WARSAW UNIVERSITY OF TECHNOLOGY Index 351733
FACULTY OF CIVIL ENGINEERING
COMMITTEE FOR CIVIL AND WATER ENGINEERING
POLISH ACADEMY OF SCIENCES ISSN 1230-2945

DOI: 10.24425/ace.2024.148920

ARCHIVES OF CIVIL ENGINEERING

Vol. LXX ISSUE 1 2024

© 2024. Waldemar Magda.

pp. 421 –442

This is an open-access article distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives License (CC BY-NC-ND 4.0, https://creativecommons.org/licenses/by-nc-nd/4.0/), which permits use, distribution, and reproduction in any medium, provided that the Article is properly cited, the use is non-commercial, and no modifications or adaptations are made.

**Research paper**

# Importance of sign conventions on analytical solutions to the wave-induced cyclic response of a poro-elastic seabed

**Waldemar Magda**[1]

**Abstract:** This paper discusses the influence of different sign conventions for strains and stresses, i.e. the solid mechanics sign convention and the soil mechanics sign convention, on the form of governing partial differential equations (the static equilibrium equations and the continuity equation) used to describe the wave-induced cyclic response of a poro-elastic seabed due to propagation of a sinusoidal surface water-wave. Some selected analytical solutions, obtained by different authors and published in specialist literature in the form of complex functions describing the wave-induced pore-fluid pressure, effective normal stress and shear stress oscillations in the seabed, have been analysed and compared with each other mainly with respect to different sign conventions for stains and stresses and also with regard to different orientations of the positive vertical axis of the two-dimensional coordinate system and different directions of surface water-wave propagation. The performed analyses of the analytical solutions has indicated many inaccuracies, or even evident errors and exemplary mistakes of wrong-signed values of basic wave-induced response parameters (the shear stress in particular), thereby disqualifying these solutions and their final equations from practical engineering applications. Most of the mistakes found in the literature must be linked to authors' lack of understanding and consistency in an uniform application of a certain sign convention for strains and stresses in the soil matrix at both stages of mathematical formulation of the governing problem and correct interpretation of equations of the final analytical solution. The present paper, based mostly on a thorough literature review, ought to draw attention and arouse interest among coastal scientists and engineers in proper identification and use of the existing analytical solutions to the wave-induced cyclic seabed response – solutions which differ very often in the applied sign convention for stresses in the soil matrix.

**Keywords:** wave-seabed interaction, wave-induced cyclic seabed response, poro-elastic seabed, sinusoidal progressive water-wave, sign conventions for strains and stresses, soil mechanics

[1]D.Sc., Ph.D., Engr., Prof. at GUT, Gdańsk University of Technology, Faculty of Civil and Environmental Engineering, Department of Geotechnical and Hydraulic Engineering, 11/12 Gabriela Narutowicza Street, 80-233 Gdańsk, Poland, e-mail: waldemar.magda@pg.edu.pl, ORCID: 0000-0003-2157-7194

# 1. Introduction

The wave-induced cyclic response of a poro-elastic seabed sediments loaded by a progressive sinusoidal surface water-wave is still an interesting subject in many coastal engineering problems. Authors of many publications on analytical solutions to the wave-induced cyclic response of seabed sediments are fixed mostly on presentations of vertical distributions of the amplitude of cyclically varying parameters only, omitting totally computational analyses of momentary values of these parameters, e.g. [1–6]. There are only a few works, e.g. [7–9], where the authors were kind to present additional information on vertical distributions of the phase-lag of wave-induced pore-fluid pressure, as an immanent part of the momentary value solution. On the other hand, it happens that momentary values of the wave-induced parameters, the shear stress component in particular, are published as wrong-signed parameters. Of course, the use of wrong-signed values of the wave-induced shear stress, for instance, will not cause problems in the residual pore-fluid pressure analyses, where only the shear stress amplitude is required in order to assess the number of cycles causing the residual liquefaction of the upper part of seabed sediments [10], but the use of wrong-signed values of the wave-induced normal effective stress components could be the reason of serious errors in analyses of the soil momentary liquefaction phenomenon. These mistakes result usually from authors' uncertainty in application of a proper sign convention for strains and stresses. Additionally, the use of unusual directions of water-wave propagation and the positive $z$-axis of the coordinate system, as presented by Yamamoto et al. [9], could also magnify difficulties in assessment of correct signs for the wave-induced cyclic seabed response parameters.

Due to many mistakes and ambiguities made in professional books and journal papers, the question of importance of application of a proper sign convention (i.e. solid or soil mechanics sign conventions) for strains and stresses in soil matrix is a fundamental issue and certainly needs a reminder, systematizing and clear explanation by means of direct comparison. This will be done in the first part of the present paper with respect to the static equilibrium and continuity equations governing the seabed cyclic response. In the second part of the paper, in order to illustrate the meaning of each of the above mentioned issues, the final form of analytical solutions, derived separately by Yamamoto et al. [9] and Madsen [5], will be taken into account and examined in details. Other analytical solutions, obtained erroneously by Hsu and Jeng [1], Jeng [2], Jeng and Hsu [4], will be also taken under investigation. Some mistakes in the final formulas, replicated by other authors, will be pointed out additionally.

A definition sketch of the governing problem is presented in Fig. 1. The considerations relate to a two-dimensional case and the governing problem is treated as a plain strain problem. A porous seabed (usually sandy seabed of finite or infinite thickness) behalves like a linear-elastic material and is loaded by a progressive sinusoidal surface water-wave. Figure 1 contains a traditional way of presentation where the following is assumed: the direction of sinusoidal water-wave propagation stays in accordance with the direction of positive $x$-axis of the Cartesian coordinate system $Oxz$, and the positive $z$-axis is directed upwards, as given by e.g.: Madsen [5] and Okusa [8]. The hydrodynamic wave loading induces a cyclic response of the porous seabed by means of an elastic deformation of the porous seabed. Simultaneously, cyclic oscillations of the wave-induced pore-fluid pressure, effective normal stress and shear stress components within the soil matrix are associated.

MOST WIEDZY Downloaded from mostwiedzy.pl

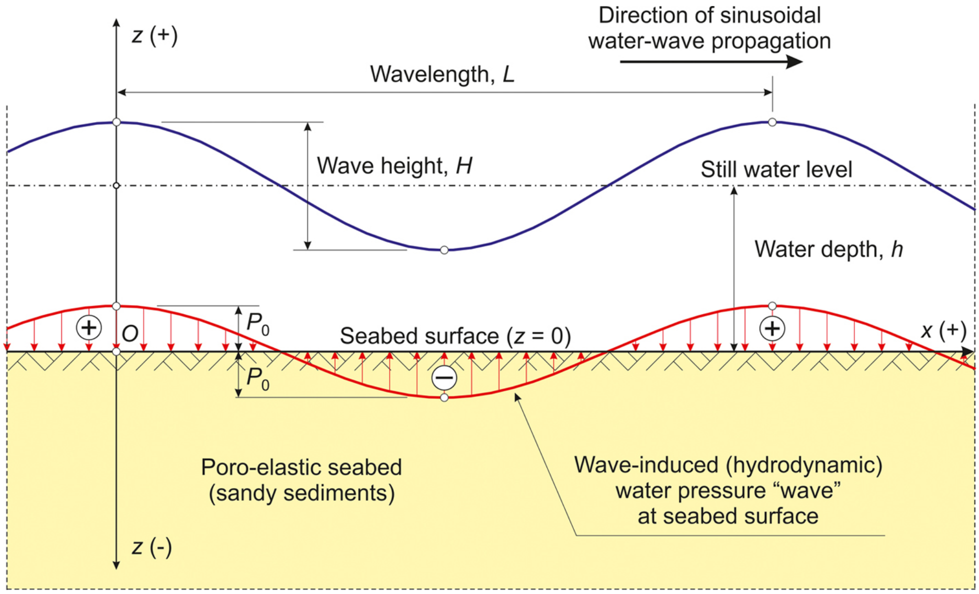


Fig. 1. Definition sketch for analyzing the two-dimensional wave-induced cyclic response of a poro-elastic sandy seabed due to a progressive sinusoidal surface water-wave

## 2. Sign conventions for strains, stresses and pressure

The following three different sign conventions, established and used in numerous calculations and presentations of strain and stress components (acting either in a solid or soil element) and pressure (acting in a fluid), have been distinguished in the literature, namely:

1. solid mechanics sign convention (for strains and stresses),
2. soil mechanics sign convention (for strains and stresses),
3. fluid mechanics sign convention (for fluid pressure).

Figure 2 illustrates the differences between the solid and soil mechanics sign conventions applied to the stress-state in a two-dimensional infinitesimal and elastically deformable element of solid or soil mass. Additionally, two cases with opposing orientations of the positive $z$-axis are presented (only the positive stress components are shown here and the negative components would be easily depicted by opposite directions of all the stress vectors).

