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Differential-algebraic systems with maxima

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Abstract

The numerical-analytic method is applied to a class of nonlinear differential-algebraic systems with maxima to find a solution assuming that functions (f, g) satisfy the Lipschitz conditions in matrix notation. This solution is given as a limit of corresponding sequences including Seidel's iterations too. Some existence results are also obtained for problems with retardations.

1. Introduction

A useful approach in studying of existence of solutions is Samoilenko's numerical-analytic method (for details, see [9,10]). The application of this technique to differential problems with boundary conditions can be found, for example, in papers [1,3,7,8,11]. In this paper we shall extend this method to differential-algebraic boundary-value problems with maxima of the form

$$\begin{cases} x'(t) = f\bigl(t, x(t), \max_{[0,t]} x(s), y(t), \max_{[0,t]} y(s)\bigr) \\ \qquad \equiv f_0(t, x, y), \quad t \in J = [0, T], \\ y(t) = g\bigl(t, \max_{[0,t]} x(s), y(t)\bigr) \equiv g_0(t, x, y), \quad t \in J, \end{cases} \tag{1}$$

$$Ax(0) + Bx(T) = d. \tag{2}$$

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Here $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times p}$, $d \in \mathbb{R}^p$ and

$$\max_{[0,t]} x(s) = \Big(\max_{[0,t]} x_1(s), \max_{[0,t]} x_2(s), \ldots, \max_{[0,t]} x_p(s)\Big).$$

Existence of solutions for initial-value differential problems with maxima is discussed, for example, in papers [2,4]; see also papers [5,6], where some applications in nonlinear mechanics are given.

The numerical-analytic method combined with the comparison one is used to formulate corresponding results under the assumption that f and g satisfy the Lipschitz conditions in matrix notation. The aim of the present paper is to discuss the conditions under which the solution exists and it is the limit of successive approximations and Seidel's iterations too. Some error estimates are given. This paper contains also some discussion for more general differential-algebraic problems with retardations and corresponding results are given in the last section of this paper.

2. Assumptions

Put

$$\mathcal{L}f(x,y)(t) = \left(1 - \frac{t}{T}\right)\int_0^t f_0(s,x,y)\,ds - \frac{t}{T}\int_t^T f_0(s,x,y)\,ds.$$

Indeed, $\mathcal{L}f(x,y)(0) = \mathcal{L}f(x,y)(T) = O_{p\times 1}$. According to the numerical-analytic method, find the vector δ such that $x(t) = \eta + \mathcal{L}f(x,y)(t) + \delta t$ satisfies condition (2). Hence (1)–(2) give the following auxiliary problem

$$\begin{cases} x(t) = \eta + \mathcal{L}f(x,y)(t) + tS(\eta) \equiv F(t,x,y;\eta), & t \in J, \\ y(t) = g_0(t,x,y), & t \in J \end{cases} \tag{3}$$

and

$$\frac{1}{T}\int_0^T f_0(s,x,y)\,ds = S(\eta)$$

with $S(\eta) = \frac{1}{T}B^{-1}[d - (A+B)\eta]$ assuming that $\det(B) \neq 0$. Note that, $F(0,x,y;\eta) = \eta$, so $x(0) = \eta$.

Let us introduce the following

Assumption H_1. (1) There are matrices $K_{p\times p}$, $L_{p\times p}$, $M_{p\times q}$, $N_{p\times q}$ with non-negative entries such that

$$\begin{aligned} &\big|f(t,x,X,y,Y) - f(t,\bar{x},\bar{X},\bar{y},\bar{Y})\big| \\ &\quad\leqslant K|x-\bar{x}| + L|X-\bar{X}| + M|y-\bar{y}| + N|Y-\bar{Y}| \end{aligned}$$

for all $t \in J$, $x, X, \bar{x}, \bar{X} \in \mathbb{R}^p$, $y, Y, \bar{y}, \bar{Y} \in \mathbb{R}^q$.

(2) There are matrices $P_{q\times p}$, $Q_{q\times q}$ with nonnegative entries, $\rho(Q) < 1$ and such that

$$\left|g(t,x,y) - g(t,\bar{x},\bar{y})\right| \leqslant P|x-\bar{x}| + Q|y-\bar{y}|$$

for all $t \in J$, $x, \bar{x} \in \mathbb{R}^p$, $y, \bar{y} \in \mathbb{R}^q$. Here $|\cdot|$ denotes the absolute value of the vector, so $|x| = (|x_1|, \ldots, |x_p|)^{\mathrm{T}}$ or $|y| = (|y_1|, \ldots, |y_q|)^{\mathrm{T}}$. Moreover, $\rho(Q)$ denotes the spectral radius of the matrix Q.

Assumption H_2. For any nonnegative function $h \in C(J \times \mathbb{R}^p, \mathbb{R}^p_+)$ there exists a unique solution $u \in C(J, \mathbb{R}^p_+)$ of the comparison equation

$$u(t) = (\Omega u)(t) + h(t,\eta), \quad t \in J, \tag{4}$$

where

$$\begin{aligned}(\Omega u)(t) = {} & \left(1 - \frac{t}{T}\right) \int_0^t \Big[Ku(s) + D \max_{[0,s]} u(\tau)\Big]\, ds \\ & + \frac{t}{T} \int_t^T \Big[Ku(s) + D \max_{[0,s]} u(\tau)\Big]\, ds, \quad t \in J\end{aligned}$$

with $D = L + (N+M)(I-Q)^{-1}P$.

