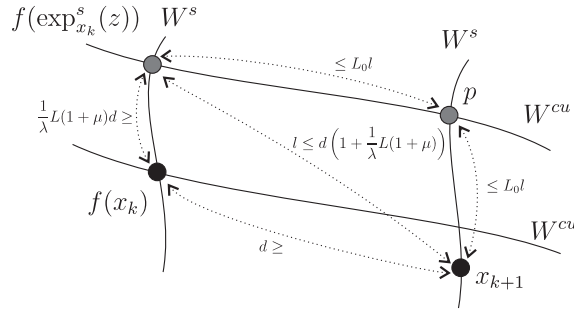


Figure 3: Illustration of the proof of Lemma 1

Statement 1 implies that point p from relation (6) is well-defined and inequality (8) implies the following

$$\text{dist}_s(p, x_{k+1}), \text{dist}_{cu}(p, f(\exp_{x_k}^s(z))) < dL_0(1 + \frac{1}{\lambda}L(1 + \mu)) < \frac{Ld}{1 + \mu}.$$

This inequality and Statement 2 imply

$$\text{dist}_{cu}(f(\exp_{x_{k+1}}^s(z)), \exp_{x_k}^s(h_k^s(z))) < Ld, \quad (9)$$

$$|h_k^s(z)| < Ld,$$

which completes the proof. \square

Let $d_0, L > 0$ are constants provided by Lemma 1. Let $d < d_0$ and $\{x_k\}$ is a d -pseudotrajectory. Denote

$$X^s = \prod_{k=-\infty}^{\infty} I_{Ld}^s(x_k).$$

This set endowed with the Tikhonov product topology is compact and convex.

Let us consider map $H : X^s \rightarrow X^s$ defined as following

$$H(\{z_k\}) = \{z'_{k+1}\}, \quad \text{where } z'_{k+1} = h_k^s(z_k).$$

By Lemma 1 this map is well-defined. Since z'_{k+1} depends only on z_k map H is continuous. Due to the Tikhonov-Schauder theorem [20], the mapping H

has a (maybe non-unique) fixed point $\{z_k^*\}$. Denote $y_k^s = \exp_{x_k}^s(z_k^*)$. Since $z_{k+1}^* = h_k^s(z_k^*)$, inequality (9) implies that

$$y_{k+1}^s \in W_{Ld}^{cu}(f(y_k^s)), \quad k \in \mathbb{Z}. \quad (10)$$

Since $|z_k^*| < Ld$ we conclude

$$\text{dist}(x_k, y_k^s) \leq \text{dist}_s(x_k, y_k^s) < (1 + \mu)Ld < 2Ld, \quad k \in \mathbb{Z}.$$

Similarly (decreasing d_0 and increasing L if necessarily) one may show that there exists a sequence $\{y_k^u \in W_{2Ld}^u(x_k)\}$ such that

$$y_{k+1}^u \in W_{Ld}^{cs}(f(y_k^u)), \quad k \in \mathbb{Z}.$$

Hence $\text{dist}(y_k^s, y_k^u) < \text{dist}(y_k^s, x_k) + \text{dist}(x_k, y_k^u) < 4Ld$. Decreasing d_0 if necessarily we can assume that $4L_0Ld < \delta_0$. Then there exists a unique point $y_k = W_{4L_0Ld}^{cu}(y_k^s) \cap W_{4L_0Ld}^s(y_k^u)$ and inclusion (10) implies that for all $k \in \mathbb{Z}$ the following holds

$$\begin{aligned} \text{dist}_{cu}(y_{k+1}, f(y_k)) < \\ \text{dist}_{cu}(y_{k+1}, y_{k+1}^s) + \text{dist}_{cu}(y_{k+1}^s, f(y_k^s)) + \text{dist}_{cu}(f(y_k^s), f(y_k)) < \\ 4L_0Ld + Ld + 4RL_0Ld = L_{cu}d, \end{aligned}$$

where $R = \sup_{x \in M} |Df(x)|$ and $L_{cu} > 1$ do not depends on d . Similarly for some constant $L_{cs} > 1$ the following inequalities hold

$$\text{dist}_{cs}(y_{k+1}, f(y_k)) < L_{cs}d, \quad k \in \mathbb{Z}.$$

Reducing d_0 if necessarily we can assume that points $y_{k+1}, f(y_k)$ satisfy assumptions of condition **A4** of Statement 2) hence

$$\text{dist}_c(y_{k+1}, f(y_k)) < (1 + \mu) \max(L_{cs}, L_{cu})d, \quad k \in \mathbb{Z}$$

and sequence $\{y_k\}$ is an L_1d -central pseudotrajectory with

$$L_1 = (1 + \mu) \max(L_{cs}, L_{cu}).$$

To complete the proof let us note that

$$\text{dist}(x_k, y_k) < \text{dist}(x_k, y_k^s) + \text{dist}(y_k^s, y_k) < 2Ld + 4L_0Ld, \quad k \in \mathbb{Z}.$$

Taking $\mathcal{L} = \max(L_1, 2L + 4L_0)$ we conclude that $\{y_k\}$ is an $\mathcal{L}d$ -central pseudotrajectory which $\mathcal{L}d$ shadows $\{x_k\}$. \square

Remark 1. Note that we do not claim uniqueness of such sequences $\{y_k^s\}$ and $\{y_k^u\}$. In fact it is easy to show (we leave details to the reader) that uniqueness of those sequences is equivalent to the plaque expansivity conjecture.

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