A proper sign convention for strains and stresses needs always a careful consideration. To the best Author's knowledge only Sawicki [11] put a kind of warning: ". . . and the soil mechanics sign convention is used (compression is positive). In many papers, the continuum mechanics sign convention is applied (extension is positive), so caution is recommended.", which should be a red flag when dealing with different sign conventions. Unfortunately, it happens that equations presented by some authors do not correspond to the initially assumed sign conventions for strains and stresses. Sometimes one can read the following, for instance: "Sign convention for shear stress . . . ", as found in [12]. Such sentence can be troublesome if

MOST WIEDZY Downloaded from mostwiedzy.pl

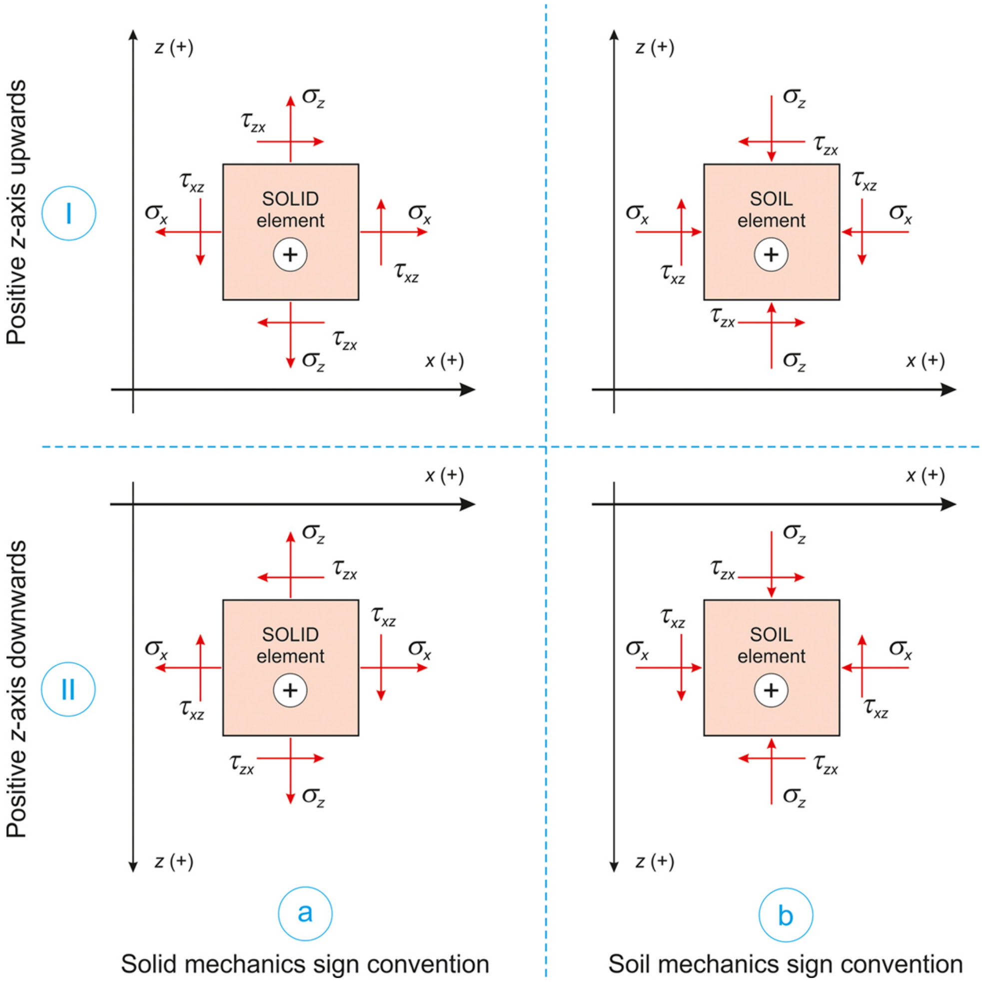


Fig. 2. Diagrams of the state-of-stress on a two-dimensional elastically deformable infinitesimal element (solid or soil) according to: (a) solid mechanics sign convention, (b) soil mechanics sign convention, and two opposing directions of the positive $z$-axis

there is not any previously given additional information on the type of mechanics, a certain paper is based on. In this case much better and more informative sentence would have the following form: "The soil (or solid) mechanics sign convention for shear stress . . . ", as given by e.g. Verruijt [13]. Some authors either do not distinguish between the two sign conventions or even do not inform about the convention used by them. Taking [12, 14] as an example of soil mechanics text books, the Reader will search in vain the terms like "sign convention", especially "soil mechanics sign convention". Such cases lead very often to many confusions, misunderstandings, and even serious errors, especially when the Reader is trying to adopt and use the existing theories and solutions into their scientific studies.

MOST WIEDZY Downloaded from mostwiedzy.pl

Following the sign convention of solid mechanics, normal strain components are considered positive for elongation (extension), and negative for contraction when induced by tension or compression of a solid element, respectively. Adequately, the normal stress components are considered positive when the force component, acting on an outer surface of the side of two-dimensional infinitesimal solid element with outward normal vector directed in a positive coordinate direction, acts in a positive direction, or when the force component, acting on an outer surface of the side of solid element with outward normal vector directed in a negative coordinate direction, acts in a negative direction. This means that unidirectional tension is considered positive, and unidirectional compression is considered negative [13, 15, 16], as illustrated in Fig. 2 (Cases I–a and II–a). The shear stress component $\tau_{xz}$ is considered positive if it is directed in the positive $z$-axis direction while acting on an outer surface of the side of solid element which outward normal has positive $x$-axis direction [12, 14], as illustrated in Fig. 2 (Cases I–a and II–a).

Because normal stresses in soils usually are compressive stresses only, it is a standard practice to use the soil mechanics sign convention for strains and stresses that is just opposite to the theoretically more balanced sign convention of classical solid mechanics. It means that normal strain and stress components are considered positive for contraction and compression, respectively, and negative for elongation and tension, respectively, in the soil matrix [16–18]. The shear stress component $\tau_{xz}$ is considered positive if it is directed in the positive $z$-axis direction while acting on an outer surface of the side of solid element which outward normal has negative $x$-axis direction [14], as illustrated in Fig. 2 (Cases I–b and II–b).

The fluid mechanics sign convention reflects probably the most simple case with respect to the others. Fluid pressure is most often the compressive stress at some point within a fluid, as it should be since fluid cannot withstand tension. Therefore, according to the fluid mechanics sign convention, it is very convenient to take the fluid pressure positive when inducing compression of a small fluid element. In case of the wave-induced pore-fluid pressure, the hydrodynamic overpressure (with respect to the initially assumed and continuously existing hydrostatic pressure) will be always treated as positive whereas the hydrodynamic underpressure will receive a negative sign. By treating the fluid pressure and the normal stress in soil skeleton as analogous physical parameters, it can be easily concluded that the fluid and soil mechanics sign conventions stay in full accordance with each other.

## 3. Equations describing the governing problem

A scientific and engineering problem of the wave-induced cyclic response of a poro-elastic seabed loaded by a progressive sinusoidal water-wave is usually sought as a solution of a set of coupled partial differential equations involving:

1. static equilibrium equations,
2. soil stress-strain and strain-displacement relations,
3. continuity equation in the form of the storage equation.

MOST WIEDZY Downloaded from mostwiedzy.pl

### 3.1. Equilibrium equations (for static force and moment)

Eliminating body forces due to gravity from the problem formulation [19] and disregarding additionally the effects of inertia, a well-known coupled system of static equilibrium partial differential equations can be found in many books, e.g. [12–14, 17]. These equations, in which the total stress satisfies the conditions of equilibrium, are independent of the type of sign convention for stresses because all the vectors of positive (or negative) stress components in the solid mechanics sign convention are opposite to those presented according to the soil mechanics sign convention (see Fig. 2).

The equations of static equilibrium can be also written in terms of the effective stress as a measure of the intergranular forces. In this case the deformation of the soil skeleton will be determined by the effective stress only. As the Terzaghi effective stress principle does not influence the shear stress, only the components of total normal stress can be decomposed according to the following recipes: $\widetilde{\sigma} = \widetilde{\sigma}' - \widetilde{p}$ [15] in case of the solid mechanics sign convention, and $\widetilde{\sigma} = \widetilde{\sigma}' + \widetilde{p}$ [16] when adopting the soil mechanics sign convention, where $\widetilde{\sigma}$ is the wave-induced total normal stress, $\widetilde{\sigma}'$ is the wave-induced effective normal stresses, and $\widetilde{p}$ is the wave-induced pore-fluid pressure. A superimposed tilde-sign $\widetilde{()}$ will always denote a dimensional form of the wave-induced parameter in the following. The minus-sign is introduced to the pore-fluid pressure in the Terzaghi principle used together with the solid mechanics sign convention, $\widetilde{\sigma} = \widetilde{\sigma}' + (-\widetilde{p}) = \widetilde{\sigma} = \widetilde{\sigma}' - \widetilde{p}$, just to keep the accordance between the usual sign convention applied in the fluid mechanics of flow through porous media, where the pore-fluid pressure is taken positive for compression, and the solid mechanics sign convention for stresses in the soil matrix, where the normal stress components and the pore-fluid pressure are negative for compression.

And thus, introducing the Terzaghi effective stress principle into the static equilibrium equations for the total stress, and adopting the fluid mechanics sign convention for pore-fluid pressure notation, the following two different coupled systems of static equilibrium equations of the governing problem are obtained:

1. for the solid mechanics sign convention for stresses (see Fig. 2 – Cases I–a and II–a):

$$\frac{\partial \widetilde{\sigma}'_x}{\partial x} + \frac{\partial \widetilde{\tau}_{zx}}{\partial z} = \frac{\partial \widetilde{p}}{\partial x} \tag{3.1}$$

$$\frac{\partial \widetilde{\tau}_{xz}}{\partial x} + \frac{\partial \widetilde{\sigma}'_z}{z} = \frac{\partial \widetilde{p}}{\partial z} \tag{3.2}$$

$$\widetilde{\tau}_{xz} = \widetilde{\tau}_{zx} \tag{3.3}$$

2. for the soil mechanics sign convention for stresses (see Fig. 2 – Cases I–b and II–b):

$$\frac{\partial \widetilde{\sigma}'_x}{\partial x} + \frac{\partial \widetilde{\tau}_{zx}}{\partial z} = -\frac{\partial \widetilde{p}}{\partial x} \tag{3.4}$$

$$\frac{\partial \widetilde{\tau}_{xz}}{\partial x} + \frac{\partial \widetilde{\sigma}'_z}{\partial z} = -\frac{\partial \widetilde{p}}{\partial z} \tag{3.5}$$

$$\widetilde{\tau}_{xz} = \widetilde{\tau}_{zx} \tag{3.6}$$

MOST WIEDZY Downloaded from mostwiedzy.pl

where: $\widetilde{\sigma}_x'$ and $\widetilde{\sigma}_z'$ – wave-induced effective normal stress components in $x$ and $z$ directions, respectively, $\widetilde{\tau}_{xz}$ and $\widetilde{\tau}_{zx}$ – wave-induced shear stress components, and $\widetilde{p}$ – wave-induced pore-fluid pressure.