Note that under Assumption H_1 we have

$$\begin{aligned}&\left|\mathcal{L}f(x,y)(t) - \mathcal{L}f(\bar{x},\bar{y})(t)\right| \\ &\quad \leqslant \left(1 - \frac{t}{T}\right) \int_0^t \left|f_0(s,x,y) - f_0(s,\bar{x},\bar{y})\right| ds \\ &\qquad + \frac{t}{T} \int_t^T \left|f_0(s,x,y) - f_0(t,\bar{x},\bar{y})\right| ds \\ &\quad \leqslant \left(1 - \frac{t}{T}\right) \int_0^t \Big[K\left|x(s) - \bar{x}(s)\right| + L \max_{[0,s]}\left|x(\tau) - \bar{x}(\tau)\right| \\ &\qquad\qquad + M\left|y(s) - \bar{y}(s)\right| + N \max_{[0,s]}\left|y(\tau) - \bar{y}(\tau)\right|\Big] ds \\ &\qquad + \frac{t}{T} \int_t^T \Big[K\left|x(s) - \bar{x}(s)\right| + L \max_{[0,s]}\left|x(\tau) - \bar{x}(\tau)\right| \\ &\qquad\qquad + M\left|y(s) - \bar{y}(s)\right| + N \max_{[0,s]}\left|y(\tau) - \bar{y}(\tau)\right|\Big] ds \\ &\quad \equiv \Omega_0\big(t, |x-\bar{x}|, |y-\bar{y}|\big).\end{aligned} \tag{5}$$

3. Lemmas

For $n = 0, 1, \ldots$ let us define the sequences $\{u_n, w_n\}$ by formulas

$$\begin{cases} u_0(t) = u(t), & t \in J, \\ u_{n+1}(t) = \Omega_0(t, u_n, w_n), & t \in J, \end{cases} \tag{6}$$

$$\begin{cases} w_0(t) = (I - Q)^{-1}\big[P \max_{[0,t]} u_0(s) + r_1(\eta)\big], & t \in J, \\ w_{n+1}(t) = P \max_{[0,t]} u_n(s) + Q w_n(t), & t \in J, \end{cases} \tag{7}$$

where u is a solution of (4) with

$$h(t, \eta) = R_1(\eta) + 2t\left(1 - \frac{t}{T}\right)(M + N)(I - Q)^{-1} r_1(\eta),$$
$$R_1(\eta) = \max_{t \in J}\big|F(t, x_0, y_0; \eta) - x_0(t)\big|,$$
$$r_1(\eta) = \max_{t \in J}\big|g_0(t, x_0, y_0) - y_0(t)\big|.$$

To obtain a solution of problem (3) we shall first establish some properties for sequences $\{u_n, w_n\}$. They are given in the next two lemmas.

Lemma 1. *Let Assumptions H_1 and H_2 be satisfied. Then,*

$$u_{n+1}(t) \leqslant u_n(t) \leqslant u_0(t), \qquad w_{n+1}(t) \leqslant w_n(t) \leqslant w_0(t) \tag{8}$$

for $t \in J$ and $n = 0, 1, \ldots$. Moreover, the sequences $\{u_n\}, \{w_n\}$ converge uniformly to zero functions, so $u_n(t) \to 0$, $w_n(t) \to 0$ on J if $n \to \infty$.

Proof. Note that the matrix $(I - Q)^{-1}$ exists and its entries are nonnegative because of the condition $\rho(Q) < 1$. Put $L_0 = (M + N)(I - Q)^{-1}$, $L_1 = L + L_0 P$. Then

$$\begin{aligned} u_1(t) &= \Omega_0(t, u_0, w_0) \\ &= \left(1 - \frac{t}{T}\right)\int_0^t \Big\{K u_0(s) + L_1 \max_{[0,s]} u_0(\tau) + L_0 r_1(\eta)\Big\}\, ds \\ &\quad + \frac{t}{T}\int_t^T \Big\{K u_0(s) + L_1 \max_{[0,s]} u_0(\tau) + L_0 r_1(\eta)\Big\}\, ds \\ &= (\Omega u)(t) + 2t\left(1 - \frac{t}{T}\right) L_0 r_1(\eta) \leqslant u(t), \\ w_1(t) &= P \max_{[0,t]} u_0(s) + Q w_0(t) \\ &= P \max_{[0,t]} u_0(s) + Q(I - Q)^{-1}\Big[P \max_{[0,t]} u_0(s) + r_1(\eta)\Big] \end{aligned}$$

$$\leqslant P \max_{[0,t]} u_0(s) + Q(I-Q)^{-1}\Big[P \max_{[0,t]} u_0(s) + r_1(\eta)\Big] + r_1(\eta)$$
$$= (I-Q)^{-1}\Big[P \max_{[0,t]} u_0(s) + r_1(\eta)\Big] = w_0(t).$$

Using the monotonicity of Ω_0, we obtain (8), by mathematical induction.

