General Description of State-Space Continuous-Time G_m-C Filters

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A general approach to analysis continuous-time analog G_m-C filters and equalizers, based on matrix description, is presented. A general transfer function formula for any G_m-C filter structure are given. The considerations accompanying this approach lead to useful relationships between the passive network of the filter and its transfer function. Based on these considerations, a new definition of continuous-time state-space G_m-C filters is given. Moreover, connections between state matrices for voltage- and current-mode state-space G_m-C filters are formulated and two canonical transformations of state-space filters into direct state-space ones, i.e. those having grounded capacitors only, are defined. Another two classes of G_m-C filters, i.e. reducible and quasi-state-space structures are also considered.

Keywords: state-space filters, G_m-C filters, matrix description, state-space representation.

1. INTRODUCTION

In recent years there has been a growing interest in continuous-time (CT) filters using transconductance amplifiers and capacitors (G_m-C). A number of attractive designs of such filters that are fully compatible with current CMOS technology have been reported in the literature [1]–[21]. The low-complexity excellent high-frequency performance and easy tunability of G_m-C filters make them suitable solutions for many applications such as hard disc drives, video filters, wireless communications, computer systems and instrumentation systems [11], [27]–[31].

In this paper, a novel and general approach to continuous-time G_m-C filters and equalizers is developed. Special attention is focused on an interesting sub-class of G_m-C filters, so-called state-space filters. This class of filters is usually identified with filters

that have only grounded capacitors [6], [25]. Such topologies are very attractive for integrated circuit (IC) realizations. In this work a novel definition of state-space filters is proposed, which comprises much wider class of G_m-C structures. According to this definition, the filter structure is called a state-space one if and only if there exist the state matrices for this structure. Of course, filter structures that have grounded capacitors satisfy this condition, however, there are many filters containing floating capacitors that are also state-space ones according to the new definition. Due to this, it is possible to apply a very useful technique of state-space description to much broader class of structures. It is important, since contemporary CMOS technologies make it possible to implement floating capacitors without any problems.

The paper is organized as follows. In Section 2, a general structure of G_m-C filter is presented. We derive a matrix description of general G_m-C filter based on a separate treatment of passive and active network of the circuit. It follows that the transfer function of any G_m-C filter can be easily calculated by means of the presented formulas. Section 3 is devoted to studying relationships between the transfer function of G_m-C filter and its passive network. In this section we also describe practical criteria of estimating the order of the filter's transmittance that can be used in computer aided filter synthesis. In Section 4 we deal with state-space G_m-C filters. A new definition of state-space G_m-C filters, based on invertibility properties of matrix corresponding to the passive network of the filter, is introduced. Relationships between state matrices of voltage- and current-mode G_m-C filters are derived. Moreover, we discuss two canonical transformations which enable us to convert any state-space filter into structures containing only grounded capacitors. In Section 5 we introduce two interesting classes of G_m-C structures, i.e. reducible and quasi-state-space ones. The paper concludes with Section 6.

2. GENERAL STRUCTURE OF G_m-C FILTER

Consider a general structure of a voltage-mode G_m-C filter shown in Fig. 1. The current-mode counterpart can be obtained by inverting all transconductors and interchanging input and output of the filter [23], [32]. The structure in Fig. 1 contains n internal nodes denoted as x_i, $i = 1,...,n$, n input transconductors G_{mbi}, an output summer consisting of transconductors $c_i G_m$ and $-G_m$ as well as a feedforward transconductor dG_m. All transconductors form *active network*, while input capacitors C_{bi}, $i = 1,...,n$ and capacitors C_{ij}, $1 \le i \le j \le n$ form *passive network*. It is easily seen that any G_m-C filter is a particular case of the general structure in Fig. 1. Note also that n is not necessarily equal to the order of the filter transmittance. In the following, we will investigate the connection between the number n of internal nodes, the order of the filter and its particular structure.

Now we would like to derive an analytical description of the considered structure. In particular, we will calculate the transmittance $H(s)$ of the filter and show that the voltage-to-voltage transmittance $H_v(s)$ of the voltage-mode filter is the same as the current-to-current transmittance $H_c(s)$ of the current-mode one. In the following con-
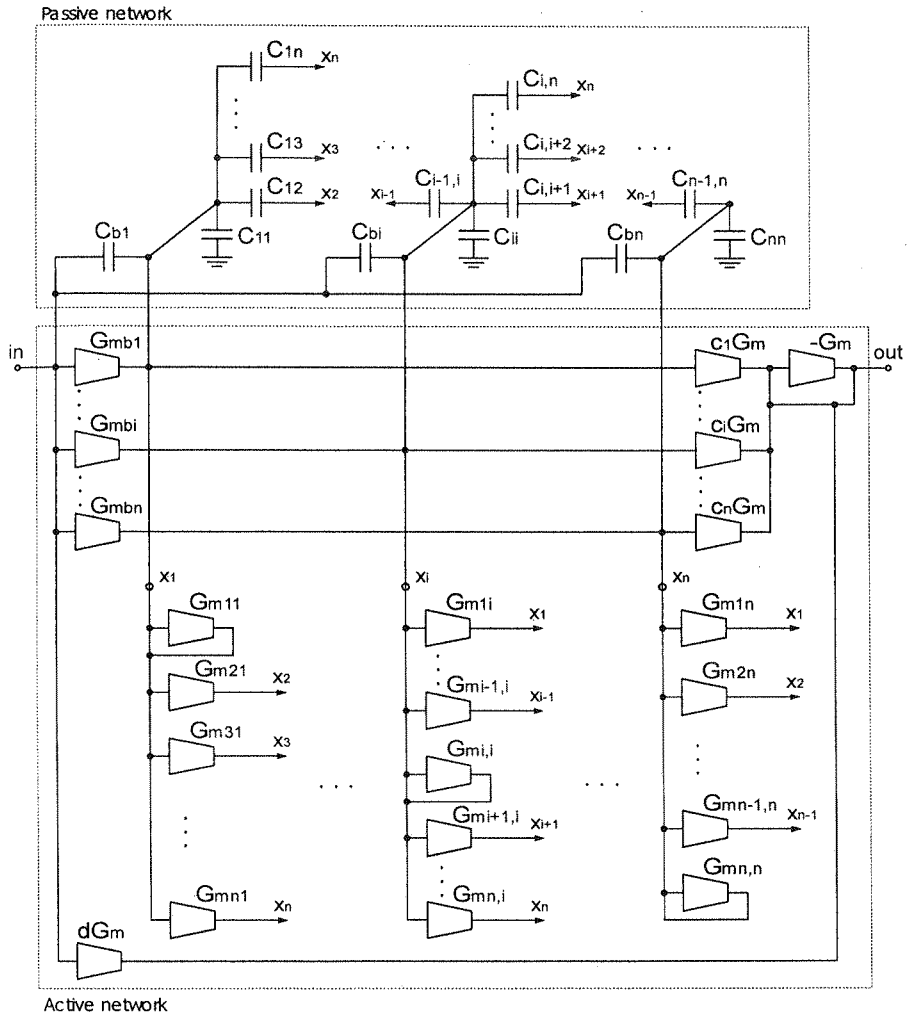
Passive network

Active network

Fig. 1. General structure of voltage-mode G_m-C filter

siderations we will denote the voltage at the i-th node x_i also by x_i, which will not lead to confusion. A general structure of the voltage-mode G_m-C filter in Fig. 1 can be described by the following matrix equations:

$$
s\begin{bmatrix}
\sum_{j=1}^{n}C_{1j} & -C_{12} & \cdots & -C_{1n} \\
-C_{12} & \sum_{j=1}^{n}C_{2j} & \cdots & -C_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-C_{1n} & -C_{2n} & \cdots & \sum_{j=1}^{n}C_{nj}
\end{bmatrix}
\cdot
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
u_i G_{mb1} + (u_i - x_1)sC_{b1} + \sum_{j=1}^{n}G_{m1j}x_j \\
u_i G_{mb2} + (u_i - x_2)sC_{b2} + \sum_{j=1}^{n}G_{m2j}x_j \\
\vdots \\
u_i G_{mbn} + (u_i - x_n)sC_{bn} + \sum_{j=1}^{n}G_{mnj}x_j
\end{bmatrix}
\tag{1}
$$

and

$$u_0 = [c_1 \ldots c_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + d \cdot u_i \tag{2}$$

where u_i, u_o are the input and output voltages, respectively. To simplify the notation, in (1) we used symbols C_{kl} ($k > l$) that denote the same elements as C_{lk}. Note that the vector on the right-hand side of (1) can be written in the form:

$$\begin{bmatrix} G_{m11} & G_{m12} & \cdots & G_{m1n} \\ G_{m21} & G_{m22} & \cdots & G_{m2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{mn1} & G_{mn2} & \cdots & G_{mnn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - s \begin{bmatrix} C_{b1} & 0 & \cdots & 0 \\ 0 & C_{b2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{bn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} G_{mb1} + sC_{b1} \\ G_{mb2} + sC_{b2} \\ \vdots \\ G_{mbn} + sC_{bn} \end{bmatrix} u_i \tag{3}$$