The assumption of solid mechanics sign convention for stresses (together with the fluid mechanics sign convention for fluid pressure) is reflected by positive wave-induced pore-fluid pressure terms in Eqs. (3.1) and (3.2), as given in [1–4, 4, 12, 15, 20–23], whereas the application of soil mechanics sign convention for stresses causes the wave-induced pore-fluid pressure terms in Eqs. (3.4) and (3.5) to be negative, as presented in [5, 8, 24, 25]. The equilibrium equations [Eqs. (3.1)–(3.3) and (3.4)–(3.6)] do not change their form when changing the direction of positive $z$-axis. This is because the change of direction of the positive $z$-axis induces the rotation of all the stress vectors about the $x$-axis (see Fig. 2 and compare Cases I–a and I–b or Cases II–a and II–b). For instance, it has to be noted that in some works, e.g. [5, 8, 13, 15–17], the positive $z$-axis is directed upwards, whereas in [9, 12, 14] is set downwards. However, in all these works the stress terms in the equilibrium equations are always the same, irrespectively of the direction of positive $z$-axis.

### 3.2. Soil strain-displacement relations

It is assumed that the deformations are small and their partial derivatives are small as well. Based on the solid mechanics sign convention, the strain components can be defined as [12–14, 26]:

$$\widetilde{\varepsilon}_x \stackrel{\text{def}}{=} \frac{\partial \widetilde{u}_x}{\partial x} \tag{3.7}$$

$$\widetilde{\varepsilon}_z \stackrel{\text{def}}{=} \frac{\partial \widetilde{u}_z}{\partial z} \tag{3.8}$$

$$\widetilde{\varepsilon} \stackrel{\text{def}}{=} \widetilde{\varepsilon}_x + \widetilde{\varepsilon}_z = \frac{\partial \widetilde{u}_x}{\partial x} + \frac{\partial \widetilde{u}_z}{\partial z} \tag{3.9}$$

$$\widetilde{\gamma}_{xz} \stackrel{\text{def}}{=} \frac{\partial \widetilde{u}_x}{\partial z} + \frac{\partial \widetilde{u}_z}{\partial x} \tag{3.10}$$

where: $\widetilde{\varepsilon}_x$ and $\widetilde{\varepsilon}_z$ – wave-induced normal strain components in $x$ and $z$ directions, respectively, $\widetilde{u}_x$ and $\widetilde{u}_z$ – wave-induced displacements of soil skeleton, respectively, $\widetilde{\varepsilon}$ – volumetric strain of soil skeleton, and $\widetilde{\gamma}_{xz}$ – engineering shear strain of soil skeleton.

On the other hand, the application of soil mechanics sign convention to the definition of strain components should indicates the following relations [17]:

$$\widetilde{\varepsilon}_x \stackrel{\text{def}}{=} -\frac{\partial \widetilde{u}_x}{\partial x} \tag{3.11}$$

$$\widetilde{\varepsilon}_z \stackrel{\text{def}}{=} -\frac{\partial \widetilde{u}_z}{\partial z} \tag{3.12}$$

$$\widetilde{\varepsilon} \stackrel{\text{def}}{=} \widetilde{\varepsilon}_x + \widetilde{\varepsilon}_z = -\left(\frac{\partial \widetilde{u}_{\partial x}}{x} + \frac{\partial \widetilde{u}_z}{\partial z}\right) \tag{3.13}$$

$$\widetilde{\gamma}_{xz} \stackrel{\text{def}}{=} -\left(\frac{\partial \widetilde{u}_x}{\partial z} + \frac{\partial \widetilde{u}_z}{\partial x}\right) \tag{3.14}$$

MOST WIEDZY Downloaded from mostwiedzy.pl

which is opposite to the solid mechanics definitions [compare with Eqs. (3.7)–(3.7)]. The soil displacement components were defined to be positive for contraction (i.e. according to the soil mechanics sign convention) only by Okusa [8]. Surprisingly, the soil mechanics definitions, presented in Eqs. (3.11)–(3.14) by Shao [17], are not widely-spread among many researchers, but it should be. Most of the soil mechanics text books, e.g. [12, 14], contain the definition which reflects only the usage of solid mechanics sign convention for strains and stresses [see Eqs. (3.7)–(3.10)]. The only explanation of this fact is that some researchers are used to apply the soil mechanics sign convention only for stresses but not for strains. However, due to undisputed stress-strain relations for soil, it seems reasonable and more elegant from the formal point of view to start using the soil mechanics sign convention already at the stage of definition of the normal and shear strain components.

## 3.3. Soil stress-strain relations

The basic equations of the theory of elasticity describe the relations between displacement, strain and stress components in an isotropic linear elastic material. For a linear elastic material the relation between stresses and strains is given by Hooke's law. In applied soil mechanics the relation between stresses and strains of an isotropic linear elastic material is usually described by Young's modulus and Poisson's ratio. Since only small departures from an initial state are considered, these coefficients are usually regarded as constants. This means that the soil is assumed to behave as a homogeneous material. And thus, the well-known stress-strain equations for the governing plain strain problem take the following form:

$$\widetilde{\sigma}'_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\widetilde{\varepsilon}_x + \nu\widetilde{\varepsilon}_z\right] \tag{3.15}$$

$$\widetilde{\sigma}'_z = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\widetilde{\varepsilon}_x + (1-\nu)\,\widetilde{\varepsilon}_z\right] \tag{3.16}$$

$$\widetilde{\tau}_{xz} = \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{1}{2}-\nu\right)\widetilde{\gamma}_{xz} \tag{3.17}$$

where, additionally: $E$ – Young modulus (modulus of elasticity) of soil and $\nu$ – Poisson ratio of soil, both coupled with each other in $G = E/[2(1+\nu)]$, where $G$ – shear modulus of soil skeleton.

Although initially derived for solid mechanics, it must be stressed that the form of Eqs. (3.15)–(3.17) are independent of the type of sign convention – the dependency can only be visible when proper strain definitions are introduced. And thus, assuming the solid mechanics sign convention for strains [see Eqs. (3.7)–(3.10)], the set of Eqs. (3.15)–(3.17), rewritten only in terms of the wave-induced soil displacement components, takes the following form [1–4, 9]:

$$\widetilde{\sigma}'_x = 2G\left[\frac{\partial\widetilde{u}_x}{\partial x} + \frac{\nu}{1-2\nu}\left(\frac{\partial\widetilde{u}_x}{\partial x} + \frac{\partial\widetilde{u}_z}{\partial z}\right)\right] \tag{3.18}$$

$$\widetilde{\partial\sigma}'_{\partial z} = 2G\left[\frac{\partial\widetilde{u}_z}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{\partial\widetilde{u}_x}{\partial x} + \frac{\partial\widetilde{u}_z}{\partial z}\right)\right] \tag{3.19}$$

MOST WIEDZY Downloaded from mostwiedzy.pl

$$\widetilde{\tau}_{xz} = G\left(\frac{\partial \widetilde{u}_x}{\partial z} + \frac{\partial \widetilde{u}_z}{\partial x}\right) \tag{3.20}$$

It is really very hard to understand why Jeng et al. [4] wrote the following sentence with respect to Eqs. (3.18)–(3.20): "A positive sign is used in the present paper, as in equations (7)–(12), i.e. compressive stresses are defined as positive". One can easily recognise that Eqs. (3.18)–(3.20), with the positively-signed right-hand sides, are derived under the assumption of solid mechanics sign convention for strains and stresses where, as explained earlier, compressive stresses must be always treated as negative from their definition.

Madsen [5] and Verruijt [13, 19] presented the following double-equations combining definitions for the horizontal and vertical normal strain components and the engineering shear strain with adequate stress-strain relations:

$$\widetilde{\varepsilon}_x \overset{\text{def}}{=} \frac{\partial \widetilde{u}_x}{\partial x} = -\frac{1-\nu^2}{E}\left(\widetilde{\sigma}'_x - \frac{\nu}{1-\nu}\widetilde{\sigma}'_z\right) \tag{3.21}$$

$$\widetilde{\varepsilon}_z \overset{\text{def}}{=} \frac{\partial \widetilde{u}_z}{\partial z} = -\frac{1-\nu^2}{E}\left(\widetilde{\sigma}'_z - \frac{\nu}{1-\nu}\widetilde{\sigma}'_x\right) \tag{3.22}$$

$$\widetilde{\gamma}_{xz} \overset{\text{def}}{=} \frac{\partial \widetilde{u}_x}{\partial z} + \frac{\partial \widetilde{u}_z}{\partial x} = -\frac{2(1+\nu)}{E}\widetilde{\tau}_{xz} \tag{3.23}$$

From the set of Eqs. (3.21)–(3.23) one can easily realize that there is a mixture of two different sign conventions in Madsen's [5] mathematical formulation of the governing problem. Firstly, the solid mechanics sign convention for strains is applied, where the normal strain components are defined to be positive for elongation, and then the soil mechanics sign convention for stresses is adopted, where the normal stress components are positive for compression. According to Jeng [2], "It is noted that a negative sign was attached to the definition of the effective stresses and strains in Madsen (1978), due to different sign notation used". This statement is only partially true because Madsen's [5] definitions of strains do not contain any negative signs and the minus-sign was "artificially" introduced only to the stress-reflecting second parts of Eqs. (3.21)–(3.23) in order to obtain the formulation consistency. Nevertheless, this is very surprising to combine the solid mechanics sign convention (for strains) with the soil mechanics sign convention (for stresses) in one and the same solution. This way of presentation leads very often to misunderstanding and can be very confusing.