Hence $u_n \to \bar{u}$, $w_n \to \bar{w}$ on J if $n \to \infty$, where $(\bar{u}, \bar{w})$ is a solution of the system

$$\bar{u}(t) = \Omega_0(t, \bar{u}, \bar{w}), \qquad \bar{w}(t) = P \max_{[0,t]} \bar{u}(s) + Q\bar{w}(t), \quad t \in J.$$

Hence $\bar{w}(t) = (I-Q)^{-1} P \max_{[0,t]} \bar{u}(s)$, $t \in J$. It is simple to see that

$$\Omega_0(t, \bar{u}, \bar{w}) = (\Omega\bar{u})(t), \quad t \in J,$$

so $\bar{u}$ is a solution of equation $\bar{u}(t) = (\Omega\bar{u})(t)$, $t \in J$. By Assumption H_2, $\bar{u}(t) = 0$ on J, and then $\bar{w}(t) = 0$ on J too. It ends the proof. □

Lemma 2. *Assume that* $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, *and* $A_{p\times p}$, $B_{p\times p}$ *and* $d_{p\times 1}$ *are given constant matrices. Assume that* $\det(B) \neq 0$. *Let Assumptions* H_1 *and* H_2 *be satisfied. Then, for* $t \in J$, $n, k = 0, 1, \ldots,$ *we have the estimates*

$$\begin{cases} \big|x_n(t) - x_0(t)\big| \leqslant u_0(t), \\ \big|x_{n+k}(t) - x_k(t)\big| \leqslant u_k(t), \end{cases} \qquad \begin{cases} \big|y_n(t) - y_0(t)\big| \leqslant w_0(t), \\ \big|y_{n+k}(t) - y_k(t)\big| \leqslant w_k(t), \end{cases} \tag{9}$$

where $x_0 \in C^1(J, \mathbb{R}^p)$, $y_0 \in C(J, \mathbb{R}^q)$ *and*

$$x_{n+1}(t) = F(t, x_n, y_n; \eta), \qquad y_{n+1}(t) = g_0(t, x_n, y_n). \tag{10}$$

Moreover,

$$Ax_{n+1}(0) + Bx_{n+1}(T) = d, \quad n = 0, 1, \ldots.$$

Proof. Indeed,

$$\big|x_1(t) - x_0(t)\big| \leqslant R_1(\eta) \leqslant h(t,\eta) \leqslant u_0(t), \quad t \in J,$$
$$\big|y_1(t) - y_0(t)\big| \leqslant r_1(\eta) \leqslant \big[Q(I-Q)^{-1} + I\big]r_1(\eta) = (I-Q)^{-1} r_1(\eta)$$
$$\leqslant w_0(t), \quad t \in J.$$

Assume that

$$\big|x_k(t) - x_0(t)\big| \leqslant u_0(t), \qquad \big|y_k(t) - y_0(t)\big| \leqslant w_0(t), \qquad t \in J$$

for some $k \geqslant 0$. Then, by (5), we have

$$\big|x_{k+1}(t) - x_0(t)\big| \leqslant \big|F(t, x_k, y_k; \eta) - F(t, x_0, y_0; \eta)\big| + R_1(\eta)$$
$$\leqslant \Omega_0(t, u_0, w_0) + R_1(\eta) = u_0(t), \quad t \in J,$$

$$\begin{aligned}\left|y_{k+1}(t)-y_0(t)\right| &\leqslant \left|g_0(t,x_k,y_k)-g_0(t,x_0,y_0)\right|+r_1(\eta)\\ &\leqslant P\max_{[0,t]}u_0(s)+Qw_0(t)+r_1(\eta)\\ &= P\max_{[0,t]}u_0(s)+Q(I-Q)^{-1}\Big[P\max_{[0,t]}u_0(s)+r_1(\eta)\Big]\\ &\quad +r_1(\eta)\\ &=(I-Q)^{-1}\Big[P\max_{[0,t]}u_0(s)+r_1(\eta)\Big]=w_0(t).\end{aligned}$$

Hence, by mathematical induction, we have

$$\left|x_n(t)-x_0(t)\right|\leqslant u_0(t) \quad \text{and} \quad \left|y_n(t)-y_0(t)\right|\leqslant w_0(t)$$

for $t\in J$, $n=0,1,\ldots$. Basing on the above, let us assume that

$$\left|x_{n+k}(t)-x_k(t)\right|\leqslant u_k(t), \qquad \left|y_{n+k}(t)-y_k(t)\right|\leqslant w_k(t), \qquad t\in J$$

for all n and some $k\geqslant 0$. Then, again by (5), we see that

$$\begin{aligned}\left|x_{n+k+1}(t)-x_{k+1}(t)\right| &= \left|F(t,x_{n+k},y_{n+k};\eta)-F(t,x_k,y_k;\eta)\right|\\ &\leqslant \Omega_0(t,u_k,w_k)=u_{k+1}(t), \quad t\in J,\\ \left|y_{n+k+1}(t)-y_{k+1}(t)\right| &= \left|g_0(t,x_{n+k},y_{n+k})-g_0(t,x_k,y_k)\right|\\ &\leqslant P\max_{[0,t]}u_k(s)+Qw_k(t)=w_{k+1}(t), \quad t\in J.\end{aligned}$$