Introducing the following notation:

$$T_C = \begin{bmatrix} C_{b1} + \sum_{j=1}^{n} C_{1j} & -C_{12} & \cdots & -C_{1n} \\ -C_{12} & C_{b2} + \sum_{j=1}^{n} C_{2j} & \cdots & -C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -C_{1n} & -C_{2n} & \cdots & C_{bn} + \sum_{j=1}^{n} C_{nj} \end{bmatrix}, \tag{4a}$$

$$G = \begin{bmatrix} G_{m11} & G_{m12} & \cdots & G_{m1n} \\ G_{m21} & G_{m22} & \cdots & G_{m2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{mn1} & G_{mn2} & \cdots & G_{mnn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \tilde{C}_v = [c_1 \ldots c_n], \tag{4b}$$

$$\tilde{C}_C = [G_{mb1} + sC_{b1} \cdots G_{mbn} + sC_{bn}], \quad D = d \tag{4c}$$

(note that T_C is symmetrical matrix, i.e. $T_C^T = T_C$), one can rewrite (1) and (2) in the form of:

$$sT_C X = GX + \tilde{C}_c^T u_i$$
$$u_o = \tilde{C}_v X + D u_i \tag{5}$$

In a similar way one can write appropriate equations for the current-mode filter. Using notation (4) we get:

$$sT_C X = G^T X + \tilde{C}_v^T i_i$$
$$i_o = \tilde{C}_c X + D i_i \tag{6}$$

where the entries x_i of the vector X denote, as before, the intrinsic node voltages, and i_i, i_o are the input and output current, respectively.

On the basis of (5) and (6) we can calculate the transmittances H_v and H_c which are:

$$H_v(s) = \frac{u_o(s)}{u_i(s)} = \tilde{C}_v(sT_C - G)^{-1}\tilde{C}_c^T + D \tag{7a}$$

$$H_c(s) = \frac{i_o(s)}{i_i(s)} = \tilde{C}_c(sT_C - G^T)^{-1}\tilde{C}_v^T + D \tag{7b}$$

It is worth noticing that using (7a) and (7b), one can easily check the reciprocal behavior of voltage- and current-mode filters:

$$H_v(s) = \frac{u_o(s)}{u_i(s)} = C_v(sT_C - G)^{-1}C_c^T + D = (C_v(sT - G)^{-1}C_c^T + D)^T =$$

$$= C_c(sT_C - G^T)^{-1}C_v^T + D = \frac{i_o(s)}{i_i(s)} = H_c(s) \tag{8}$$

where we made use of the facts that $H^T = H$ (since H is a scalar), $(A^{-1})^T = (A^T)^{-1}$ for any invertible matrix A [22], and T_C is symmetrical. Since $H_v(s) = H_c(s)$, we will use general symbol $H(s)$ to denote a filter transmittance. Now, let us denote adjoint matrix of $sT_C - G$ as $\tilde{A}$ where

$$\tilde{A}(s) = adj(sT_C - G) = adj(sT_C - G^T)^T = \begin{bmatrix} \tilde{A}_{11}(s) & \tilde{A}_{12}(s) & \cdots & \tilde{A}_{1n}(s) \\ \tilde{A}_{21}(s) & \tilde{A}_{22}(s) & \cdots & \tilde{A}_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}_{n1}(s) & \tilde{A}_{n2}(s) & \cdots & \tilde{A}_{nn}(s) \end{bmatrix} \tag{9}$$

This allows us to rewrite H in the form:

$$H(s) = \frac{1}{det(sT_C - G)} \sum_{i,j=1}^{n} c_i(G_{mbj} + sC_{bj})\tilde{A}_{ij}(s) + d \tag{10}$$

Note that many filter structures have only one input transconductor (i.e. no input signal distribution), a trivial output summer (i.e. one of the internal nodes is the output of the filter), and no input capacitors. This means that $\tilde{C}_C = [0 \cdots 0 \; G_{mbk} \; 0 \cdots 0]$, $\tilde{C}_v = [0 \cdots 0 \; 1 \; 0 \cdots 0] - 1$ at l-th position and $C_{bi} = 0$ for $i = 0,1,...n$. In such case, expression (13) reduces to the form:

(2)

notation, in
the vector

$$\Big] u_i \quad (3)$$

(4a)

(4b)

(4c)

(2) in the

(5)

lter. Using

$$H(s) = \frac{G_{mbk}\tilde{A}_{lk}(s)}{det(sT_C - G)} \qquad (11)$$

Similar expression can be written for two slightly more general cases. The first one is the case with no input capacitors, input signal distribution (i.e. $\tilde{C}_c = [G_{mb1} \cdots G_{mbn}]$), and trivial output summer ($\tilde{C}_v = [0 \cdots 0 \ 1 \ 0 \cdots 0] - 1$ at l-th position). Then, the filter transmittance takes the form:

$$H(s) = \frac{1}{det(sT_C - G)} \sum_{i=1}^{n} G_{mbi}\tilde{A}_{li}(s) \qquad (12)$$

Another case is when there are no input capacitors, no input signal distribution (i.e. $\tilde{C}_c = [0 \cdots 0 \ G_{mbk} \ 0 \cdots 0]$), however, there is non-trivial output summer (e.g. in follow--the-leader-like structures). Then, we have:

$$H(s) = \frac{G_{mbk}}{det(sT_C - G)} \sum_{i=1}^{n} c_i \tilde{A}_{ik}(s) \qquad (13)$$

On the basis of the above expressions one can easily calculate the transmittance of any particular structure of G_m-C filter. In the next section we deal with the sensitivity of our general G_m-C structure.

We end this section with an example of a particular filter structure considered in the general setting presented above. Fig. 2 shows a structure of third order elliptic G_m-C filter based on a LC ladder simulation.
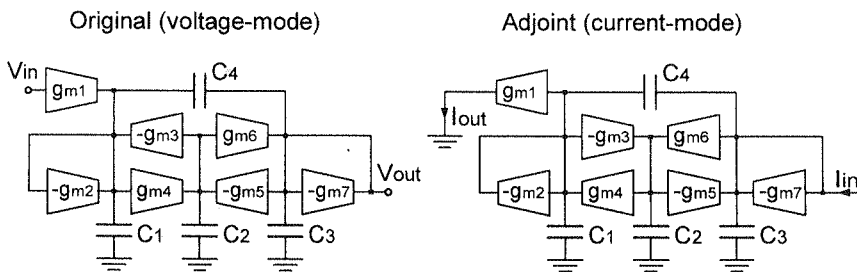

Fig. 2. Third order elliptic G_m-C filter in voltage- and current-mode

Matrices T_C, G, $\tilde{C}_c$, $\tilde{C}_v$ and D for this circuit are the following (internal nodes are indexed from left to right):

$$T_C = \begin{bmatrix} C_1 + C_4 & 0 & -C_4 \\ 0 & C_2 & 0 \\ -C_4 & 0 & C_3 + C_4 \end{bmatrix}, \quad G = \begin{bmatrix} -g_{m2} & -g_{m3} & 0 \\ g_{m4} & 0 & -g_{m5} \\ 0 & g_{m6} & -g_{m7} \end{bmatrix}, \quad \begin{array}{l} \tilde{C}_c = [g_{m1} \ 0 \ 0] \\ \tilde{C}_v = [0 \ 0 \ 1] \\ D = 0 \end{array} \qquad (14)$$

Transmittance of the filter can be calculated using (11) with $k = 1$ and $l = 3$. Assuming for simplicity that all transconductances are equal, i.e. $g_{mi} = g_m$ for $i = 1,...,7$, we obtain:

$$\det(sT_C - G) =$$
$$= C_2(C_1C_3 + C_1C_4 + C_3C_4)s^3 + C_2(C_1 + C_3 + 2C_4)g_ms^2 + (C_1 + C_2 + C_3)g_m^2s + 2g_m^3 \quad (15)$$

$$\tilde{A}_{31}(s) = C_2C_4s^2 + g_m^2 \quad (16)$$

Let $C_x^2 = C_1C_3 + C_1C_4 + C_3C_4$. Then, filter transmittance $H(s)$ has the form:

$$H(s) = \left(\frac{C_4g_m}{C_x^2}\right) \cdot \frac{s^2 + \dfrac{g_m^2}{C_2C_4}}{s^3 + \dfrac{(C_1 + C_3 + 2C_4)g_m}{C_x^2}s^2 + \dfrac{(C_1 + C_2 + C_3)g_m^2}{C_x^2}s + \dfrac{2g_m^3}{C_2C_x^2}} \quad (17)$$

The above example shows that the presented general approach provides as with a very elegant and easy way to calculate transfer function of any G_m-C filter structure.

