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On the theory and numerical simulation of acoustic and heat modes interaction in a liquid with bubbles: acoustic quasi-solitons

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Abstract

The projecting technique is used to split one-dimensional disturbances into the rightwards progressive, leftwards progressive and stationary modes. Three independent modes are specified by set of orthogonal projectors. The system of coupled nonlinear Korteweg–de Vries (KdV) type equations for the three modes is derived and analysed. The non-singular perturbations method is used to show that for the case of one mode initial excitation the evolution of this mode is approximated by KdV–MKdV (Modified KdV) equation. A version of the numerical Lax–Wendroff (LW) scheme is applied to compute evolution described by the system. Analytical and numerical evaluations are compared. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction. Separating of the modes and coupled nonlinear equations

The idea of field presentation as a combination of physically specific components is common in the wave theory [1]. For weak nonlinear dynamics of essentially one-dimensional evolution the idea to separate waves according to direction of propagation appeared in the pioneering papers by

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Korteweg and de Vries [2], who used a method of slowly varying time. Using this method, a problem of nonlinear dynamics may be solved effectively but it only permits derivation of evolution equations for a single mode. That is a result of asymmetric choice of variables: either $t + r/c$, μt or $t - r/c$, μt and the use of the small parameter μ [3] (t is time, and r, c denote space coordinate and wave velocity, respectively). The problem of nonlinear interaction of non-collinear acoustic waves is discussed in [4] and later papers of the authors, where the stationary component is ignored.

Indeed, there are three basic types of motion relating to eigenvectors of a linear problem. It is no reason to neglect one of them in nonlinear dynamics problems.

It seems to be more effective to separate modes on the level of the basic system of equations [5] and to include a stationary mode at the same level of description. We show how to do it in a vector form by considering a complete set of eigenspaces of certain well-defined operator. A procedure of an overall field separating by matrix projectors that correspond to these subspaces is then mathematically unique. By the projection procedure the main system is transformed to a system of coupled nonlinear equations. The procedure is algorithmic: one should not take care about presentation of every mode, but apply corresponding projectors to the basic system of equations. Generally, the modes splitting by projectors is often used in quantum mechanics, but this technique is not common for gas and liquid flow problems. One can also iterate by small parameter inside the general (nonlinear) evolution operator, leading to the consequence of more and more exact equations [6].

Recently, this method was applied to gas dynamics in a stratified atmosphere and to the dynamics of viscous gases and liquids [7]. Here a hydrodynamic problem of an acoustic pulse propagating in bubbly liquid is investigated. We follow the basic ideas of Foldy [8] who started with individual scatters (gas bubbles) and involved subsequent ensemble averaging in analysis of dynamics of a bubbly liquid. The presence of bubbles causes dispersion of the medium and it is expected to yield interesting features of dynamics like soliton solutions.

We present:

matrix projectors for liquid with weak dispersion caused by the presence of bubbles;

the derivation of coupled KdV system in a compressible liquid with bubbles (both for directed and stationary components);

analytical stationary solution for the case of one dominant mode;

numerical simulations of the system of KdV type coupled equations based on the LW scheme and comparison of analytical and numerical solutions.

2. Dynamics of a compressible liquid with bubbles: basic equations

The dynamics of an incompressible bubbly liquid was originally studied by van Wijngaarden. A KdV equation for a progressive wave in the case of self-action only has been derived in [9]. Bubble dynamics law is included in the basic system of equations. The pioneer in this area was Rayleigh who studied bubble dynamics in an incompressible liquid. Note that recent numerical investigations have taken into account of the processes of heat transfer and vaporization [10]. Formulae were also modified for the surrounding compressible liquid [11].

Let us consider in this section a mixture of compressible liquid and a perfect gas. We assume that: all bubbles are of the same radii at equilibrium, there is no heat and mass transfer between liquid and gas. The characteristic wavelength is much more than a bubble radius, so that the mixture can be treated as homogeneous continuum.

All values, relating to liquid phase are marked by index ld, with gas – by index g, with mixture – by m. It is assumed also, that pressure in the mixture is the same as in the liquid phase [9,11]. Background values are marked by additional zero. Disturbed values are primed. Use of mass concentration of gas x is preferable than volume α used in [9,11]. A mass concentration is a constant in contrast to volume one that needs an additional equation. The only condition we add, x being a constant, is equivalent to an assumption of liquid and gas having the same velocity. An assumption of homogeneous continuum implies also the continuity of medium so that the laws of conservation may be written on in the differential form with continuous coefficients of the equations. So, we find the solutions over class of continuous (with all required derivatives) functions. The basic hydrodynamic system is

$$\begin{aligned}
&\partial v/\partial t + v\partial v/\partial r + (1/\rho_{\mathrm{m}})\partial p'_{\mathrm{ld}}/\partial r = 0,\\
&\partial p'_{\mathrm{ld}}/\partial t - c_{\mathrm{ld}}^2 \partial \rho'_{\mathrm{ld}}/\partial t - \left(c_{\mathrm{ld}}^2(\gamma_{\mathrm{ld}} - 1)/\rho_{\mathrm{ld0}}\right)\rho'_{\mathrm{ld}}\partial \rho_{\mathrm{ld}}/\partial t = 0, \qquad (2.1)\\
&\partial \rho'_{\mathrm{m}}/\partial t + \partial(\rho_{\mathrm{m}} v')/\partial r = 0.
\end{aligned}$$