Hence, by mathematical induction, the estimates (9) hold. It is quite simple to verify that x_{n+1} satisfies integral condition (2) for any $n=0,1,\ldots$. It ends the proof. □

4. Existence results

Combining Lemmas 1 and 2 we have

Theorem 1. *Assume that* $f\in C(J\times\mathbb{R}^p\times\mathbb{R}^p\times\mathbb{R}^q\times\mathbb{R}^q,\mathbb{R}^p)$, $g\in C(J\times\mathbb{R}^p\times\mathbb{R}^q,\mathbb{R}^q)$, *and* $A_{p\times p}$, $B_{p\times p}$ *and* $d_{p\times 1}$ *are given constant matrices. Assume that* $\det(B)\neq 0$. *Let Assumptions* H_1 *and* H_2 *be satisfied. Then, for every* $\eta\in\mathbb{R}^p$, *there exists a unique solution* $(\bar{x},\bar{y})$ *of problem* (3) *where* $x_n(t)\to\bar{x}(t)$, $y_n(t)\to\bar{y}(t)$ *on* J *as* $n\to\infty$ *and we have the estimates*

$$\left|x_n(t)-\bar{x}(t)\right|\leqslant u_n(t), \qquad \left|y_n(t)-\bar{y}(t)\right|\leqslant w_n(t)$$

for $t\in J$, *and* $n=0,1,\ldots$.

Moreover, $(\bar{x},\bar{y})$ *is the solution of problem* (1)–(2) *iff*

$$\frac{1}{T}\int_0^T f_0(s,\bar{x},\bar{y})\,ds=S(\eta).$$

Remark 1. Note that Assumption H_2 holds if we assume that

$$\rho(W) < 1 \quad \text{for } W = \frac{T}{2}(K + D).$$

To get this condition we apply the Banach fixed point theorem. Denote the right-hand side of Eq. (4) by $\mathcal{D}$. Then, for $u, \bar{u} \in C(J, \mathbb{R}_+^p)$ we have

$$\begin{aligned}
&\left|\mathcal{D}u(t) - \mathcal{D}\bar{u}(t)\right| \\
&\quad \leqslant \left(1 - \frac{t}{T}\right) \int_0^t \left[K\left|u(s) - \bar{u}(s)\right| + D \max_{[0,s]}\left|u(\tau) - \bar{u}(\tau)\right|\right] ds \\
&\qquad + \frac{t}{T} \int_t^T \left[K\left|u(s) - \bar{u}(s)\right| + D \max_{[0,s]}\left|u(\tau) - \bar{u}(\tau)\right|\right] ds \\
&\quad \leqslant 2t\left(1 - \frac{t}{T}\right)(K + D) \max_{t \in J}\left|u(t) - \bar{u}(t)\right| \leqslant W \max_{t \in J}\left|u(t) - \bar{u}(t)\right|.
\end{aligned}$$

Hence, operator $\mathcal{D}$ is a contraction mapping, so Eq. (4) has a unique solution.

In place of iterations (10), it is sometimes convenient to use Seidel's method described by

$$\begin{cases} \tilde{x}_{n+1}(t) = F(t, \tilde{x}_n, \tilde{y}_n; \eta), \\ \tilde{y}_{n+1}(t) = g_0(t, \tilde{x}_{n+1}, \tilde{y}_n), \end{cases} \quad \text{or} \quad \begin{cases} \bar{y}_{n+1}(t) = g_0(t, \bar{x}_n, \bar{y}_n), \\ \bar{x}_{n+1}(t) = F(t, \bar{x}_n, \bar{y}_{n+1}; \eta) \end{cases} \tag{11}$$

for $t \in J$, and $n = 0, 1, \ldots$.

For $t \in J$ and $n = 0, 1, \ldots,$ let us define the sequences:

$$\begin{cases} \tilde{u}_0(t) = u_0(t), \qquad \tilde{w}_0(t) = w_0(t), \\ \tilde{u}_{n+1}(t) = \Omega_0(t, \tilde{u}_n, \tilde{w}_n), \\ \tilde{w}_{n+1}(t) = P \max_{[0,t]} \tilde{u}_{n+1}(s) + Q\tilde{w}_n(t), \end{cases}$$
$$\begin{cases} \bar{u}_0(t) = u_0(t), \qquad \bar{w}_0(t) = w_0(t), \\ \bar{w}_{n+1}(t) = P \max_{[0,t]} \bar{u}_n(s) + Q\bar{w}_n(t), \\ \bar{u}_{n+1}(t) = \Omega_0(t, \bar{u}_n, \bar{w}_{n+1}). \end{cases}$$

Now, by mathematical induction, we are able to show the following result

Lemma 3. *Let Assumptions H_1 and H_2 hold. Then*

$$\bar{u}_n(t) \leqslant u_n(t), \qquad \bar{w}_n(t) \leqslant w_n(t), \qquad t \in J,\ n = 0, 1, \ldots,$$
$$\tilde{u}_n(t) \leqslant u_n(t), \qquad \tilde{w}_n(t) \leqslant w_n(t), \qquad t \in J,\ n = 0, 1, \ldots,$$

and $\bar{u}_n(t) \to 0$, $\bar{w}_n(t) \to 0$, $\tilde{u}_n(t) \to 0$, $\tilde{w}_n(t) \to 0$ on J if $n \to \infty$.