3. PASSIVE NETWORK OF GENERAL G_m-C FILTER

In this section we investigate how the passive network of general of G_m-C filter influences its properties, especially the order of filter transfer function $H(s)$. First, we would like to introduce necessary notation and give some general remarks.

Recall that matrix T_C representing the passive network of the filter is symmetrical. By construction of T_C, for any $i = 1,...,n$ we have (see (4a)):

$$|(T_C)_{ij}| \geq \sum_{k=1, k \neq i}^{n} |(T_C)_{ik}|, \quad \text{and} \quad |(T_C)_{ii}| \geq \sum_{k=1, k \neq i}^{n} |(T_C)_{ki}| \quad (18)$$

i.e. matrix T_C is diagonally dominant. From linear algebra (Gershgorin's theorem), T_C is then positively semidefinite (that is $x^T T_C x \geq 0$ for any nonzero $n \times 1$ vector x). Moreover, T_C is invertible ($\det(T_C) \neq 0$) if and only if T_C is positively definite (i.e. $x^T T_C x > 0$ for any nonzero $n \times 1$ vector x). Since T_C is symmetrical, there exists an orthogonal matrix T (i.e. $TT^T = T^TT = I$, which implies $T^{-1} = T^T$) such that $T^{-1}T_CT$ is diagonal (with non-negative elements, since T_C is positively semidefinite).

It is useful to introduce graphs representing passive network of the G_m-C filter. Let $G_t(V_t, E_t)$ be a graph in which V_t is the set of nodes $V_t = \{v_0, v_1,..., v_n\}$ and E_t is a set of edges. Node v_0 represents both the common node x_0 and input node x_{in} of the filter, while $v_1,..., v_n$ represent internal nodes $x_1,..., x_n$ of the filter. Edge $e_{ij} \in E_t$ if and only if x_i and x_j are connected by the floating capacitor $C_{ij}(j > i > 0)$ or x_0 and x_i are connected by

the grounded capacitor $C_{ii}(i > 0)$. Edge $e_{in,j} \in E_t$ if and only if x_{in} and x_j are connected by input capacitor $C_{bj}, j = 1,...,n$. We define also graph $G_f(V_f, E_f)$ as the subgraph of G_t, in which $V_f = \{v_1,...,v_n\}$ represents internal nodes of the filter and $E_f \subset E_t$ such that $e_{ij} \in E_f$ if and only $j > i > 0$. Fig. 3 gives an example of the passive network and corresponding graphs G_t and G_f. It is seen from Fig. 3 that our example graph G_t has three disjoint components, while G_f has four disjoint components.
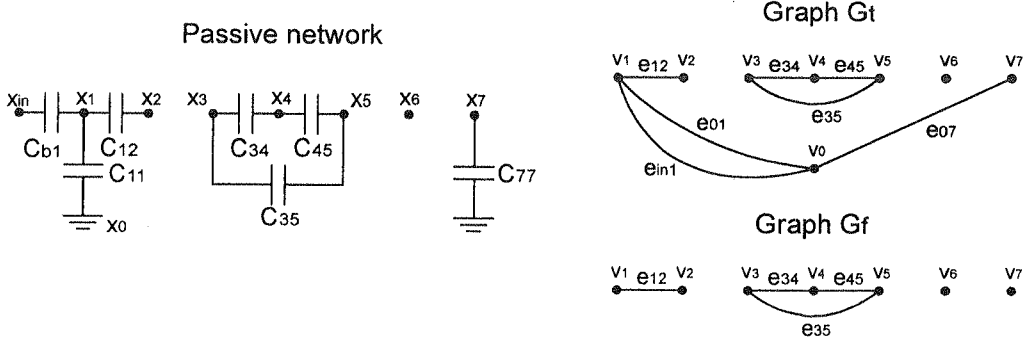

Fig. 3. Example of filter's passive network and related graphs G_t and G_f

Having defined graphs G_t and G_f, we can consider the structure of the matrix T_C. It is seen from (4a) and the definition of graphs that T_C consists of square submatrices (*blocks*) situated along its diagonal (up to the renumbering of node indexes). The number of blocks is equal to the number of disjoint components of graph G_f reduced by the number of disjoint components of graph G_t having cardinality one and different from v_0. The latter represent just those internal nodes of the filter which are not connected to any other node by capacitance elements (we will call them *floating nodes*). The structure of matrix T_C for the passive network shown in Fig. 3 is the following:

$$T_C \approx \begin{bmatrix} X & X & 0 & 0 & 0 & 0 & 0 \\ X & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & X & X & 0 & 0 \\ 0 & 0 & X & X & X & 0 & 0 \\ 0 & 0 & X & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix} \qquad (19)$$
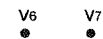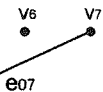
It is evident that T_C cannot be invertible if filter has floating nodes. It is important for further applications to formulate necessary and sufficient conditions for invertibility of matrix T_C. In the subsequent considerations we will identify the block of matrix T_C with the corresponding sets of capacitors. We have the following proposition:

Proposition 1. Let $\tilde{T}_C$ be the block of matrix T_C. Then, $\tilde{T}_C$ is invertible if and only if the corresponding set of capacitors contains at least one grounded capacitor or input capacitor.

Proof. Matrix $\tilde{T}_C$ is symmetrical and diagonally dominant since it inherits these properties from T_C. Thus, $\tilde{T}_C$ is invertible if and only if it is positively definite. Assume that $\tilde{T}_C$ is $k \times k$ matrix. The general form of $\tilde{T}_C$ is then analogous to those of T_C given by (4a) (with possible renumbering of node indexes if necessary). Let $\boldsymbol{y} = (y_1, ..., y_k)^T$ be any nonzero vector. Then, we have (recall that $C_{ji} = C_{ij}$):

$$
\boldsymbol{y}^T \tilde{T}_C \boldsymbol{y} = \boldsymbol{y}^T
\begin{bmatrix}
C_{b1} + C_{11} + \sum_{j \neq 1}^{k} C_{1j} & -C_{12} & \cdots & -C_{1k} \\
-C_{21} & C_{b2} + C_{22} + \sum_{j \neq 2}^{k} C_{2j} & \cdots & -C_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
-C_{k1} & -C_{k2} & \cdots & C_{bk} + C_{kk} + \sum_{j \neq k}^{n} C_{kj}
\end{bmatrix}
\cdot
\begin{bmatrix}
y_1 \\ y_2 \\ \vdots \\ y_k
\end{bmatrix}
=
$$

$$
= \boldsymbol{y}^T
\begin{bmatrix}
y_1(C_{b1} + C_{11}) + \sum_{j \neq 1}^{k} C_{1j}(y_1 - y_j) \\
y_2(C_{b2} + C_{22}) + \sum_{j \neq 2}^{k} C_{2j}(y_2 - y_j) \\
\vdots \\
y_k(C_{bk} + C_{kk}) + \sum_{j \neq k}^{k} C_{kj}(y_k - y_j)
\end{bmatrix}
= \sum_{i=1}^{k} y_i^2 (C_{bi} + C_{ii}) + \sum_{i=1}^{k} \sum_{j \neq 1}^{k} C_{ij} y_i (y_i - y_j) =
$$

$$
= \sum_{i=1}^{k} y_i^2 (C_{bi} + C_{ii}) + \sum_{i=1}^{k} \left(\sum_{j > i}^{k} C_{ij} y_i (y_i - y_j) + \sum_{i > j}^{k} C_{ij} y_i (y_i - y_j) \right) =
$$

$$
= \sum_{i=1}^{k} y_i^2 (C_{bi} + C_{ii}) + \sum_{i=1}^{k} \left(\sum_{j > i}^{k} C_{ij} y_i (y_i - y_j) + \sum_{j > i}^{k} C_{ij} y_j (y_j - y_i) \right) =
$$

$$
= \sum_{i=1}^{k} y_i^2 (C_{bi} + C_{ii}) + \sum_{i=1}^{k} \sum_{j > i}^{k} C_{ij} (y_i - y_j)^2
$$

It is evident from (20) that $\boldsymbol{y}^T \tilde{T}_C \boldsymbol{y} = 0$ if and only if $y_i = y_j$ for any $i, j = 1, ..., k$ and $C_{bi} + C_{ii} = 0$ for $i = 1, ..., k$. This proves our proposition since if $\tilde{T}_C$ has at least one grounded or input capacitor then it is positively definite (so it is invertible). Otherwise, there exists $\boldsymbol{y}$ for which $\boldsymbol{y}^T \tilde{T}_C \boldsymbol{y} = 0$ so $\tilde{T}_C$ is not invertible.