Here ρ_{m} is the mixture density,

$$\rho_{\mathrm{m}} = \rho_{\mathrm{g}}\rho_{\mathrm{ld}}/(x\rho_{\mathrm{ld}} + (1-x)\rho_{\mathrm{g}}). \qquad (2.2)$$

The second equation of (2.1) expresses the dependence of pressure on density in an adiabatic process and consists of the second-order nonlinear terms only.

System (2.1) should be completed with the density and pressure relation for an adiabatic behavior of gas:

$$p_{\mathrm{g}}\rho_{\mathrm{g}}^{-\gamma_{\mathrm{g}}} = p_{\mathrm{g0}}\rho_{\mathrm{g0}}^{-\gamma_{\mathrm{g}}}, \qquad (2.3)$$

where γ_{g} is the specific heat ratio, $\gamma_{\mathrm{g}} = C_{\mathrm{p,g}}/C_{\mathrm{v,g}}$ and a single bubble mass conservation:

$$R^3\rho_{\mathrm{g}} = R_0^3\rho_{\mathrm{g0}}, \qquad (2.4)$$

with R being a bubble radius. Another completing equation is the Rayleigh equation for a bubble radius. The framework of continuum theory supposes that the relation between pressure in gas phase and liquid phase are the same as exists between the pressure in an isolated oscillating bubble in an infinite fluid and the pressure far away. The Rayleigh equation has been improved by Prosperetti following Keller [11,12] with an assumption of liquid compressibility:

$$R\frac{\partial^2 R}{\partial t^2} + \frac{3}{2}\left(\frac{\partial R}{\partial t}\right)^2 - \frac{1}{c_{\mathrm{ld}}}\left(R^2\frac{\partial^3 R}{\partial t^3} + 6R\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial t^2} + 2\left(\frac{\partial R}{\partial t}\right)^3\right) = \frac{p'_{\mathrm{g}} - p'_{\mathrm{ld}}}{\rho_{\mathrm{ld}}}. \qquad (2.5)$$

The surface tension and viscosity are left out of account. Eqs. (2.1)–(2.5) form a complete system that allows the second equation of (2.1) to be expressed in terms of ρ_{m}, p_{ld}, v. Let us introduce the dimensionless variables $v_*, p'_*, \rho'_*, r_*, t_*$: $v = \varepsilon c_{\mathrm{m}} v_*$, $p'_{\mathrm{ld}} = \varepsilon c_{\mathrm{m}}^2 \rho_{\mathrm{m0}} p'_*$, $\rho'_{\mathrm{m}} = \varepsilon\rho_{\mathrm{m0}}\rho'_*$, $r = r_*\lambda/\beta$; $t = t_*\lambda/(\beta c_{\mathrm{m}})$, λ being a characteristic scale of disturbance, c_{m} is the speed of sound in mixture derived from linear equations corresponding to (2.1)–(2.5)

$$\frac{1}{c_{\mathrm{m}}^2}=\frac{(1-\alpha_0)^2}{c_{\mathrm{ld}}^2}+\frac{\alpha_0(1-\alpha_0)\rho_{\mathrm{ld0}}}{\gamma_{\mathrm{g}}p_{\mathrm{g0}}}. \tag{2.6}$$

We will omit asterisks for dimensionless variables everywhere later. That is the final form of the system

$$\begin{aligned}
&\frac{\partial v}{\partial t}+\frac{\partial p'}{\partial r}=-\varepsilon\left(v\frac{\partial}{\partial r}v-\rho'\frac{\partial}{\partial r}p'\right)+\mathrm{O}(\varepsilon^2),\\
&\frac{\partial p'}{\partial t}+\frac{\partial v}{\partial r}-\beta^2\frac{\alpha_0(1-\alpha_0)R_0^2\rho_{\mathrm{d10}}^2c_{\mathrm{m}}^4}{3(\gamma_{\mathrm{g}}p_{\mathrm{g0}})^2\lambda^2}\frac{\partial^3p'}{\partial t^3}\\
&\quad=\varepsilon(1-\alpha_0)c_{\mathrm{m}}^2\left[-\frac{\gamma_{\mathrm{ld}}+1}{c_{\mathrm{ld}}^2}\rho'\frac{\partial}{\partial r}v-c_{\mathrm{m}}^2\frac{\alpha_0(1-\alpha_0)\rho_{10}^2(\gamma_{\mathrm{g}}+1)}{(\gamma_{\mathrm{g}}p_{\mathrm{g0}})^2}p'\frac{\partial}{\partial r}v-\frac{v(\partial/\partial r)\rho'-\rho'(\partial/\partial r)v}{(1-\alpha_0)c_{\mathrm{m}}^2}\right]\\
&\qquad+\mathrm{O}(\varepsilon\beta^2,\varepsilon^2),\\
&\frac{\partial\rho'}{\partial t}+\frac{\partial v}{\partial r}=-\varepsilon\left(v\frac{\partial}{\partial r}\rho'+\rho'\frac{\partial}{\partial r}v\right).
\end{aligned} \tag{2.7}$$