It is also not a good habit to mix the two considered sign conventions for strains and stresses in one and the same solution where, for instance, the storage equation is presented according to the solid mechanics sign convention for strains, the equilibrium equations are written based on the soil mechanics sign convention for stresses and the wave-induced effective normal stress and shear stress components are given again according to the solid mechanics sign convention for strains and stresses, as found in e.g. [27, 28]. Wong et al. [29] decided to make a slight difference and presented the storage equation following the soil mechanics sign convention for strains. All of this could cause serious problems especially with interpretation of the results obtained, even leading occasionally to totally wrong solutions of the problem under consideration.

Many authors mixed the both sign conventions which can be often confusing for the Reader trying to apply this information into their research work. And to make matters

MOST WIEDZY Downloaded from mostwiedzy.pl

worse, it concerns soil mechanics text books. A book by Das [12] is a typical example where the soil mechanics sign convention for stresses was initially assumed in Chapter 2 of his book ("The normal stresses are considered positive when they are directed onto the surface.", and particularly in Fig. 2.2 where "... all shear stresses are positive ..."). Afterwards, in the following text of Chapter 2, normal and shear strain definitions, originally numbered as (2.22)–(2.26) in [12], and the stress-strain relations, originally numbered as (2.27)–(2.35) in [12], are indicated adequately to the solid mechanics sign convention. It is clear that such confusing situations should be avoided.

Taking the above into account, it can be stated that a much more elegant, theoretically correct and unambiguous way of mathematical formulation of the governing problem would be to use the soil mechanics sign convention in total, i.e. for both strains and stresses. This obvious goal can be easily achieved by using the definition equations for strains utilizing the notation of soil mechanics sign convention [Eqs. (3.11)–(3.14)], as pointed out by Shao et al. [17]. Then, the only thing to be done is to apply these equations to the general stress-strain relations independently of the type of sign convention [see Eqs. (3.15)–(3.17)].

### 3.4. Storage equation (continuity equation)

In the theory of linear consolidation, only small incremental deformations of a soil element are considered, taking into account small deviations from an initial steady state (i.e. geostatic volumetric strain of soil and hydrostatic pore-fluid pressure). Adopting the continuity principle for a two-dimensional problem, incorporating the Darcy law of a compressible pore-fluid flow through a compressible porous medium, and assuming anisotropic conditions as far as the soil permeability is concerned, the continuity equation of the pore-fluid [9] or the mass conservation equation [5, 8] can be written – based on Biot's three-dimensional theory of consolidation (or poro-elasticity) – in the form of so-called the storage equation, expressing that a volume change of soil can be caused by either a pressure change or by a net outflow of pore-fluid from the pores [15]

$$\frac{k_x}{\gamma}\frac{\partial^2\widetilde{p}}{\partial x^2}+\frac{k_z}{\gamma}\frac{\partial^2\widetilde{p}}{\partial z^2}=n\beta\frac{\partial\widetilde{p}}{\partial t}+\gamma\frac{\partial\widetilde{\varepsilon}}{\partial t} \tag{3.24}$$

where: $\widetilde{p}$ – wave-induced pore-fluid pressure, $\widetilde{\varepsilon}$ – wave-induced volumetric strain of porous soil skeleton, $k_x$ and $k_z$ – coefficients of soil permeability in $x$- and $z$-directions, respectively, $n$ – porosity of soil, $\beta$ – compressibility of pore-fluid, $\gamma$ – unit weight of seawater, $x$ and $z$ – horizontal and vertical coordinates, respectively, and $t$ – time.

The incremental volumetric strain of soil, $\widetilde{\varepsilon}$, and thereby the wave-induced normal strain components, $\widetilde{\varepsilon}_x$ and $\widetilde{\varepsilon}_z$, are defined to be positive for elongation, which reflects the use of the solid mechanics sign convention, as adopted by: Hsu et al. [1], Jeng [2, 3], Jeng et al. [4] and Madsen [5] – for a hydraulically anisotropic porous seabed, and Yamamoto et al. [9] – for a hydraulically isotropic ($k_x = k_z$) porous seabed. Only Okusa [8] used the soil mechanics sign convention when presenting the storage equation, adequate to Eq. (3.24), in which the soil strain term has a negative sign.

MOST WIEDZY Downloaded from mostwiedzy.pl

## 4. Discussion on some existing analytical solutions

Different basic shapes of deformation of a two-dimensional infinitesimal element are illustrated in Fig. 3 (for a solid element with the solid mechanics sign convention applied) and in Fig. 4 (for a soil element with the soil mechanics sign convention applied), depending on the sign of shear stress components and the orientation of positive $z$-axis.

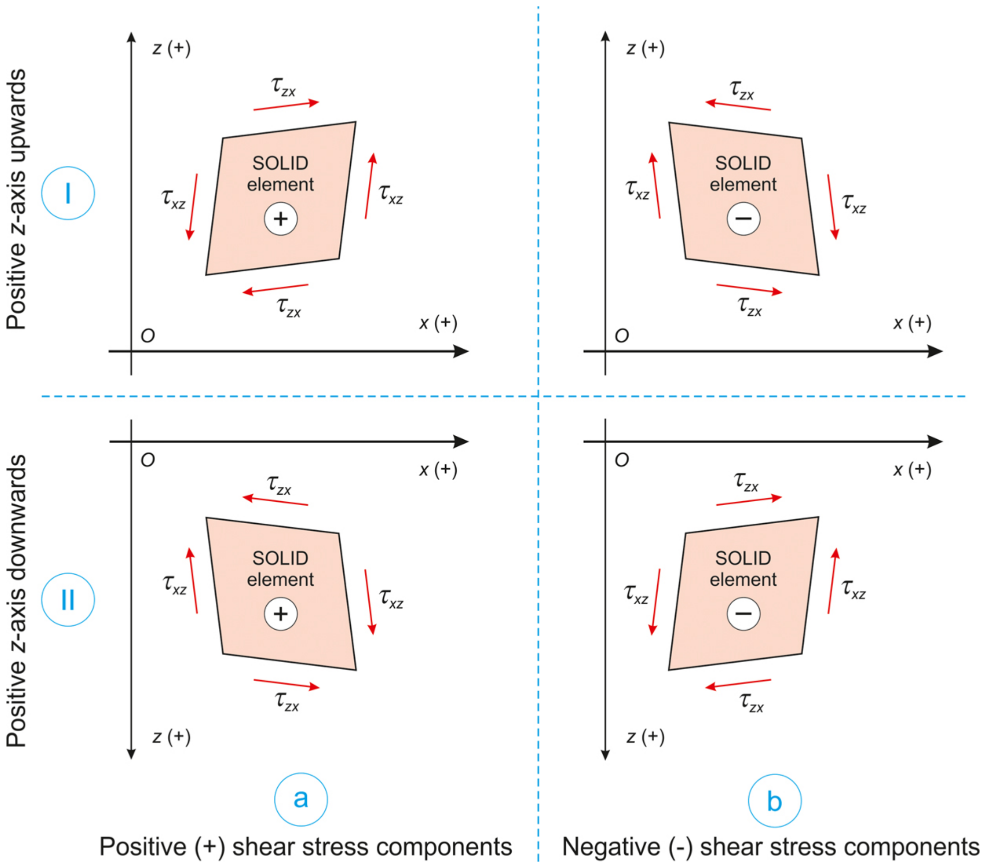


Fig. 3. Diagrams of the shape of elastic deformation of a two-dimensional infinitesimal SOLID element, according to: different signs of the shear stress components and two opposing orientations of the positive $z$-axis

Figures 3 and 4, together with Fig. 2, will be very helpful in understanding the comparison analysis performed in the following.

Two groups of resultant equations for the wave-induced cyclic response of the seabed of infinite thickness due to a progressive sinusoidal water-wave loading are presented below, where the first one was obtained by Yamamoto et al. [9] and the second one by Madsen [5]. For clarity of presentation, the results presented below refer only to a fully saturated soil (i.e.

MOST WIEDZY Downloaded from mostwiedzy.pl

$S_r$ = 1.0, where $S_r$ denotes the degree of soil saturation). The important differences in basic assumptions for obtaining both solutions are as follows:

1. Yamamoto's [9] solution: solid mechanics sign convention for strains and stresses, positive $z$-axis directed downwards, water-wave propagates from right to left according to the direction of negative $x$-axis,
2. Madsen's [5] solution: soil mechanics sign convention for stresses, positive $z$-axis directed upwards, water-wave travels from left to right according to the direction of positive $x$-axis.

Although Madsen [5] applied the solid mechanics sign convention for strains in his mathematical formulation, some informal operations, obtained by "attaching a negative sign" to the right-hand sides of stress-strain equations, allowed for interpretation of his analytical solution and all the final formulas for the wave-induced stress components under the assumption of the soil mechanics sign convention.