Corollary 1. Let $\tilde{T}_C$ be the block of matrix T_C. Then, $\tilde{T}_C$ is invertible if and only if the sum of its elements if positive.

Proof. It is apparent from (20) that the sum of elements of $\tilde{T}_C$ equals $\sum_{i=1}^{k} (C_{bi} + C_{ii})$ and is positive if and only if $\tilde{T}_C$ contains at least one grounded or input capacitor. Our assertion follows now from Proposition 1.

Corollary 2. Let $\tilde{T}_C$ be the block of matrix T_C. Suppose that the size of $\tilde{T}_C$ is $k \times k$. Then, $rank(\tilde{T}_C) = k$ if and only if at least one of the capacitors corresponding to $\tilde{T}_C$ is grounded or input one. Otherwise, $rank(\tilde{T}_C) = k - 1$.

Proof. If at least one of the capacitors corresponding to $\tilde{T}_C$ is grounded or input one, then it is invertible by Proposition 1, hence $rank(\tilde{T}_C) = k$. Suppose now that $\tilde{T}_C$ contains only floating capacitors $C_{ij}(j > i > 0)$. Then the sum of elements of $\tilde{T}_C$ is zero by definition (see (20)). Consider any submatrix $(\tilde{T}_C)_i$ of $\tilde{T}_C$ defined by removing i-th row and i-th column of $\tilde{T}_C$, $i = 1, ..., k$. We assert that $(\tilde{T}_C)_i$ is invertible. Obviously, $(\tilde{T}_C)_i$ is symmetrical and diagonally dominant. Thus, it is enough to show that $(\tilde{T}_C)_i$ is positively definite. Without loss of generality we can assume that $i = k$. Proceeding similarly as in the proof of Proposition 1 we obtain for any nonzero vector $y = (y_1, ..., y_{k-1})^T$:

$$y^T(\tilde{T}_C)_k y = \sum_{i=1}^{k-1} y_i^2 C_{ik} + \sum_{i=1}^{k-1}\sum_{j>1}^{k-1} C_{ij}(y_i - y_j)^2 \qquad (21)$$

This sum is positive since $C_{ik} > 0$ for at least one $i = 1, ..., k-1$ by connectivity of the component of graph G_f corresponding to $\tilde{T}_C$. We have proved that the $k-1 \times k-1$ submatrix $(\tilde{T}_C)_k$ of $\tilde{T}_C$ is invertible so $rank(\tilde{T}_C) = k - 1$.

Corollary 3. Let T_C be $n \times n$ matrix. Then, $rank(T_C) = n + 1 - k$, where k is the number of disjoint components of the corresponding graph G_t.

Proof. It is apparent that $rank(T_C) = \sum_{i=1}^{n} rank((\tilde{T}_C)_i)$, where $(\tilde{T}_C)_i$ is i-th block of T_C and N is the total number of blocks. Denote the number of floating nodes of the filter as n_f and number of blocks connected to the common node or to the input node (by at least one capacitor) as N_c. It is seen from the definition of graphs G_t and G_f that the number of disjoint components of graph G_t equals $k = N - N_c + 1 + n_f$. Indeed, components of G_t are the following: blocks not connected to the common node $(N - N_c)$, one component consisting of all blocks connected to the common node and n_f floating nodes. Now, $rank(T_C) = n - n_f - (N - N_c)$ since each floating node 'produces' zero row and zero column in matrix T_C and, by Corollary 2, each block of T_C not connected to the common node reduces $rank(T_C)$ by one. Thus $rank(T_C) = n - n_f - (N - N_c) = n + 1 - (N - N_c + 1 + n_f) = n + 1 - k$ which concludes the proof.

It is easy to observe that matrix T_C is invertible if and only if the filter has no floating nodes and all blocks of T_C are invertible. Now, we formulate it in terms of graph G_t.

Proposition 2. Matrix T_C is invertible if and only if the corresponding graph G_t is connected.

Proof. Suppose that T_C is invertible. From the preceding paragraph there are no floating nodes and each block of T_C contains a grounded capacitor or input capacitor. This means that each component of graph G_f is connected as a subgraph of G_t to node v_0 by at least one edge. Thus, G_t is connected. Suppose now that G_t is connected. This implies in particular that the filter has no floating nodes. Moreover, if there exists a block without a grounded (or input) capacitor then G_t cannot be connected, which leads to contradiction. Thus, all blocks are invertible and so is T_C.

Let us now turn to relationships between the passive network of the G_m-C filter and the order of the filter transmittance which we will denote as n_H. Recall that $H(s)$ is given by (11). Let T be any invertible $n \times n$ matrix (so $TT^{-1} = T^{-1}T = I$ — identity matrix). Then, we have:

$$H(s) = \tilde{C}_v(sT_C - G)^{-1}\tilde{C}_c^T + D = \tilde{C}_v TT^{-1}(sT - G)^{-1}TT^{-1}\tilde{C}_c^T + D =$$
$$= \tilde{C}_v T(sT^{-1}T_C T - T^{-1}GT)^{-1} T^{-1}\tilde{C}_c^T + D \qquad (22)$$

This means that $H(s)$ is invariant under the transformation:

$$\tilde{C}_v \to \tilde{C}_v T, \quad \tilde{C}_c^T \to T^{-1}\tilde{C}_c^T, \quad T_C \to T^{-1}T_C T, \quad G \to T^{-1}GT \qquad (23)$$

Using this fact we can prove the proposition which is the main goal of the section:

Proposition 3. Order n_H of the transmittance of a general G_m-C filter is not larger than the rank of matrix T_C.

Proof. Transmittance of the filter is given by $H(s) = \tilde{C}_v(sT_C - G)^{-1}\tilde{C}_c^T + D$. It follows from the preceding paragraph that $H(s)$ is invariant under the transformation given by (23) for any invertible matrix T. However, matrix T_C is symmetrical, hence there exists such T that $T^{-1}T_C T$ is diagonal. Since $rank(T_C)$ is invariant under similarity transformation, the number of nonzero (diagonal) elements of $T^{-1}T_C T$ is exactly equal to $rank(T_C)$. It follows that $det(sT_C - G)$ is a polynomial in s of order $rank(T_C)$. This means that $n_H \leq rank(T_C)$ (a strict inequality may occur when $H(s)$ has common poles and zeros).

Using Corollary 3 we can reformulate assertion of Proposition 3 in terms of graph G_t corresponding to matrix T_C. Such a formulation may be more convenient for practical applications.

Proposition 4. Order n_H of the transmittance of a general G_m-C filter is not larger than $n+1-k$, where n is the number of internal nodes of the filter and k is the number of disjoint components of graph G_t corresponding to matrix T_C.

Proof. The proof is straightforward and emerges from Proposition 3 and Corollary 3.

Note that the estimate similar to that given in Proposition 4 is known in literature (e.g. [33]), however here it is formulated in terms of matrix T_C, hence more convenient for practical applications.

The results formulated in this section have both theoretical and practical significance. For example, Proposition 2 can be used to define some important class of G_m-C filters, which is done in the next section. Proposition 4 can be useful from the point of view of computer aided synthesis of G_m-C filters. This is because it allows us to estimate a priori the possible order of transmittance of a filter generated by any design algorithm without calculating this transmittance (which is usually a time consuming task) so it can be a convenient criterion while choosing of a particular filter structure within some design strategies.

We end this section with the following remark. It has been mentioned in the discussion preceding Proposition 3 that the transformation given by (23) does not influence transmittance of the filter for any invertible matrix T. It is worth noticing that

such a transformation may determine a new realization of $H(s)$. Suppuse, in particular, that T is similarity matrix for which $T^{-1}T_CT$ is diagonal, and that there are no input capacitors in the filter (i.e. $C_{bi} = 0$ for $i = 1,...,n$). Then, transformed matrices $\tilde{C}_vT$, $T^{-1}\tilde{C}_c^T$, $T^{-1}T_CT$, $T^{-1}GT$ determine a new structure that realizes the same transmittance (of course, the considered transformation is nontrivial if T_C is not diagonal). This new structure has only grounded capacitors with values equal to the values of nonzero diagonal elements of $T^{-1}T_CT$.