The simple consequence of Lemma 3 is the following

Theorem 2. *Assume that all assumptions of Theorem* 1 *are satisfied. Let* $\bar{x}_0(t) = \tilde{x}_0(t) = x_0(t)$, $\bar{y}_0(t) = \tilde{y}_0(t) = y_0(t)$, $t \in J$. *Then,* $\bar{x}_n(t) \to \bar{x}(t)$, $\bar{y}_n(t) \to \bar{y}(t)$, $\tilde{x}_n(t) \to \bar{x}(t)$, $\tilde{y}_n(t) \to \bar{y}(t)$ *on* J *as* $n \to \infty$.

Moreover, we have the estimates

$$\left|\bar{x}_n(t) - \bar{x}(t)\right| \leqslant \bar{u}_n(t), \qquad \left|\bar{y}_n(t) - \bar{y}(t)\right| \leqslant \bar{w}_n(t),$$
$$\left|\tilde{x}_n(t) - \bar{x}(t)\right| \leqslant \tilde{u}_n(t), \qquad \left|\tilde{y}_n(t) - \bar{y}(t)\right| \leqslant \tilde{w}_n(t)$$

for $t \in J$, *and* $n = 0, 1, \ldots$.

Note that iterations (10) and (11) converge to $(\bar{x}, \bar{y})$ under the same conditions but basing on Lemma 3 we see that the error estimates for (11) are better in comparing with the corresponding estimates for (10). This notice is important since $\{x_n, y_n\}$, $\{\bar{x}_n, \bar{y}_n\}$ and $\{\tilde{x}_n, \tilde{y}_n)$ are approximated solutions of problem (3).

Theorem 3. *Assume that all assumptions of Theorem* 1 *are satisfied. Then*

$$\left|\Delta(\bar{x}, \bar{y}; \eta) - \Delta(x_n, y_n; \eta)\right| \leqslant \bar{\Omega}(u_n, w_n),$$
$$\left|\Delta(\bar{x}, \bar{y}; \eta) - \Delta(\tilde{x}_n, \tilde{y}_n; \eta)\right| \leqslant \bar{\Omega}(\tilde{u}_n, \tilde{w}_n),$$
$$\left|\Delta(\bar{x}, \bar{y}; \eta) - \Delta(\bar{x}_n, \bar{y}_n; \eta)\right| \leqslant \bar{\Omega}(\bar{u}_n, \bar{w}_n),$$
$$\left|\Delta(x_n, y_n; \eta) - \Delta(\tilde{x}_n, \tilde{y}_n; \eta)\right| \leqslant \bar{\Omega}(u_n, w_n) + \bar{\Omega}(\tilde{u}_n, \tilde{w}_n),$$
$$\left|\Delta(x_n, y_n; \eta) - \Delta(\bar{x}_n, \bar{y}_n; \eta)\right| \leqslant \bar{\Omega}(u_n, w_n) + \bar{\Omega}(\bar{u}_n, \bar{w}_n),$$
$$\left|\Delta(\tilde{x}_n, \tilde{y}_n; \eta) - \Delta(\bar{x}_n, \bar{y}_n; \eta)\right| \leqslant \bar{\Omega}(\tilde{u}_n, \tilde{w}_n) + \bar{\Omega}(\bar{u}_n, \bar{w}_n)$$

for $t \in J$, $n = 0, 1, \ldots$, *where*

$$\Delta(x, y; \eta) = \int_0^T f_0(s, x, y)\, ds - T B^{-1}\left[d - (A + B)\eta\right),$$

$$\bar{\Omega}(u, w) = \int_0^T \left[K u(s) + L \max_{[0,s]} u(\tau) + M w(s) + N \max_{[0,t]} w(\tau)\right] ds.$$

5. Differential-algebraic systems with retardations

Let $\alpha, \beta, \gamma, \delta, \mu \in C(J, J)$. For $t \in J$, let us consider the following problem

$$\begin{cases} x'(t) = f\big(t, x(\alpha(t)), \max_{[0,\beta(t)]} x(s), y(\gamma(t)), \max_{[0,\delta(t)]} y(s)\big) \\ \qquad \equiv f_1(t, x, y), \\ y(t) = g\big(t, \max_{[0,\mu(t)]} x(s), y(t)\big) \equiv g_1(t, x, y) \end{cases} \tag{12}$$

with condition (2), where $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$. Let $\det(B) \neq 0$. According to the numerical-analytic method find the vector δ such that

$$x(t) = \eta + \mathcal{P}z(t) + \delta t$$

$$\text{with} \quad \mathcal{P}z(t) = \left(1 - \frac{t}{T}\right) \int_0^t z(s)\,ds - \frac{t}{T} \int_t^T z(s)\,ds$$