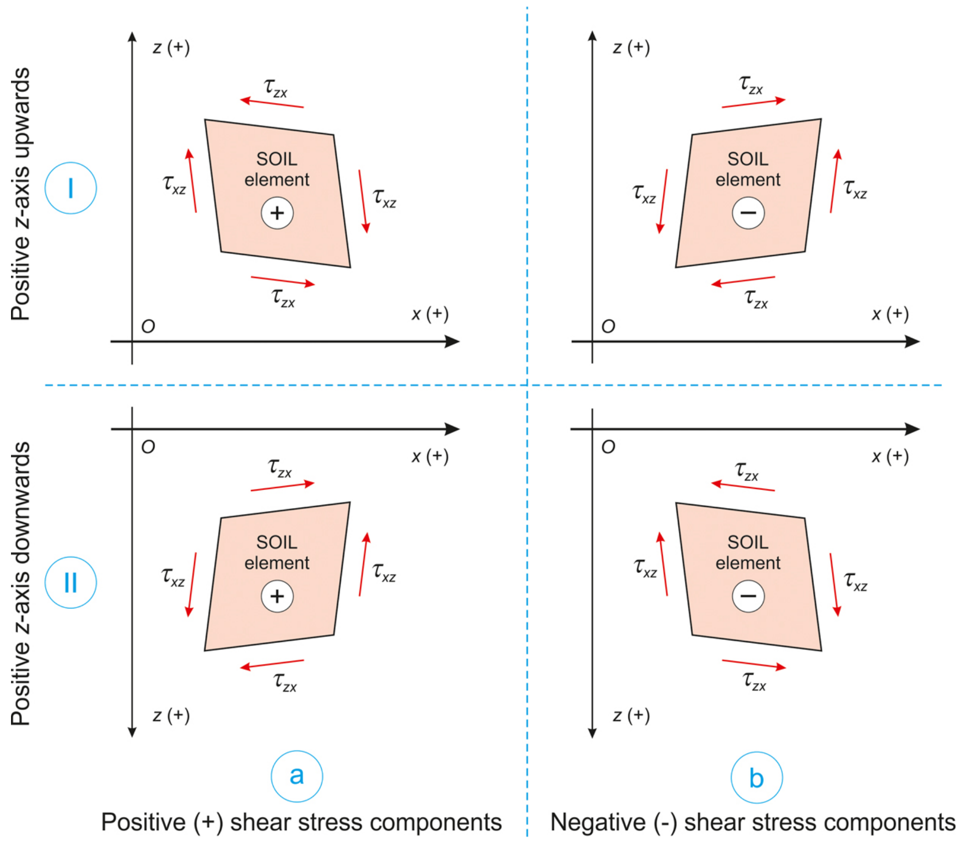


Fig. 4. Diagrams of the shape of elastic deformation of a two-dimensional infinitesimal SOIL element, according to: different signs of the shear stress components and two opposing orientations of the positive $z$-axis

MOST WIEDZY Downloaded from mostwiedzy.pl

The analytical solutions by Yamamoto et al. [9] and Madsen [5] we obtained in terms of six complex-valued wave-induced parameters. For simplicity, the main resulting parameters are usually presented in their relative and dimensionless forms denoted by a superimposed bar-sign $\overline{()}$, whereas dimensional forms of the wave-induced parameters are denoted by a superimposed tilde-sign $\widetilde{()}$. And thus, by introducing the recipes for relative and dimensionless forms: $\overline{u} = \widetilde{u}/[P_0/(2kG)]$ and $[\overline{p}, \overline{\sigma}, \overline{\tau}] = \{\widetilde{p}, \widetilde{\sigma}, \widetilde{\tau}\}/P_0$, applying Euler's formulas to the phase-angle of oscillation terms, and taking into account only the case of fully saturated, isotropic and dense soil (meaning practically incompressibility of the two-phase soil-fluid medium), the analytical solutions take the following form of real-valued functions:

1. solution obtained by Yamamoto et al. [9] (see stress diagrams – Case II–a in Fig. 2 and Cases II–a and II–b in Fig. 3):

$$\overline{u}_x = kz \exp(-kz) \sin(kx + \omega t) \tag{4.1}$$

$$\overline{u}_z = (1 + kz) \exp(-kz) \cos(kx + \omega t) \tag{4.2}$$

$$\overline{p} = \exp(-kz) \cos(kx + \omega t) \tag{4.3}$$

$$\overline{\sigma}'_x = kz \exp(-kz) \cos(kx + \omega t) \tag{4.4}$$

$$\overline{\sigma}'_z = -kz \exp(-kz) \cos(kx + \omega t) \tag{4.5}$$

$$\overline{\tau}_{xz} = -kz \exp(-kz) \sin(kx + \omega t) \tag{4.6}$$

2. solution obtained by Madsen [5] (see stress diagrams – Case I–b in Fig. 2 and Cases I–a and I–b in Fig. 4):

$$\overline{u}_x = -kz \exp(kz) \sin(kx - \omega t) \tag{4.7}$$

$$\overline{u}_z = -(1 - kz) \exp(kz) \cos(kx - \omega t) \tag{4.8}$$

$$\overline{p} = \exp(kz) \cos(kx - \omega t) \tag{4.9}$$

$$\overline{\sigma}'_x = kz \exp(kz) \cos(kx - \omega t) \tag{4.10}$$

$$\overline{\sigma}'_z = -kz \exp(kz) \cos(kx - \omega t) \tag{4.11}$$

$$\overline{\tau}_{xz} = kz \exp(kz) \sin(kx - \omega t) \tag{4.12}$$

where: $\overline{u}_x$ and $\overline{u}_z$ – wave-induced soil displacement components in $x$ and $z$ directions, respectively, $\overline{p}$ – pore-water pressure, $\sigma'_x$ and $\sigma'_z$ – normal stress components in $x$ and $z$ directions, respectively, $\tau_{xz}$ – shear stress component, $P_0$ – wave-induced pore-water pressure amplitude at the seabed surface, $k$ – wave number, $G$ – shear modulus of soil, $\omega$ – wave angular frequency, $i$ – imaginary unit ($i = \sqrt{-1}$), $t$ – time, $x$ and $z$ – horizontal and vertical coordinates, respectively.

At this point a small digression is strongly needed. Many researchers are used to operate with the following terms: "pore pressure" [8, 10, 13, 16], "wave-induced pore pressure" [1–5] or "excess pore pressure" [7] in order to describe the pressure (with respect to the hydrostatic conditions) of a medium existing in pores of the seabed sediments. It is worth noting that these terms are simplified too far and they should not be used at all. The soil pores, being geometrical structures, can be characterised only by geometrical parameters (e.g. a volume) and not by a pressure – the soil pores do not have any pressure! Instead of it, the term

MOST WIEDZY Downloaded from mostwiedzy.pl

"pore-fluid pressure" should be used in general, especially when partially saturated soils are concerned ($S_r < 1.0$); in this case the pore-fluid is treated as a two-phase medium consisting of pore-water and gas (e.g. air, methane) bubbles entrapped in it. On the other hand, when the soil is fully saturated ($S_r = 1.0$) and the soil pores are filled only with a pore-water (a one-phase medium), the generally accepted terms like "pore-fluid" [1–6, 8, 13, 16] and "pore-fluid pressure" should be successfully replaced by the terms "pore-water" [18, 22] and "pore-water pressure" [9, 10, 18, 33], respectively. This fine distinction is very helpful and logical but only few researchers prefer to follow this idea.

It has to be remembered that, as indicated by the presence of attenuation term $\exp(-kz)$, the positive $z$-axis is directed downwards in the solution obtained by Yamamoto et al. [9], which is opposite to Madsen's [5] solution. Another aspect that is worthy of attention is, as indicated by the phase-angle term $\omega(kx + t)$, the direction of surface sinusoidal wave propagation in accordance with the negative $x$-axis in the solution obtained by Yamamoto et al. [9], which is opposite to Madsen's [5] way of solution. It can be easily examined that the direction of wave propagation influences the wave-induced displacement $\widetilde{u}_x$ (or $\overline{u}_x$) and the shear stress $\widetilde{\tau}_{xz}$ (or $\overline{\tau}_{xz}$), as it can be deducted from Fig. 5.

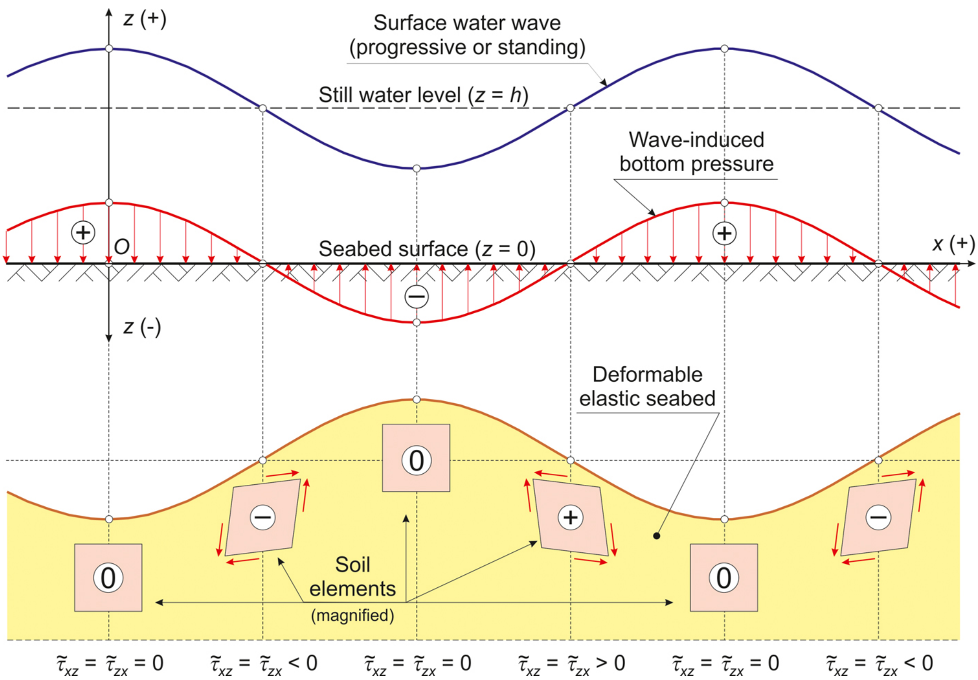


Fig. 5. Qualitative shapes of elastic deformations of the poro-elastic seabed under a surface progressive sinusoidal water-wave together with the soil mechanics sign convention for the wave-induced shear stress components, $\widetilde{\tau}_{xz} = \widetilde{\tau}_{zx}$, acting on small two-dimensional soil elements

MOST WIEDZY Downloaded from mostwiedzy.pl

Figure 5 illustrates characteristic possible shapes of elastic deformations (strains) of the porous seabed under a progressive sinusoidal water-wave together with a commonly used in soil mechanics sign convention for the wave-induced shear stress of a small two-dimensional elastic soil element. A deformed shape of a soil element in the upper part of the seabed was presented qualitatively only by a few of researchers, among them: Kirca et al. [31], Sawicki and Mierczyński [32], Sumer and Fredsøe [33] and Sumer [10]. However, their presentations are very general because they are limited only to one possible case and they did not contain any information about the sign of the shear stress (just a shape). Kirca et al. [31] (see the original Fig. 1) and Sawicki and Mierczyński [32] (see the original Fig. 1) illustrated schematically only one basic deformation shape of a soil element, where the directions of stress components indicate their positive values according to the soil mechanics sign convention for strains and stresses, assuming additionally the positive $z$-axis directed upwards [31] or downwards [32]. Unfortunately there is an obvious discrepancy between the definition sketch presented by Sawicki and Mierczyński [32], who adopted the solution obtained by Yamamoto et al. [9] where the solid mechanics sign convention was customized. Illustrations given by Sumer and Fredsøe [33] (see the original Fig. 10.1) and Sumer [10] (see the original Fig. 1.5), based on the solid mechanics sign convention, are also not very much informative.