4. STATE-SPACE G_m-C FILTERS

In this section we consider a certain class of filters which we call state-space G_m-C filters. Recall that a general description of linear filtering circuits can be obtained using the state-space representation. The appropriate state equations can be written as follows [7], [25]:

$$sX(s) = AS(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s) \tag{24}$$

where X, U, Y are, respectively: state vector consisting of output signals of n internal integrators (thus, $X \in M_{n \times 1}$, where $M_{n \times m}$ denotes the set of $n \times m$ matrices), input signal and output signal; $A \in M_{n \times n}$, $B \in M_{n \times 1}$, $C \in M_{1 \times n}$, $D \in M_{1 \times 1}$.

Using (50), the overall transfer function of the filter can be expressed in the form:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \tag{25}$$

where $I \in M_{n \times m}$ stands for the identity matrix. It is useful to introduce internal transfer functions f_i and g_i, i.e. the transfer function from the input of the filter to the output of integrator i (i.e. to x_i) and the transfer function from the input of integrator i to the output of the filter, respectively [7]. Using the following notation:

$$F = (f_1\ f_2 \cdots f_n)^T, \quad G = (g_1\ g_2 \cdots g_n) \tag{26}$$

one can define the so-called observability and controllability Gramian matrices W and K [7]:

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} G * G d\omega, \quad K = \frac{1}{2\pi}\int_{-\infty}^{\infty} FF * d\omega \tag{27}$$

One can also prove that the matrices W and K satisfy the following equations:

$$A^TW + WA = -C^TC, \tag{28a}$$

$$AK + KA^T = -BB^T \tag{28b}$$

Now, we turn to the general description of a G_m-C filter presented in Section 2. Assume that matrix T_C, corresponding to the passive network of the filter, is invertible, and that there are no input capacitors in the filter (i.e. $C_{bi} = 0$ for $i = 1,...,n$ and $\tilde{C}_c = [G_{mb1} \cdots G_{mbn}]$). Then, we can rewrite (5) and (6) in the following form:

$$sX = T_C^{-1}GX + T_C^{-1}\tilde{C}_c^T u_i$$
$$u_o = \tilde{C}_v X + D u_i \tag{29}$$

$$sX = T_C^{-1}G^T X + T_C^{-1}\tilde{C}_v^T i_i$$
$$i_o = \tilde{C}_c X + D i_i \tag{30}$$

We compare (29) and (30) to (24). It is apparent that all those equations are the same provided that $A = T_C^{-1}G$, $B = T_C^{-1}\tilde{C}_c^T$, $C = \tilde{C}_v$ in (29), $A = T_C^{-1}G^T$, $B = T_C^{-1}\tilde{C}_v^T$, $C = \tilde{C}_c$ in (30), and that intrinsic node voltages are chosen as state variables. Note that absence of input capacitors is crucial, since otherwise we would have a complex variable s in the right-hand sides of (29) and (30); in such a case (29) and (30) could not be compared to (24). Due to this fact we can introduce the following definition:

Definition 1. Any G_m-C filter without input capacitors and such that corresponding matrix T_C is invertible is called *a state-space G_m-C filter*. The state variables of such a filter, relating to equations (29)–(30), are just internal node voltages.

Thus, invertibility of matrix T_C and absence of input capacitors are the necessary and sufficient conditions for the existence of the state matrices. From the discussion preceding Definition 1 we obtain that the state matrices of voltage-mode filter denoted as A_v, B_v, C_v, D_v and the state matrices of current-mode filter denoted as A_c, B_c, C_c, D_c are the following:

$$A_v = T_C^{-1}G \quad B_v = T_C^{-1}\tilde{C}_c^T \quad C_v = \tilde{C}_v \quad D_v = D \tag{31}$$

$$A_c = T_C^{-1}G^T \quad B_c = T_C^{-1}\tilde{C}_v^T \quad C_c = \tilde{C}_c \quad D_c = D \tag{32}$$

Note that if the G_m-C filter has only grounded capacitors and no floating nodes, then the corresponding matrix T_C is diagonal and, of course, invertible. In literature (e.g. [7], [26]), only such filters are usually considered as state-space ones. We will refer to them as *direct state-space filters*. Our definition comprises a much wider class of filters. Using Proposition 2 (Section 4) one can easily reformulate Definition 1 to the form: any G_m-C filter for which graph G_t, corresponding to matrix T_C is connected, is called a state-space filter.

Using (31) and (32) we can obtain the following relationships between the state matrices for voltage- and current-mode filter:

$$A_c = T_C^{-1}A_v^T T_C \quad B_c = T_C^{-1}C_v^T \quad C_c = B_v^T T_C \quad D_c = D_v \tag{33}$$

Having (33) we can also derive analogous relations for the observability and controllability matrices W and K. Rewriting (28a) for the voltage-mode filter and using (33) we obtain:

$$A_v^T W_v + W_v A_v = -C_v^T C_v \tag{34}$$

which yields:

$$(T_C A_c T_C^{-1}) W_v + W_v (T_C^{-1} A_c^T T_C) = -T_C B_c B_c^T T_C \tag{35}$$

$$A_c (T_C^{-1} W_v T_C^{-1}) + (T_C^{-1} W_v T_C^{-1}) A_c^T = -B_c B_c^T \tag{36}$$

However, rewriting (28b) for the current-mode filter we have:

$$A_c K_c + K_c A_c^T = -B_c B_c^T \tag{37}$$

Thus, we get the following equality:

$$K_c = T_C^{-1} W_v T_C^{-1} \tag{38}$$

A similar reasoning that starts from (54b), yields the following for voltage-mode filter:

$$W_c = T_C K_v T_C \tag{39}$$

Relations (33), (38) and (39) are important in applications. For example, in [32] the authors considered dynamic range (*DR*) of voltage- and current-mode state-space G_m-C filters. It turned out that the general *DR* formulas for a current-mode filter can be (due to relations (33), (38) and (39)) easily expressed in terms of a voltage-mode filter, which significantly facilitates further comparison of both modes of the filter.

It is well known (e.g. [2]) that structures containing only grounded capacitors and no floating nodes (i.e. direct state-space filters in our terminology) are most suitable for monolithic implementation. The reason is that parasitic capacitances can be absorbed into the circuit capacitances in these structures and that grounded capacitors need smaller chip areas than floating ones. General description of G_m-C filters introduced in Section 2 gives an easy way to transform any state-space filter (i.e. containing, in general, also floating capacitors) into direct state-space structure. Below, we describe two 'canonical' transformations from a state-space structure into a direct one.

Let $\hat{T}_C$ be any diagonal matrix with positive elements, i.e. $(\hat{T}_C)_{ii} = \hat{C}_i$, $\hat{C}_i > 0$, $i = 1, ..., n$. Multiplying the first equation in (29) by $\hat{T}_C$, one gets:

$$s\hat{T}_C X = \hat{T}_C T_C^{-1} G X + \hat{T}_C T_C^{-1} \tilde{C}_c^T u_i$$
$$u_o = \tilde{C}_v X + D u_i \tag{40}$$

Introducing the following notation:

$$\hat{G} = \hat{T}_C T_C^{-1} G, \qquad \hat{C}_c = \tilde{C}_c T_C^{-1} \hat{T}_C, \qquad \hat{C}_v = \tilde{C}_v, \qquad \hat{D} = D \qquad (41)$$

we can rewrite (40) in the form

$$s\hat{T}_C X = \hat{G} X + \hat{C}_c^T u_i$$

$$u_o = \hat{C}_v X + \hat{D} u_i \qquad (42)$$

which is the same as the equations (5) describing the general structure of a voltage-mode G_m-C filter. This means that matrices $\hat{T}_C$, $\hat{G}$, $\hat{C}_c$, $\hat{C}_v$ and $\hat{D}$ determine the new filter structure. Due to the assumptions concerning matrix $\hat{T}_C$, this structure is a direct state-space one and has the same transfer function as the original filter. Indeed:

$$\hat{H}(s) = \hat{C}_v (s\hat{T}_C - \hat{G})^{-1} \hat{C}_c^T + \hat{D} = \tilde{C}_v (s\hat{T}_C - \hat{T}_C T_C^{-1} G)^{-1} \hat{T}_C T_C^{-1} \tilde{C}_c^T + D =$$

$$= \tilde{C}_v (T_C \hat{T}_C^{-1} (s\hat{T}_C - \hat{T}_C T_C^{-1} G))^{-1} \tilde{C}_c^T + D = \tilde{C}_v (sT_C - G)^{-1} \tilde{C}_c^T + D = H(s) \qquad (43)$$

Note that invertibility of matrix $\hat{T}_C$ is crucial in proving above equality. All elements of a new filter structure are given by the appropriate matrix elements, namely: $C_i = (\hat{T}_C)_{ii}$, $G_{mij} = (\hat{G})_{ij}$, $G_{mbi} = (\hat{C}_c)$ and $c_i = (\hat{C}_v)$. It is seen that changing elements $\hat{C}_i$ of matrix $\hat{T}_C$ means changing the values of filter elements without changing its structure. This is very convenient since it makes possible to optimize the filter, e.g. with respect to the sensitivity or the dynamic range.