satisfies condition (2). Then, this and (12) give the following auxiliary problem

$$\begin{cases} z(t) = f\Big(t, \eta + \mathcal{P}z\big(\alpha(t)\big) + \alpha(t)S(\eta), \max_{[0,\beta(t)]}\big[\eta + \mathcal{P}z(s) + sS(\eta)\big], \\ \qquad\qquad y\big(\gamma(t)\big), \max_{[0,\delta(t)]} y(s)\Big) \equiv \mathcal{F}(t, z, y; \eta), \\ y(t) = g\big(t, \max_{[0,\mu(t)]}\big[\eta + \mathcal{P}z(s) + sS(\eta)\big], y(t)\big) \equiv \mathcal{G}(t, z, y; \eta) \end{cases} \tag{13}$$

and

$$\int_0^T z(s)\,ds = TS(\eta).$$

Assumption H_3. For any nonnegative function $H \in C(J \times \mathbb{R}^p, \mathbb{R}^p_+)$ there exists a unique solution $V \in C(J, \mathbb{R}^p_+)$ of the comparison equation

$$V(t) = (\Omega_1 V)(t) + H(t, \eta)$$

with

$$\begin{aligned} (\Omega_1 V)(t) = {} & K\Lambda\big(\alpha(t), V\big) + L \max_{[0,\beta(t)]} \Lambda(s, V) \\ & + M(I - Q)^{-1} P \max_{[0,\mu(\gamma(t))]} \Lambda(s, V) \\ & + N(I - Q)^{-1} P \max_{[0,\delta(t)]} \max_{[0,\mu(s)]} \Lambda(\tau, V) \end{aligned}$$

$$\Lambda(t, V) = \left(1 - \frac{t}{T}\right) \int_0^t V(s)\,ds + \frac{t}{T} \int_t^T V(s)\,ds.$$

Put

$$\begin{aligned} \bar{\Lambda}(t, u, w) = {} & K\Lambda\big(\alpha(t), u\big) + L \max_{[0,\beta(t)]} \Lambda(s, u) + Mw\big(\gamma(t)\big) \\ & + N \max_{[0,\delta(t)]} w(s). \end{aligned}$$

For $t \in J$, and $n = 0, 1, \ldots,$ let us define the sequences $\{U_n\}$, $\{W_n\}$ by relations

$$\begin{cases} U_0(t) = V(t), \\ U_{n+1}(t) = \bar{\Lambda}(t, U_n, W_n), \end{cases} \tag{14}$$

$$\begin{cases} W_0(t) = (I - Q)^{-1}\big[P \max_{[0,\mu(t)]} \Lambda(s, U_0) + r_2(\eta)\big], \\ W_{n+1}(t) = P \max_{[0,\mu(t)]} \Lambda(s, U_n) + Q W_n(t), \end{cases} \tag{15}$$

where V is defined as in Assumption H_3 with

$$\begin{aligned} H(t,\eta) &= (M+N)(I-Q)^{-1} r_2(\eta) + R_2(\eta), \\ r_2(\eta) &= \max_{t\in J} \big|\mathcal{G}(t, Z_0, Y_0; \eta) - Y_0(t)\big|, \\ R_2(\eta) &= \max_{t\in J} \big|\mathcal{F}(t, Z_0, Y_0; \eta) - Z_0(t)\big|. \end{aligned}$$

Lemma 4. *Let Assumptions H_1 and H_3 be satisfied. Then the sequences $\{U_n\}$, $\{W_n\}$ satisfy the relations*

$$U_{n+1}(t) \leqslant U_n(t) \leqslant U_0(t), \qquad W_{n+1}(t) \leqslant W_n(t) \leqslant W_0(t) \tag{16}$$

for $t \in J$, $n = 0, 1, \ldots$. Moreover U_n, W_n converge uniformly to zero functions if $n \to \infty$.

Proof. Note that

$$\begin{aligned} U_1(t) &= \bar{\Lambda}(t, U_0, W_0) = (\Omega_1 U_0)(t) + (M+N)(I-Q)^{-1} r_2(\eta) \leqslant U_0(t), \\ W_1(t) &= P \max_{[0,\mu(t)]} \Lambda(s, U_0) + Q(I-Q)^{-1}\Big[P \max_{[0,\mu(t)]} \Lambda(s, U_0) + r_2(\eta)\Big] \\ &\leqslant \big[(I-Q)(I-Q)^{-1} + Q(I-Q)^{-1}\big]\Big[P \max_{[0,\mu(t)]} \Lambda(s, U_0) + r_2(\eta)\Big] \\ &= (I-Q)^{-1}\Big[P \max_{[0,\mu(t)]} \Lambda(s, U_0) + r_2(\eta)\Big] = W_0(t). \end{aligned}$$

By mathematical induction, it is simple to show that (16) holds. Hence $U_n \to U$, $W_n \to W$ on J if $n \to \infty$. Indeed, the pair (U, W) is a solution of the system

$$U(t) = \bar{\Lambda}(t, U, W), \qquad W(t) = P \max_{[0,\mu(t)]} \Lambda(s, U) + Q W(t), \qquad t \in J.$$