And thus, introducing the negative $z$-axis directed downwards and the phase-angle term $(kx - \omega t)$ (a minus-sign informs that the direction of wave propagation stays in accordance with the direction of positive $x$-axis; in other words, the wave propagates "from left to right", as presented in Fig. 2) into Eqs. (4.1)–(4.6), one should obtain as follows:

$$\overline{u}_x = -kz \exp(kz) \sin(kx - \omega t) \tag{4.13}$$

$$\overline{u}_z = -(1 - kz) \exp(kz) \cos(kx - \omega t) \tag{4.14}$$

$$\overline{p} = \exp(kz) \cos(kx - \omega t) \tag{4.15}$$

$$\overline{\sigma}'_x = -kz \exp(kz) \cos(kx - \omega t) \tag{4.16}$$

$$\overline{\sigma}'_z = kz \exp(kz) \cos(kx - \omega t) \tag{4.17}$$

$$\overline{\tau}_{xz} = -kz \exp(kz) \sin(kx - \omega t) \tag{4.18}$$

Comparing Eqs. (4.7)–(4.12), obtained by Madsen [5], and keeping in mind that Yamamoto's [9] Eqs. (4.13)–(4.18) resulted from the procedure of the above performed adaptation so that they can reflect the same geometry (i.e. two-dimensional Cartesian coordinate system with the negative $z$-axis directed downwards) and wave conditions (i.e. direction of sinusoidal water-wave propagation along with the positive $x$-axis) as given in Madsen's [5] solution [see Eqs. (4.7)–(4.12)], it is clearly visible that:

1. the equations for wave-induced soil displacement components, $\overline{u}_x, \overline{u}_z$ [compare Eqs. (4.7) and (4.8) with Eqs. (4.13) and (4.14)], and pore-fluid pressure, $\overline{p}$ [compare Eq. (4.9) with Eq. (4.15)] are identical in Madsen's [5] and Yamamoto's [9] solutions, respectively,
2. the wave-induced stress components, $\overline{\sigma}'_x$, $\overline{\sigma}'_z$ and $\overline{\tau}_{xz}$ [compare Eqs. (4.10)–(4.12) with Eqs. (4.16)–(4.18)] have opposite signs in Madsen's [5] and Yamamoto's [9] solutions, respectively.

Only on the face of it, this looks like the two solutions, obtained by Yamamoto et al. [9] and Madsen [5], differ from each other. But the opposite signs of the wave-induced normal

MOST WIEDZY Downloaded from mostwiedzy.pl

stress components, $\overline{\sigma}'_x$ and $'_z$, and shear stress component, $\overline{\tau}_{xz}$, result from the assumptions of two different sign conventions for stresses: the solid mechanics sign convention in [9], and the soil mechanics sign convention in [5], used throughout their papers.

As illustrated by Ishikara and Yamazaki [34] (see the original Fig. 1 and note that the positive $z$-axis is directed downwards), the total normal stress components acting on a soil element, $\sigma_h \equiv \sigma_x$ and $\sigma_v \equiv \sigma_z$ indicate the use of the soil mechanics sign convention, whereas the directions of the wave-induced shear stress components stay surprisingly in accordance with the solid mechanics sign convention. Additionally, Ishikara and Towhata [35] and Ishikara and Yamazaki [34] followed the work by Madsen [5] and Yamamoto et al. [9] and presented the following equations (the original notation is preserved where: $\sigma_h \equiv \widetilde{\sigma}'_x$, $\sigma_v \equiv \widetilde{\sigma}'_z$ and $\tau_{vh} \equiv \widetilde{\tau}_{xz}$, $p_0 \equiv P_0$, $L$ is the wavelength):

$$\sigma_h = p_0 \left(1 - \frac{2\pi z}{L}\right) \mathrm{e}^{-2\pi z/L} \cos(kx - \omega t) \tag{4.19}$$

$$\sigma_v = p_0 \left(1 + \frac{2\pi z}{L}\right) \mathrm{e}^{-2\pi z/L} \cos(kx - \omega t) \tag{4.20}$$

$$\tau_{vh} = p_0 \frac{2\pi z}{L} \mathrm{e}^{-2\pi z/L} \sin(kx - \omega t) \tag{4.21}$$

A closer look into Eqs. (4.19) and (4.20), and analysing the signs of the wave-induced effective parts of the total stress components, reveals identity with Madsen's [5] solution, thereby certifying that Ishikara and Yamazaki [34] used the soil mechanics sign convention in their effective stress equations. If so, the consistency of the solution would require the usage of the same sign convention for the equation describing the shear stress component. Transforming the solution for $\tau_{vh} = \tau_{xz}$, obtained by Ishikara and Yamazaki [34], to Madsen's [5] definition conditions, Eq. (4.21) needs a double action: the first one means the change of $z$ sign, and the second one means the change of $\tau_{vh}$ sign according to the idea of sign change for the shear stress component due to the change of direction of positive $z$-axis, as formerly illustrated in Fig. 4 (from Case II–b to Case I–a). This double action leads again to exactly the same relation as given in Eq. (4.21). Concluding, the directions of vectors of the shear stress components in the stress diagram illustrated by Ishikara and Yamazaki [34] contradict the mathematical function describing the shear stress component $\tau_{vh}$.

Citing Yamamoto's [9] solution, Sumer [10] and Sumer and Fredsøe [33] presented a wrong equation for the wave-induced shear stress component, $\widetilde{\tau}_{xz}$, assuming: (1) the solid mechanics sign convention for stresses, (2) the wave propagation from right to left (i.e. according to the direction of negative $x$-axis) and (3) the positive $z$-axis directed downwards. This erroneous equation, also found in [32], has the following complex-valued form

$$\widetilde{\tau}_{xz} = -iP_0 kz \exp(-kz) \exp[i(kx + \omega t)] \tag{4.22}$$

where the real part thereof is

$$\Re\{\widetilde{\tau}_{xz}\} = P_0 kz \exp(-kz) \sin(kx + \omega t) \tag{4.23}$$

MOST WIEDZY Downloaded from mostwiedzy.pl

and this is of course not the same (the sign is quite the contrary) as derived correctly by Yamamoto et al. [9] in the complex-number domain

$$\widetilde{\tau}_{xz} = -\frac{1}{i} P_0 kz \, \exp(-kz) \exp[i(kx + \omega t)] \tag{4.24}$$

where the real part thereof is

$$\Re\{\widetilde{\tau}_{xz}\} = -P_0 kz \, \exp(-kz) \sin(kx + \omega t) \tag{4.25}$$

Usually, presentations of vertical distributions of amplitudes of the wave-induced seabed response parameters do not cause any serious problems. Even though, strange equations can be still found in the literature, e.g. [10] – see the original Eq. (3.66), page 100 (the original notation is preserved, where $\tau \equiv \widetilde{\tau}_{xz}$, $p_b \equiv P_0$, $\lambda \equiv k$, and the positive $z$-axis is directed downwards) in which

$$\tau = -p_b \lambda z \exp(-\lambda z) \tag{4.26}$$

is described as the amplitude of the wave-induced shear stress in the seabed of infinite thickness under a progressive sinusoidal water-wave. Noting that $z$ cannot be negative in Eq. (4.26), nothing could be further from the truth. From the definition, any amplitude of oscillations cannot be negative.

Referring to Yamamoto's [9] solution, Sumer [10] and Sumer and Fredsøe [33] showed also a wrong graphical presentation of the wave-induced shear stress oscillating response of the seabed of infinite thickness; see the original Fig. 10.6 in [33] and Fig. 2.3 in [10]. According to Yamamoto's [9] analytical solution (the solid mechanics sign convention is applied), correct functions of momentary values of the dimensionless wave-induced parameters [see Eqs. (4.4)–(4.6)]: pore-water pressure, $\overline{p}/\exp(-kz)$, effective normal stress components, $\overline{\sigma}'_x/[kz \exp(-kz)]$ and $\overline{\sigma}'_z/[kz \exp(-kz)]$, and shear stress component, $\overline{\tau}_{xz}/[kz \exp(-kz)]$, are presented in Fig. 6 (solid lines) as a cyclic variation with time just for one period of water-wave oscillations ($\omega t = 0$–$2\pi$). A short comparison of the graphs of wave-induced oscillations of the shear stress component, $\overline{\tau}_{xz}$, presented in Fig. 6 and illustrated in Sumer's [10, 33] books (a dashed line in Fig. 6), reveals that they differ by a sign, putting into question the correctness of Sumer's [10, 33] illustrations. It is highly probable that this misunderstanding is caused by wrong interpretation of the sign convention used.