Now we describe another transformation of a state-space G_m-C filter to direct state-space one. Selecting any diagonal matrix $\check{T}_C$ with positive elements $(\check{T}_C)_{ii} = \check{C}_i$, $\check{C}_i > 0$, $i = 1,...,n$, we can repeat, for the current-mode filter, the procedure presented in the preceding paragraph. Then, matrices $\check{G}$, $\check{C}_c$, $\check{C}_v$ and $\check{D}$, defined as:

$$\check{G} = GT_C^{-1} \check{T}_C, \qquad \check{C}_c = \tilde{C}_c, \qquad \check{C}_v = \tilde{C}_v T_C^{-1} \check{T}_C, \qquad \check{D} = D \qquad (44)$$

as well as matrix $\check{T}_C$, determine another structure of the G_m-C filter which is, of course, a direct state-space one. Filter elements are defined similarly as in the previous case.

In order to see better the difference between filter structures generated by the two described above transformation procedures, consider the case when the original filter has only one input transconductor (no input signal distribution) and a trivial output summer, i.e. $\tilde{C}_c = [0 \cdots 0 \, G_{mbk} \, 0 \cdots 0]$ and $\tilde{C}_v = [0 \cdots 0 \, 1 \, 0 \cdots 0] - 1$ at l-th position. Comparing (41) and (44) one can see that the filter structure determined by matrices $\hat{T}_C$, $\hat{G}$, $\hat{C}_c$, $\hat{C}_v$ and $\hat{D}$ has (in general) input signal distribution and a trivial output summer, however, the structure generated by matrices $\check{T}_C$, $\check{G}$, $\check{C}_c$, $\check{C}_v$ and $\check{D}$ has no input signal distribution and (in general) a non-trivial output summer. This is very convenient, since it gives us an additional degree of freedom while designing the filter for particular application.

It is worth noticing that the described above transformation procedures can be easily generalized. Suppose that $\hat{T}_C$ is any invertible matrix, which is symmetrical, diagonally dominant, with positive diagonal and negative non-diagonal elements. Then, the transformation generated by $\hat{T}_C$ determines a new filter structure, since any $n \times n$ matrix satisfying the above conditions represents the passive network of some state-space filter (see Section 3). Indeed, any non-diagonal element $(\hat{T}_C)_{ij}$, $j > i$, can be interpreted as a floating capacitor and the element $(\hat{T}_C)_{ij} - \sum_{j=1,\,j\neq i}^{n} (\hat{T}_C)_{ij}$, $i = 1,...,n$ as a grounded capacitor. Other filter elements are given as usual: $G_{mij} = (\hat{G})_{ij}$, $G_{mbi} = (\hat{C}_C)_i$ and $c_i = (\hat{C}_v)_i$. The same is true for the transformation generated by matrix $\check{T}_C$. Thus, we have proved the following result:

Proposition 5. Any state-space Gm-C filter can be transformed (without changing its transfer function) to the structure that has any passive network, provided that the corresponding matrix T_C is invertible. For each such passive network there exist two canonical transformations that yields different state-space filter structures.

The above result creates new possibilities of filter optimization. For instance, given any structure realizing the desired transfer function, by changing the passive network of the filter we can look for another structure with as small number of transconductors as possible. It is also possible to look for design trade-offs concerning complexity of passive and active network.
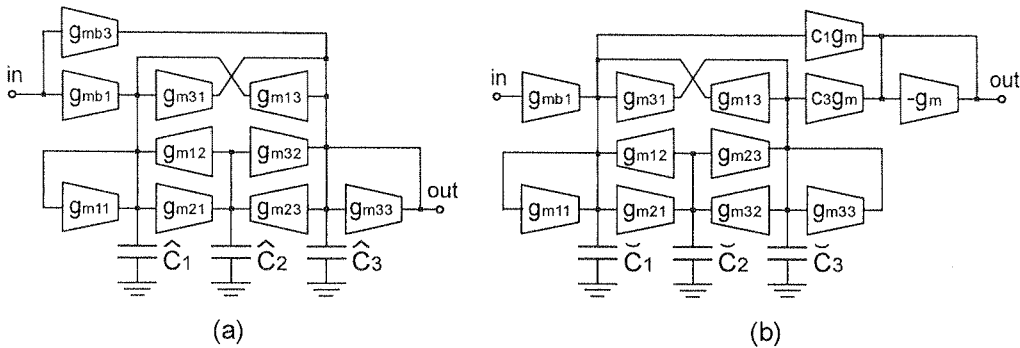

Fig. 4. Third order elliptic filter of Fig. 2 transformed to direct state-space structure by the transformation generated by matrix $\hat{T}_C$ (a) and $\check{T}_C$ (b)

Now we would like to give an example of the state-space filter transformation into the direct state-space structure. Consider the third order elliptic filter shown in Fig. 2 (Section 2). Fig. 4a shows the (voltage-mode) filter structure obtained by means of transformation generated by matrix $\hat{T}_C$. Filter elements are the following: $\hat{C}_i = (\hat{T}_C)_{ii}$, $g_{mij} = (\hat{G})_{ij}$ and $g_{mgi} = (\hat{T}_C)_i$ with matrices $\hat{G}$ and $\hat{C}_c$ calculated according to (41), where matrices T_C, G and $\tilde{C}_c$ of the original filter are given by (14). Fig. 4b shows the (voltage-mode) filter structure obtained using the transformation generated by matrix $\check{T}_C$. Filter elements are the following: $\tilde{C}_i = (\check{T}_C)_{ii}$, $g_{mij} = (\check{G})_{ij}$, $g_{mbi} = (\check{C}_c)_i$ and $c_i = \check{C}_v)_i$ with

matrices $\breve{G}$, $\breve{C}_c$, $\breve{C}_v$ calculated according to (44). In particular, matrices $\hat{T}_C$ and $\breve{T}_C$ can be identical, which means that both resulting direct state-space structures have the same passive network.

Note that the structure in Fig. 4a has two input transconductors and no output summer (output signal is taken directly from one of the internal nodes), while structure in Fig. 4b has one input transconductor and nontrivial output summer consisting of two transconductors.