It gives $U(t) = (\Omega_1 U)(t)$, $t \in J$ because $W(t) = (I-Q)^{-1} P \max_{[0,\mu(t)]} \Lambda(s, U)$. Hence, by Assumption H_3, we see that $U = 0$ on J, and then $W = 0$ on J too. It ends the proof. □

Lemma 5. *Assume that $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, $\alpha, \beta, \gamma, \delta, \mu \in C(J, J)$. Moreover, $A_{p\times p}$, $B_{p\times p}$ and $d_{p\times 1}$ are given constant matrices. Assume that* $\det(B) \neq 0$. *Let Assumptions H_1 and H_3 be satisfied. Then*

$$\begin{cases} \big|Z_n(t) - Z_0(t)\big| \leqslant U_0(t), \\ \big|Z_{n+k}(t) - Z_k(t)\big| \leqslant U_k(t), \end{cases} \qquad \begin{cases} \big|Y_n(t) - Y_0(t)\big| \leqslant W_0(t), \\ \big|Y_{n+k}(t) - Y_k(t)\big| \leqslant W_k(t) \end{cases} \tag{17}$$

for $t \in J$ *and* $n = 0, 1, \ldots,$ *where*

$$Z_{n+1}(t) = \mathcal{F}(t, Z_n, Y_n; \eta), \qquad Y_{n+1}(t) = \mathcal{G}(t, Z_n, Y_n; \eta)$$

with $Z_0 \in C(J, \mathbb{R}^p)$, $Y_0 \in C(J, \mathbb{R}^q)$.

Proof. Note that

$$\begin{aligned}
\left|Z_1(t) - Z_0(t)\right| &\leqslant R_2(\eta) \leqslant H(t, \eta) \leqslant U_0(t), \\
\left|Y_1(t) - Y_0(t)\right| &\leqslant r_2(\eta) \leqslant \left[(I - Q)(I - Q)^{-1} + Q(I - Q)^{-1}\right] r_2(\eta) \\
&= (I - Q)^{-1} r_2(\eta) \leqslant W_0(t).
\end{aligned}$$

If we assume that $|Z_k(t) - Z_0(t)| \leqslant U_0(t)$, $|Y_k(t) - Y_0(t)| \leqslant W_0(t)$, $t \in J$ for some $k \geqslant 1$, then we see that

$$\begin{aligned}
&\left|Z_{k+1}(t) - Z_0(t)\right| \\
&\quad \leqslant \left|\mathcal{F}(t, Z_k, Y_k; \eta) - \mathcal{F}(t, Z_0, Y_0; \eta)\right| + R_2(\eta) \\
&\quad \leqslant K\Omega\big(\alpha(t), U_0\big) + L \max_{[0,\beta(t)]} \Lambda(s, U_0) \\
&\qquad + M(I - Q)^{-1} P \max_{[0,\mu(\gamma(t))]} \Lambda(s, U_0) \\
&\qquad + N(I - Q)^{-1} P \max_{[0,\delta(t)]} \max_{[0,\mu(s)]} \Lambda(\tau, U_0) \\
&\qquad + (M + N)(I - Q)^{-1} r_2(\eta) + R_2(\eta) \\
&\quad = (\Omega_1 U_0)(t) + H(t, \eta) = U_0(t), \\
&\left|Y_{k+1}(t) - Y_0(t)\right| \\
&\quad \leqslant \left|\mathcal{G}(t, Z_k, Y_k; \eta) - \mathcal{G}(t, Z_0, Y_0; \eta)\right| + r_2(\eta) \\
&\quad \leqslant P \max_{[0,\mu(t)]} \Lambda(s, U_0) + Q(I - Q)^{-1}\Big[P \max_{[0,\mu(t)]} \Lambda(s, U_0) + r_2(\eta)\Big] + r_2(\eta) \\
&\quad = \left[(I - Q)(I - Q)^{-1} + Q(I - Q)^{-1}\right]\Big[P \max_{[0,\mu(t)]} \Lambda(s, U_0) + r_2(\eta)\Big] \\
&\quad = W_0(t).
\end{aligned}$$

Hence, by mathematical induction, we have the assertion of this lemma. It ends the proof. □

Lemma 5 follows

Theorem 4. *Assume that all assumptions of Lemma* 4 *are satisfied. Then, for every* $\eta \in \mathbb{R}^p$, *the pair* $\{Z_n, Y_n\}$ *converges to the unique solution* $(\bar{Z}, \bar{Y})$ *of problem* (13), *so* $Z_n(t) \to \bar{Z}(t)$, $Y_n(t) \to \bar{Y}(t)$ *on* J *if* $n \to \infty$, *and for* $t \in J$ *we have the error estimates*

$$\left|Z_n(t) - \bar{Z}(t)\right| \leqslant U_n(t), \qquad \left|Y_n(t) - \bar{Y}(t)\right| \leqslant W_n(t), \qquad n = 0, 1, \ldots.$$