Hsu and Jeng [1] and Jeng and Hsu [4] presented the three-dimensional governing partial differential equations (the static force equilibrium equations and the storage equation), clearly indicating the use of the solid mechanics sign convention throughout their papers; this was also certified by the stress block in the original Fig. 2 in [1]. The form of equations for wave-induced effective normal stress components and the shear stress components also indicate the use of the solid mechanics sign convention. Surprisingly, Jeng and Hsu [4] wrote: "A positive sign is used in the present paper, as in equations (7)–(12), i.e. compressive stresses are defined as positive". Nothing could be further from the truth. The form of equilibrium equations and equations for the wave-induced stress components, given in [4], indicates clearly that the solid mechanics sign convention was used to derive these equations. Therefore, the solution obtained by Jeng

MOST WIEDZY Downloaded from mostwiedzy.pl

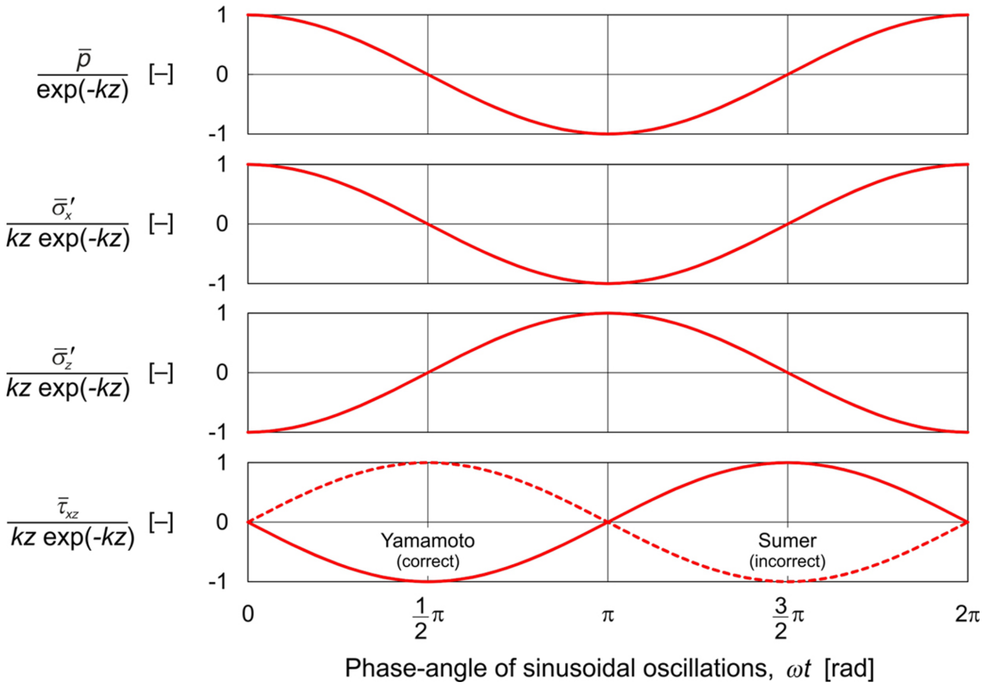


Fig. 6. Graph of oscillations of the relative and dimensionless wave-induced parameters: pore-water pressure at the seabed surface, $\overline{p}$, effective normal stress components, $\overline{\sigma}'_x$ and $\overline{\sigma}'_z$, and the shear stress component, $\overline{\tau}_{xz}$, with time (Yamamoto's [9] analytical solution – solid lines, and Sumer's [10] illustration – dashed line)

and Hsu [4] for fully saturated and isotropic soil of infinite thickness should follow strictly the solution presented by Yamamoto et al. [9]. The solution by Jeng and Hsu [4] has the following real-valued form (only a two-dimensional case is shown here and the notation of the present paper is used):

$$\overline{\sigma}'_x = -kz\exp(kz)\cos(kx-\omega t) \tag{4.27}$$

$$\overline{\sigma}'_z = kz\exp(kz)\cos(kx-\omega t) \tag{4.28}$$

$$\overline{\tau}_{xz} = -kz\exp(kz)\sin(kx-\omega t) \tag{4.29}$$

which is exactly the same as Yamamoto's [9] solution transformed for the positive $z$-axis directed upwards and the water-wave propagation direction consistent with the positive $x$-axis [see Eqs. (4.16)–(4.18)], confirming thereby the use of the solid mechanics sign convention (tensile stresses are positive) by Jeng and Hsu [4]. And thus, there is a clear contradiction between what they wrote in the text (please recall the citation mentioned above) and in Eqs. (4.27) and (4.28).

And last but not least, Jeng [2] used exactly the same assumptions as Hsu and Jeng [1] and Jeng and Hsu [4]. However, among seven basic assumptions indicated in his book there is no any information regarding the sign convention applied. Again, it can only be deducted

MOST WIEDZY Downloaded from mostwiedzy.pl

from the form of the equilibrium equations that the solid mechanics sign convention is used throughout Chapter 3 of the book. The wave-induced effective normal vertical stress solution, obtained for a fully saturated isotropic seabed of infinite thickness, after transforming it into the two-dimensional case and real-valued functions, and adopting the notation used in the present paper, can be presented as

$$\overline{\sigma}'_z = -kz \exp(kz) \cos(kx - \omega t) \tag{4.30}$$

The stress component, $\overline{\sigma}'_z$, which should obviously be compressive under the wave crest, e.g. for $\omega(kx - t) = 0$, obtains non-negative values for $z \leq 0$, as it is always expected in the case of soil mechanics sign convention. But this is not the sign convention used by Jeng [2]. Another contradiction? It can be checked that the signs of other stresses, $\overline{\sigma}'_x$ and $\overline{\tau}_{xz}$, are in line with the solid mechanics sign convention. Such mixture of two different sign conventions in one set of solution equations is unacceptable, leading very often to many misunderstandings and mistakes.

## 5. Conclusions

A careful review of some selected analytical solutions to the wave-induced cyclic response of elastic seabed sediments due to surface water-wave loading is presented mainly with respect to different sign conventions used throughout the solution procedures. Only few of the authors named explicitly the type of the sign convention used in their formulations of the governing problem, leaving the Readers with their doubts and compelling them very often to guess correct meanings of the parameters presented in the papers.

Yamamoto et al. [9] and Okusa [8] presented the most sophisticated ways of analytical solution as far as the application of one and the same sign convention (i.e. the solid mechanics sign convention in [9] and the soil mechanics sign convention in [8]) from the very beginning till the very end of the solution procedure. It means that their sets of the static equilibrium equations and the storage equation, each of them individually, are consistent from the sign convention point of view.

An "attaching a negative sign" to the right-hand sides of stress equations, as done by Madsen [5] in order to achieve the consistency between the two sign conventions within one and the same formulation of the governing problem, does not seem to be a proper mathematical operation, especially when a certain solution is derived analytically. Therefore, instead of this, it is highly recommended to apply the strain definitions respectively to either solid or soil mechanics sign convention, as introduced by Shao et al. [17] [see Eqs. (3.11)–(3.14) for the soil mechanics sign convention], into the solution procedure rather than the so-called "hybrid method" used by Madsen [5].

Based on the analytical solutions to the wave-induced cyclic seabed response, obtained by Madsen [5] and Yamamoto et al. [9] for the case of infinite thickness of seabed sediments, the question of use of different sign conventions to one and the same problem is illustrated together with a discussion on additional influence of the positive $z$-axis direction and the direction of water-wave propagation on the form of final solution. Based on the final equations for the

MOST WIEDZY Downloaded from mostwiedzy.pl

wave-induced soil displacement, effective normal stress and shear stress components, obtained for fully saturated and dense soil conditions, ostensible differences have been analysed and explained in details.

It has been shown that the sign of wave-induced shear stress component, in particular, is very often misinterpreted by many authors in their computations and illustrations as well. In order to clarify this problem, the Author has presented two qualitative illustrations of the shape of infinitesimal soil element deformed elastically due to positive and negative shear stress loadings (see Figs. 3 and 4), taking into account different orientations of the positive $z$-axis and directions of the water-wave propagation. Additionally, by extracting an infinitesimal soil element from the upper part of seabed loaded by a surface progressive sinusoidal water-wave, correct signs of the wave-induced shear stress component have been correlated with some characteristic water-wave phase-angles, as presented in Fig. 5.

Some authors, e.g. Jeng [2], use the solid mechanics sign convention, although they works deal with pure geotechnical problems. Such treatment could be an another reason of incorrect interpretations of the solution obtained for momentary values of the seabed cyclic response parameters, particularly when the wave-induced shear stress is of a great importance.

Taking the above into account, the reader must be very careful when comparing and adopting analogous mathematical formulations of a certain engineering problem based on different sign conventions. In the face of many mistakes, found in scientific journal papers, e.g. [1, 4, 32, 34], and text books, e.g. [2, 10, 33], probably as a wrong interpretation of the sing convention for strains and stresses, the present paper illustrates the absolute need to be more sensitive to the question of mixing different sign conventions for strains and stresses, especially in relation to the wave-induced cyclic seabed response. This must be also reflected in consequent notation (only one type of sign convention is used) at both stages, i.e.: the initial stage of mathematical formulation when constituting governing equations, and the final stage of post-processing when the stress-state in the seabed is calculated and interpreted.

## References

[1] J.R.C. Hsu and D.-S. Jeng, "Wave-induced soil response in an unsaturated anisotropic seabed of finite thickness", *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 18, no. 11, pp. 785–807, 1994, doi: 10.1002/nag.1610181104.

[2] D.-S. Jeng, *Porous models for wave-seabed interactions*. Berlin-Heidelberg: Springer-Verlag, 2012, doi: 10.1007/978-3-642-33593-8.

[3] D.-S. Jeng, *Mechanics of wave-seabed-structure interactions. Modelling, processes and applications*. Cambridge University Press, 2018, doi: 10.1017/9781316672266.

[4] D.-S. Jeng and J.R.C. Hsu, "Wave-induced soil response in a nearly saturated sea-bed of finite thickness", *Gèotechnique*, vol. 46, no. 3, pp. 427–440, 1996, doi: 10.1680/geot.1996.46.3.427.

[5] O.S. Madsen, "Wave-induced pore pressures and effective stresses in a porous bed", *Gèotechnique*, vol. 28, no. 4, pp. 377–393, 1978, doi: 10.1680/geot.1978.28.4.377.