5. REDUCIBLE AND QUASI-STATE-SPACE STRUCTURES

In this section we focus our attention on certain interesting classes of G_m-C structures which we call reducible structures and quasi-state-space structures. Suppose that we have G_m-C filter with n internal nodes x_1, x_2,...,x_n and corresponding matrices T_C, G, $\tilde{C}_c$, $\tilde{C}_v$ and D defined by (4a)-(4c). Moreover, let the filter has a floating node, say x_k (recall that floating node is the node not connected to any other node by capacitance elements). It follows that if this k-th node has inner loop (i.e. $G_{mkk} \neq 0$) then node x_k can be removed, i.e. the filter structure can be reduced to the structure with n-1 internal nodes. The reduction does not influence the transmittance of the filter. Below, we give the detailed description of the reduction procedure.
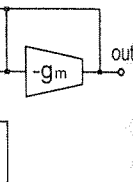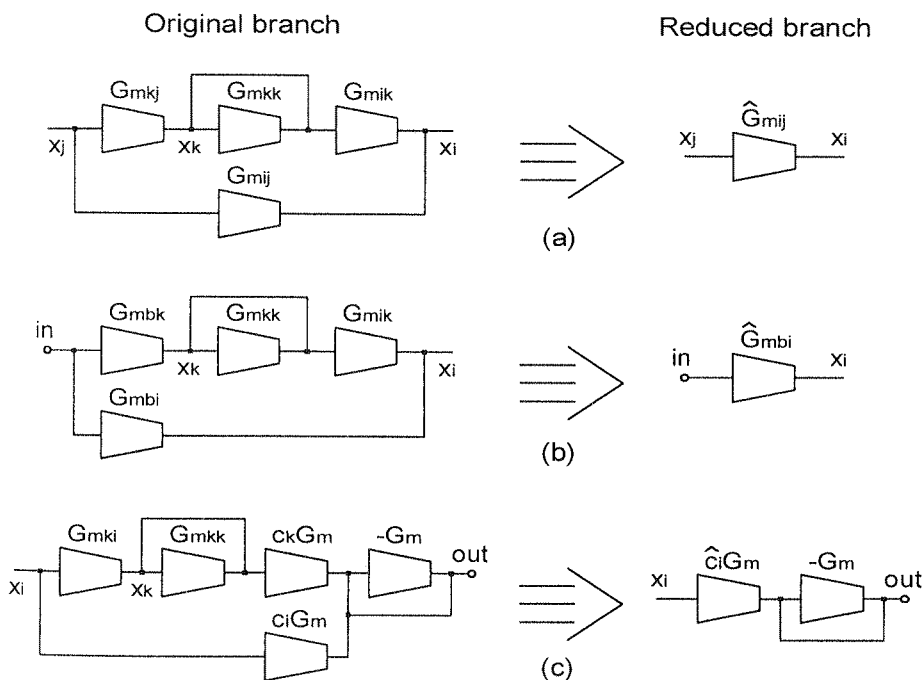

Fig. 5. Filter structure reduction: connection between input node and i-th internal node (a), j-th and i-th internal nodes (b), i-th internal node and output of the filter (c)

Fig. 5a shows the fragment of the active network of the filter concerning connections between input node and i-th internal node ($i \neq k$). It is seen that the current at i-th node due to the voltage u_i at the input of G_{mbi} and G_{mbk} is $u_i(G_{mbk} - G_{mbk}G_{mik}/G_{mkk})$. Thus, the two branches shown in Fig. 5a can be replaced by one transconductor $\hat{G}_{mbi}$ such that:

$$\hat{G}_{mbi} = G_{mbi} - \frac{G_{mbk}G_{mik}}{G_{mkk}} \tag{45}$$

Similar reasoning concerning connections between j-th and i-th internal nodes ($j \neq i \neq k$) and between i-th internal node and output of the filter ($i \neq k$) results in the reduction of appropriate branches shown in Figs. 5b and 5c. The new transconductance value $\hat{G}_{mij}$ and new coefficient $\hat{c}_i$ are the following:

$$\hat{G}_{mij} = G_{mij} - \frac{G_{mkj}G_{mik}}{G_{mkk}} \tag{46}$$

$$\hat{c}_i = c_i - \frac{c_k G_{mik}}{G_{mkk}} \tag{47}$$

Described procedure enables us to eliminate k-th internal node from the filter structure. This results in a new structure with $n-1$ internal nodes and corresponding matrices $\hat{T}_C$, $\hat{G}$, $\hat{C}_c$, $\hat{C}_v$ and $\hat{D}$. Appropriate matrix elements are given below:

$$(\hat{T}_C)_{ij} = \begin{cases} (T_C)_{ij} & i,j < k \\ (T_C)_{i,j-1} & i < k, j \geq k \\ (T_C)_{i-1,j} & i \geq k, j < k \\ (T_C)_{i-1,j-1} & i,j \geq k \end{cases} \qquad (\hat{G})_{ij} = \begin{cases} \hat{G}_{m_{ij}} & i,j < k \\ \hat{G}_{m_{i,j-1}} & i < k, j \geq k \\ \hat{G}_{m_{i-1,j}} & i \geq k, j < k \\ \hat{G}_{m_{i-1,j-1}} & i,j \geq k \end{cases} \tag{48a}$$

$$(\hat{C}_c)_i = \begin{cases} \hat{G}_{mb_i} + sC_{b_i} & i < k \\ \hat{G}_{mb_{i-1}} + sC_{bi-1} & i \geq k \end{cases} \qquad (\hat{C}_v)_i = \begin{cases} \hat{c}_i & i < k \\ \hat{c}_{i-1} & i \geq k \end{cases} \qquad \hat{D} = d \tag{48b}$$

It is seen that due to removing k-th node, some shift of indexes is necessary. Obviously, if the original filter structure has more than one floating node with inner loop, then all such nodes can be removed by iteration of described procedure.

On the basis of the above considerations we can formulate the formal definitions:

Definition 2. The G_m-C filter structure is called *reducible* if it has at least one floating node x_k with inner loop (i.e. $G_{mkk} \neq 0$).

Definition 3. The G_m-C filter structure is called *completely reducible* if each of its floating nodes has inner loop.

Reduction procedure can be helpful in computer aided synthesis of G_m-C filters since it allows to remove some redundancy in filter structure. Note, however that reduction is not always desirable — more complex structures are often used for their

flexibility (there exist universal structures making possible independent control of filter poles and zeros, e.g. [2]). Note also that some design techniques, e.g. Bruton Transformation [2], lead to the structures which are not reducible.

Now, we would like to introduce another class of G_m-C structures.

Definition 4. The G_m-C filter structure is called *quasi-state-space* structure if it has no input capacitors (i.e. $C_{bi} = 0$, $i = 1, \ldots, n$) and matrix T_C corresponding to filter's passive network is not invertible but each block of T_C is invertible.

Although for quasi-state-space filters do not exists state matrices, it follows that such structures exhibits similar properties as state-space structures. In particular, any quasi-state-space filter can be transformed to the direct structure (i.e. having only grounded capacitors). To show this, suppose that $n \times n$ matrix T_C has the following structure:

$$
T_C = \begin{bmatrix} (\tilde{T}_C)_1 & & & & & 0 \\ & \ddots & & & & \\ & & (\tilde{T}_C)_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix} \tag{49}
$$

where $(\tilde{T}_C)_i$, $i = 1, \ldots, k$ are invertible blocks of sizes $n_i \times n_i$ and the rest of elements are zeros. Let $n^\# = n_1 + n_2 + \ldots + n_k$. Note, that matrix T_C of any quasi-state-space filter has this structure up to renumbering of indexes. Let $T_C^\#$ be the matrix defined as below:

$$
T_C^\# = \begin{bmatrix} (\tilde{T}_C)_1 & & & & & 0 \\ & \ddots & & & & \\ & & (\tilde{T}_C)_k & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix} \tag{50}
$$

Then, $T_C^\#$ is invertible and $(T_C^\#)^{-1} \cdot T_C = T_C \cdot (T_C^\#)^{-1} = I^\#$, where $I^\#$ is diagonal matrix such that $I_{ii}^\# = 1$ for $1 < i \leq n^\#$ and $I_{ii}^\# = 0$ for $n^\# < i \leq n$. Now, multiplying equation (5) from the left by $(T_C^\#)^{-1}$ we obtain an analog of equation (29) for quasi-state-space filters:

$$
sI^\# X = (T_C^\#)^{-1} GX + (T_C^\#)^{-1} \check{C}_c^T u_i
$$
$$
u_o = \tilde{C}_v X + D u_i \tag{51}
$$

Let $\hat{T}_C^{\#}$ be any diagonal matrix such that, i.e. $(\hat{T}_C^{\#})_{ii} = \hat{C}_i$, $\hat{C}_i > 0$, for $1 \le i \le n^{\#}$ and $(\hat{T}_C^{\#})_{ii} = 1$ for $n^{\#} < i \le n$. Multiplying first equation in (51) by $\hat{T}_C^{\#}$ one gets:

$$s\hat{T}_C^{\#\#}X = \hat{T}_C^{\#}(T_C^{\#})^{-1}GX + \hat{T}_C^{\#}(T_C^{\#})^{-1}\check{C}_c^T u_i$$

$$u_o = \tilde{C}_v X + D u_i \tag{52}$$

where $\hat{T}_C^{\#\#} = \hat{T}_C^{\#}I^{\#}$ is diagonal matrix such that $(\hat{T}_C^{\#\#})_{ii} = \hat{C}_i$, $\hat{C}_i > 0$, for $1 \le i \le n^{\#}$ and $(\hat{T}_C^{\#\#})_{ii} = 0$ for $n^{\#} < i \le n$. It is seen, when comparing (52) to (5), that $\hat{T}_C^{\#\#}$ and matrices $\hat{G}_C^{\#}$, $\hat{C}_c^{\#}$, $\hat{C}_v^{\#}$ and $\hat{D}^{\#}$ defined as:

$$\hat{G}^{\#} = \hat{T}_C^{\#}(T_C^{\#})^{-1}G, \quad \hat{C}_c^{\#} = \tilde{C}_c(T_C^{\#})^{-1}\hat{T}_C^{\#}, \quad \hat{C}_c^{\#} = \tilde{C}_v, \quad \hat{D}^{\#} = D \tag{53}$$

determine new structure which is a direct (quasi-state-space) one. Of course, elements $\hat{C}_i$ are new grounded capacitances of the filter. Note that conclusions comprised in the discussion following (42) remain valid in this case. In particular, transmittance of the original and transformed filters are the same, since $\hat{T}_C^{\#}$ is invertible matrix.