Moreover, $(\bar{X}, \bar{Y})$ *with* $\bar{X}(t) = \eta + \int_0^t \bar{Z}(s)\,ds$, $t \in J$ *is the solution of problem* (12) *with condition* (2) *iff*

$$\int_0^T \bar{Z}(s)\,ds = T S(\eta).$$

Remark 2. Note that Assumption H_3 holds if we assume that $\rho(W) < 1$, where

$$W = 2\max_{t\in J}\left\{K\alpha(t)\left[1 - \frac{\alpha(t)}{T}\right] + L\max_{[0,\beta(t)]} s\left(1 - \frac{s}{T}\right) + M(I-Q)^{-1}P\max_{[0,\mu(\gamma(t))]} s\left(1 - \frac{s}{T}\right) + N(I-Q)^{-1}P\max_{[0,\delta(t)]}\max_{0,\mu(s)]}\tau\left(1 - \frac{\tau}{T}\right)\right\}.$$

The matrix W can be obtained in the same way as in Remark 1. Note that condition $\rho(W) < 1$ can be replaced by $\rho(\bar{W}) < 1$, where

$$\bar{W} = 2K\max_{t\in J}\alpha(t)\left[1 - \frac{\alpha(t)}{T}\right] + \frac{T}{2}\big[L + (M+N)(I-Q)^{-1}P\big].$$

Indeed, $\rho(W) < 1$ and $\rho(\bar{W}) < 1$ hold if we assume that $\|\bar{W}\| < 1$, where $\|\cdot\|$ denotes any norm of a matrix.

Similarly as before to find a solution $(\bar{Z}, \bar{Y})$ of problem (13) we can apply Seidel's method to use iterations:

$$\begin{cases} \tilde{Z}_{n+1}(t) = \mathcal{F}(t, \tilde{Z}_n, \tilde{Y}_n; \eta), \\ \tilde{Y}_{n+1}(t) = \mathcal{G}(t, \tilde{Z}_{n+1}, \tilde{Y}_n; \eta), \end{cases} \qquad \begin{cases} \bar{Y}_{n+1}(t) = \mathcal{G}(t, \bar{Z}_n, \bar{Y}_n; \eta), \\ \bar{Z}_{n+1}(t) = \mathcal{F}(t, \bar{Z}_n, \bar{Y}_{n+1}; \eta) \end{cases}$$

for $t \in J$, $n = 0, 1, \ldots$.

For $t \in J$, $n = 0, 1, \ldots$, we put

$$\begin{cases} \tilde{U}_0(t) = U_0(t), \\ \tilde{U}_{n+1}(t) = \bar{\Lambda}(t, \tilde{U}_n, \tilde{W}_n), \end{cases}$$

$$\begin{cases} \tilde{W}_0(t) = W_0(t), \\ \tilde{W}_{n+1}(t) = P\max_{[0,\mu(t)]}\Lambda(s, \tilde{U}_{n+1}) + Q\tilde{W}_n(t), \end{cases}$$

$$\begin{cases} \bar{W}_0(t) = W_0(t), \\ \bar{W}_{n+1}(t) = P\max_{[0,\mu(t)]}\Lambda(s, \bar{U}_n) + Q\bar{W}_n(t), \end{cases}$$

$$\begin{cases} \bar{U}_0(t) = U_0(t), \\ \bar{U}_{n+1}(t) = \bar{\Lambda}(t, \bar{U}_n, \bar{W}_{n+1}). \end{cases}$$

Note that under Assumptions H_1 and H_3 we have $\tilde{U}_n(t) \leqslant U_n(t)$, $\bar{U}_n(t) \leqslant U_n(t)$, $\tilde{W}_n(t) \leqslant W_n(t)$, $\bar{W}_n(t) \leqslant W_n(t)$ on J and $\tilde{U}_n \to 0$, $\bar{U}_n \to 0$, $\tilde{W}_n \to 0$, $\bar{W}_n \to 0$ on J too.

Using the method of mathematical induction we are able to prove the following

Theorem 5. *Let all assumptions of Lemma* 4 *be satisfied. Let* $\bar{Z}_0(t) = \tilde{Z}_0(t) = Z_0(t)$, $\bar{Y}_0(t) = \tilde{Y}_0(t) = Y_0(t)$, $t \in J$. *Then the results of Theorem* 4 *hold and* $\tilde{Z}_n(t) \to \bar{Z}(t)$, $\bar{Z}_n(t) \to \bar{Z}(t)$, $\tilde{Y}_n(t) \to \bar{Y}(t)$, $\bar{Y}_n(t) \to \bar{Y}(t)$ *on* J *if* $n \to \infty$.

Moreover we have the error estimates

$$\begin{cases} \left|\bar{Z}_n(t) - \bar{Z}(t)\right| \leqslant \bar{U}_n(t), \\ \left|\tilde{Z}_n(t) - \bar{Z}(t)\right| \leqslant \tilde{U}_n(t), \end{cases} \qquad \begin{cases} \left|\bar{Y}_n(t) - \bar{Y}(t)\right| \leqslant \bar{W}_n(t), \\ \left|\tilde{Y}_n(t) - \bar{Y}(t)\right| \leqslant \tilde{W}_n(t) \end{cases}$$

for $t \in J$, $n = 0, 1, \ldots.$

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