[6] C.C. Mei and M.A. Foda, "Wave-induced responses in a fluid-filled poro-elastic solid with a free surface – a boundary layer theory", *Geophysical Journal of the Royal Astronomical Society*, vol. 66, no. 3, pp. 597–631, 1981, doi: 10.1111/j.1365-246X.1981.tb04892.x.

[7] W. Magda, "Analytical solution for the wave-induced excess pore-pressure in a finite-thickness seabed layer", in *Proceedings of the $24^{th}$ Conference on Coastal Engineering, ICCE 1994, 23–28 October 1994, Kobe, Japan*, B. Edge, Ed. ASCE, 1994, pp. 3111–3125, doi: 10.1061/9780784400890.225.

MOST WIEDZY Downloaded from mostwiedzy.pl

[8] S. Okusa, "Wave-induced stresses in unsaturated submarine sediments", *Gèotechnique*, vol. 35, no. 4, pp. 517–532, 1985, doi: 10.1680/geot.1985.35.4.517.

[9] T. Yamamoto, H.L. Koning, H. Sellmeijer, and E. Hijum, "On the response of a poro-elastic bed to water waves", *Journal of Fluid Mechanics*, vol. 87, no. 1, pp. 193–206, 1978, doi: 10.1017/S0022112078003006.

[10] B.M. Sumer, *Liquefaction around marine structures*. Hoboken, New Jersey: World Scientific, 2014, doi: 10.1142/7986.

[11] A. Sawicki, "Mechanics of seabed liquefaction and resolidification", *Archives of Mechanics*, vol. 66, no. 5, pp. 307–328, 2014.

[12] B.M. Das, *Advanced soil mechanics*. New York: Taylor & Francis, 2008.

[13] A. Verruijt, *An introduction to soil mechanics*. Springer, 2018, doi: 10.1007/978-3-319-61185-3.

[14] R.F. Craig, *Soil mechanics*. London: Chapman & Hall, 1992, doi: 10.1007/978-1-4899-3772-8.

[15] A. Verruijt, "Elastic storage of aquifers", in *Flow through porous media*, R. J. M. de Wiest, Ed. New York, London: Academic Press, pp. 331–376, 1969.

[16] A. Verruijt, *Theory and problems of poroelasticity*. Delft University of Technology, 2016.

[17] L. Shao, X. Guo, S. Liu, and G. Zheng, *Effective stress and equilibrium equations for soil mechanics*. London: Taylor & Francis, 2018, doi: 10.1201/9781315107554.

[18] B.S. Olek, "Some remarks on the pore water pressure dissipation patterns from the one-dimensional consolidation test", *Archives of Civil Engineering*, vol. 68, no. 4, pp. 147–162, 2022, doi: 10.24425/ace.2022.143031.

[19] A. Verruijt, *Soil mechanics*. Delft University of Technology, 2012.

[20] B. Shabani and D.-S. Jeng, "Three-dimensional analysis of momentary liquefaction near submarine pipelines", in *Proceedings of the 16$^{th}$ Australian Fluid Mechanics Conference, 2–7 December 2007, Gold Coast, Australia*. Brisbane, 2007, pp. 1083–1090.

[21] B. Shabani, "Wave associated seabed behaviour near submarine buried pipelines", M.Sc. thesis, The University of Sydney, Australia, 2008.

[22] T. Yamamoto, "Wave-induced pore pressures and effective stresses in inhomogeneous seabed foundations", *Ocean Engineering*, vol. 8, no. 1, pp. 1-16, 1981, doi: 10.1016/0029-8018(81)90002-0.

[23] Y. Zhang, D.-S. Jeng, F.P. Gao, and J.-S. Zhang, "An analytical solution for response of a porous seabed to combined wave and current loading", *Ocean Engineering*, vol. 57, pp. 240–247, 2013, doi: 10.1016/j.oceaneng.2012.09.001.

[24] R. Shiddharthan, "Stability of buried pipelines subjected to wave loading", M.Sc. thesis, The University of British Columbia, Vancouver, Canada, 1981, doi: 10.14288/1.0062677.

[25] J. Zhang, Q. Zhang, T. Han, and C-R. Qin, "Numerical simulation of seabed response and liquefaction due to nonlinear waves", *China Ocean Engineering*, vol. 19, pp. 497–507, 2005.

[26] S.S. Rao, *Vibration of continuous systems*. Hoboken, New Jersey: John Wiley & Sons, Inc., 2007, doi: 10.1002/9780470117866.

[27] J.H. Ye and D.-S. Jeng, "Responses of porous seabed to nature loadings: waves and currents", *Journal of Engineering Mechanics, Proceedings of the American Society of Civil Engineers (ASCE)*, vol. 138, no. 6, pp. 601–613, 2012, doi: 10.1061/(ASCE)EM.1943-7889.0000356.

[28] D.-S. Jeng and L. Cheng, "Wave-induced seabed response around a pipe laid on a poro-elastic seabed", *Journal of Offshore Mechanics and Arctic Engineering*, vol. 121, no. 4, pp. 227–236, 1999, doi: 10.1115/1.2829572.

[29] Z.S. Wong, C.C. Liao, and D.-S. Jeng, "3-D poro-elastoplastic model for short-crested wave-induced pore pressures in a porous seabed", *The Open Civil Engineering Journal*, vol. 9, pp. 408–416, 2015, doi: 10.2174/1874149501509010408.

[30] G. Wrzesiński, "Anisotropy of soil shear strength parameters caused by the principal stress rotation", *Archives of Civil Engineering*, vol. 67, no. 1, pp. 163–187, 2021, doi: 10.24425/ace.2021.136467.

[31] V.S.O. Kirca, B.M. Sumer, and J. Fredsøe, "Seabed liquefaction under standing waves", *Journal of the Waterways, Harbors and Coastal Engineering Division*, vol. 139, no. 6, pp. 489–501, 2013, doi: 10.1061/(ASCE)WW.1943-5460.0000208.

[32] A. Sawicki and J. Mierczyński, "Wave-induced stresses and liquefaction in seabed according to the Biot-type approach", *Archives of Hydro-Engineering and Environmental Mechanics*, vol. 52, no. 2, pp. 131–145, 2005.

MOST WIEDZY Downloaded from mostwiedzy.pl

[33] B.M. Sumer and J. Fredsøe, *The mechanics of scour in the marine environment*. Singapore: World Scientific Publishing, 2002, doi: 10.1142/4942.

[34] K. Ishikara and A. Yamazaki, "Analysis of wave-induced liquefaction in seabed deposits of sand", *Soils and Foundations*, vol. 24, no. 3, pp. 85–100, 1984, doi: 10.3208/sandf1972.24.3_85.

[35] K. Ishikara and I. Towhata, "Sand response to cyclic rotation of principal stress directions as induced by wave loads", *Soils and Foundations*, vol. 23, no. 4, pp. 11–26, 1983, doi: 10.3208/sandf1972.23.4_11.

## Znaczenie konwencji znaków dla rozwiązań analitycznych cyklicznej odpowiedzi porowatego i sprężystego dna morskiego poddanego oddziaływaniu falowania morskiego

**Słowa kluczowe:** oddziaływanie falowania na dno morskie, cykliczna odpowiedź dna morskiego, ośrodek porowato-sprężysty, progresywna fala sinusoidalna, konwencje znaków dla odkształceń i naprężeń, mechanika gruntów

### Streszczenie:

W artykule poddano dyskusji wpływ przyjętej konwencji znaków dla odkształceń i naprężeń, tj. klasycznej konwencji mechaniki ośrodka ciągłego oraz konwencji mechaniki gruntów (ośrodka porowatewgo), na postać równań równowagi oraz równania ciągłości stosowanych do opisu cyklicznej odpowiedzi porowatego i sprężystego dna morskiego na działanie powierzchniowej progresywnej sinusoidalnej fali wodnej. Istniejące rozwiązania analityczne rozważanego problemu, opublikowane w literaturze fachowej, a otrzymane w postaci funkcji zespolonych opisujących cykliczne oscylacje ciśnienia wody w porach gruntu, efektywnych naprężeń normalnych oraz naprężenia stycznego w szkielecie gruntowym, zostały przeanalizowane i porównane ze sobą głównie z punktu widzenia zastosowanych konwencji znaków dla odkształceń i naprężeń, a także biorąc pod uwagę przyjmowane różnie kierunki dodatniej pionowej osi płaskiego układu odniesienia oraz propagacji fali powierzchniowej. Analiza opublikowanych rozwiązań analitycznych wykazała szereg nieścisłości, a nawet ewidentnych błędów polegających na nieprawidłowym znakowaniu parametrów cyklicznej reakcji dna morskiego (w szczególności naprężenia stycznego), dyskwalifikujących te rozwiązania z ich dalszego stosowania w praktyce inżynierskiej. Większość zauważonych usterek należy powiązać z brakiem zrozumienia oraz konsekwencji autorów poszczególnych publikacji w jednolitym stosowaniu wybranej konwencji znaków dla odkształceń i naprężeń w elemencie gruntowym zarówno na etapie matematycznego formułowania zagadnienia, jak i na etapie właściwej interpretacji otrzymanych wzorów końcowych danego rozwiązania analitycznego. Zadaniem prezentowanego artykułu, opracowanego głównie na podstawie skrupulatnego przeglądu literatury, jest przyciągnięcie uwagi i wzbudzenie zainteresowania wśród naukowców i inżynierów z szeroko pojętej inżynierii morskiej i brzegowej w celu poprawnego „odczytywania" i stosowania istniejących analitycznych rozwiązań dla cyklicznej odpowiedzi dna morskiego na falowanie, które to rozwiązania często różnią się m.in. przyjętą konwencją znaków dla naprężeń w szkielecie gruntowym.

Received: 2023-03-21, Revised: 2023-08-17

MOST WIEDZY Downloaded from mostwiedzy.pl