Similarly as in Section 5, we can define second transformation starting from description of the current-mode filter. If $\check{T}_C^{\#}$ is any diagonal matrix such that, i.e. $(\check{T}_C^{\#})_{ii} = \check{C}_i$, $\check{C}_i > 0$, for $1 \le i \le n^{\#}$ and $(\check{T}_C^{\#})_{ii} = 1$ for $n^{\#} < i \le n$, then a new filter structure is determined by the following matrices:

$$\check{T}_C^{\#\#} = \check{T}_C^{\#}I^{\#}, \quad \check{G}^{\#} = G(T_C^{\#})^{-1}\check{T}_C^{\#}, \quad \check{C}_v^{\#} = \tilde{C}_v, \quad \check{C}_v^{\#} = \tilde{C}_v(T_C^{\#})^{-1}\check{T}_C^{\#}, \quad \check{D}^{\#} = D \tag{54}$$

The above transformations can be generalized similarly as in the state-space filters case, i.e. diagonal matrix $\hat{T}_C^{\#}$ can be replaced by any matrix $\hat{T}_C^{\#}$ such that its $n^{\#} \times n^{\#}$ submatrix is symmetrical, diagonally dominant, having positive diagonal and negative non-diagonal elements and $(\hat{T}_C^{\#})_{ij} = 1$ for $i, j > n^{\#}$. By the argument similar to that given in paragraph preceding Proposition 5, transformation generated by $\hat{T}_C^{\#}$ determine new filter structure with passive network corresponding to matrix $\hat{T}_C^{\#} = \hat{T}_C^{\#}I^{\#}$ and the rest of elements given by $G_{mij} = (\hat{G}^{\#})_{ij}$, $G_{mbi} = (\hat{C}_c^{\#})_i$ and $c_i = (\hat{C}_v^{\#})$. We have also the result analogous to Proposition 5:

Proposition 6. Any quasi-state-space G_m-C filter can be transformed (without changing its transfer function) to the structure having any passive network provided that corresponding matrix T_C has the structure such as described in the preceding paragraph. For each such passive network there exist two canonical transformations that give different quasi-state-space filter structures.

We end this section with the remark that every quasi-state-space G_m-C filter that is completely reducible can be reduced to the state-space structure. Indeed, let the filter has k floating nodes, i.e. graph G_t corresponding to its passive network has $k+1$ connected components. Removing all floating nodes causes that graph $\tilde{G}_t$ representing passive network of the reduced filter becomes connected, hence corresponding matrix $\tilde{T}_C$ is invertible. This means that the reduced filter is a state-space one.

6. CONCLUSION

In the paper, a novel and general approach to G_m-C filters based on matrix description has been presented. There have been shown general transfer function formulas for any G_m-C filter structure. We have investigated relationships between the passive network of the filter and its transfer function as well as conditions of invertibility of matrix corresponding to the filter's passive network. Based on these considerations we have given a new definition of state-space G_m-C filters and investigated their properties. In particular, we formulated relations between state matrices for voltage- and current-mode state-space G_m-C filters and defined two canonical transformations of state-space filters into direct state-space ones, i.e. those having only grounded capacitors. It has been shown that any state-space filter can be realized with any passive network (provided that the corresponding matrix is invertible). All the results presented in the paper can be used in computer aided filter synthesis and optimization. The reason is that matrix description can be easily handled by computer. Moreover, based on matrix description of G_m-C filter one can derive general formulas for sensitivity function of any G_m-C filter structure. The reduction procedures discussed in Section 5 can be used in automated filter design to simplify filter structures generated by computer programs within some design strategies.

It seems that the presented consistent approach to G_m-C filters can be interesting not only from a theoretical point of view but also it may be useful for filter designers. The matter of future work is to construct tools for automatic filter design based on the results drawn in this paper, with special care to the sensitivity optimization.

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ANALIZA OGÓLNYCH STRUKTUR FILTRÓW G_m-C ZMIENNYCH STANU CZASU CIĄGŁEGO

Streszczenie

W pracy przedstawiono nowe podejście do ważnej i szeroko stosowanej klasy filtrów aktywnych czasu ciągłego, wykorzystujących wzmacniacze transkonduktancyjne i pojemności (tzw. filtry G_m-C). Zaproponowano ogólny opis macierzowy filtrów G_m-C, który obejmuje wszystkie możliwe realizacje filtrów w rozważanej klasie elementów. Zaletą tego opisu jest między innymi to, że wszystkie macierze występujące w równaniach mogą być tworzone bezpośrednio przez wgląd w schemat układu. Podano ogólne wzory określające transmitancję filtru, a także wykazano wzajemną odwracalność struktur napięciowych i prądowych. Jakkolwiek jest to cecha powszechnie znana, w ramach przedstawianego podejścia otrzymuje się ją jako prosty i bezpośredni wniosek z ogólnych równań opisujących filtr. W dalszej kolejności, przedstawiono zależności pomiędzy strukturą pasywną (tj. pojemnościową) filtru a rzędem transmitancji filtru. Sformułowano szereg twierdzeń ustalających warunki konieczne i dostateczne odwracalności macierzy odpowiadającej sieci pasywnej filtru, a także górne ograniczenia rzędu transmitancji filtru w zależności od spójności grafu opisującego sieć pasywną.

W drugiej części pracy wprowadzono nową definicję filtrów zmiennych stanu, zgodnie z którą filtrem zmiennych stanu jest taki filtr, dla którego macierz odpowiadająca sieci pasywnej jest odwracalna, co jest równoważne istnieniu macierzy zmiennych stanu dla takiej struktury. Jest to istotne uogólnienie istniejących definicji, ponieważ dotychczas, za filtry zmiennych stanu uważano te, które posiadają wyłącznie uziemione pojemności. Zgodnie z nową definicją do podklasy filtrów zmiennych stanu należy również szereg struktur zawierających pojemności nieuziemione. Pozwala to na użycie aparatu opisu zmiennych stanu w odniesieniu do szerszej klasy filtrów. Zaproponowano również dwa kanoniczne przekształcenia umożliwiające transformację dowolnego filtru zmiennych stanu do struktury zawierającej wyłącznie pojemności uziemione, co może mieć znaczenie w przypadku realizacji monolitycznych. Wykazano również, że każdy filtr zmiennych stanu może być przetransformowany do struktury zawierającej dowolną sieć pasywną (o ile tylko odpowiadająca macierz jest odwracalna).

W pracy wprowadzono również kolejne dwie klasy filtrów, tzw. filtry redukowalne oraz filtry pseudo — zmiennych stanu. Pierwsza z nich obejmuje struktury zawierające węzły wewnętrzne nie dołączone do żadnej pojemności w układzie ale zawierające wewnętrzną pętlę transkonduktancyjną. Struktury tego typu można zredukować poprzez usunięcie takich węzłów bez zmiany charakterystyk filtru, modyfikując jednocześnie wartości elementów układu. Druga klasa filtrów, to struktury o nieodwracalnej macierzy podukładu pasywnego, która ma tę własność, że jej dowolna kwadratowa podmacierz o niezerowych elementach diagonalnych jest odwracalna. Pokazano, że struktury takie mają w pewnej mierze własności analogiczne do filtrów zmiennych stanu. Należy także zauważyć, że podklasa struktur pseudo — zmiennych stanu zawiera nietrywialne przykłady filtrów, np. realizacje układowe powstające przy użyciu transformacji Brutona.

Wyniki zawarte w omawianej pracy mogą być one zastosowane do automatycznego projektowania i analizy filtrów G_m-C. W szczególności, należy zauważyć, że opis macierzowy może być łatwo implementowany w postaci programu komputerowego, podobnie jak wszelkie transformacje i przekształcenia realizowane z jego pomocą.

Słowa kluczowe: filtry zmiennych stanu, filtry G_m-C, opis macierzowy, reprezentacja zmiennych stanu