The third term in the second equation of (2.7) accounts for dispersion in the first approximation. Two small parameters are introduced: ε responsible for nonlinear processes and β responsible for dispersive ones. Here and further we consider the traditional coupling between the small parameters $\beta^2=\varepsilon$, that means a possibility of an equilibrium between nonlinearity and dispersion.

Generally, equations retaining the higher-order nonlinear terms as well as terms responsible for nonlinear dispersion may be written as well. There appears a first approach for solution of these equations [13]. It is clear, nevertheless, that in the frames of the perturbation theory the effects due to these corrections are small and do not lead to the essential changes of shape of solitons, though some peculiarities may arise. These peculiarities may concern with non-elasticity of solitons interaction, or slow destroying of soliton shape due to radiation.

For convenience of comparison with previous results of other authors the mass concentration is expressed through the initial value of volume one: $x=\alpha_0\rho_{\mathrm{g0}}/\rho_{\mathrm{m0}}$. The final formulae (2.7) have been somewhat simplified assuming that $\rho_{\mathrm{g0}}/\rho_{\mathrm{ld0}}\ll 1$ and $c_{\mathrm{g}}<c_{\mathrm{ld}}$. So, $\rho_{\mathrm{m0}}\approx(1-\alpha_0)\rho_{10}$ etc. Naturally, our formula go to incompressible liquid limit when $c_{\mathrm{ld}}\to\infty$.

3. Coupled KdV equations for the interacting modes in compressible liquid with bubbles

3.1. *Projectors in the linear problem with weak dispersion*

A linear analogue of (2.7) looks as $(\partial/\partial t)\psi+L\psi=0$, where

$$\psi=\begin{pmatrix}v\\p'\\\rho'\end{pmatrix}$$

is column of perturbations,

$$L = \begin{pmatrix} 0 & \partial/\partial r & 0 \\ \partial/\partial r + \varepsilon D\partial^3/\partial r^3 & 0 & 0 \\ \partial/\partial r & 0 & 0 \end{pmatrix}, \quad D = \frac{\alpha_0(1-\alpha_0)R_0^2\rho_{\mathrm{ld0}}^2 c_{\mathrm{m}}^4}{3(\gamma_{\mathrm{g}} p_{\mathrm{g0}})^2\lambda^2}.$$

An estimation $\partial^3 p'/\partial t^3 = -\partial^3 v/\partial r^3 + \varepsilon D(\partial^5 p'/\partial t^5 + \partial^5 p'/\partial t^3\partial r^2)$ following from the first and the second linearized equations of (2.7), was used to exclude time derivatives in L. Since the linear problem is investigated, we can find a solution as a sum of plane waves, or, the same, to use Fourier transformation to solve the problem strictly. Let us take the plane waves $\sim \exp(\mathrm{i}\omega t - \mathrm{i}kr)$ with amplitudes $V_k,\ P_k,\ R_k$. The eigenvalues of the corresponding system of equations for Fourier-transformed components are $\omega_{1,2} = \pm k\sqrt{1-\varepsilon Dk^2}$, $\omega_3 = 0$. These connections serve as the dispersion relations for the rightwards and leftwards progressive (acoustic) and the stationary (heat) modes. Note that there exists only one stationary mode in the one-dimensional flow that is called also the heat mode. In the general three-dimensional flow, there appear two stationary vorticity modes as well.

The corresponding eigenvectors are

$$\psi_{1,2} = (\pm(1-\varepsilon Dk^2)^{-1/2} \quad 1 \quad 1)^{\mathrm{T}} R_{k1,2}, \quad \psi_3 = (0 \quad 0 \quad 1)^{\mathrm{T}} R_{k3}.$$

The orthogonal projectors are evaluated immediately from these formulae, in r-space they look

$$P_{1,2} = \begin{pmatrix} 1/2 & \pm\left(1-\varepsilon(D/2)\partial^2/\partial r^2\right)/2 & 0 \\ \pm\left(1+\varepsilon(D/2)\partial^2/\partial r^2\right)/2 & 1/2 & 0 \\ \pm\left(1-\varepsilon(D/2)\partial^2/\partial r^2\right)/2 & \left(1-\varepsilon D\partial^2/\partial r^2\right)/2 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1+\varepsilon D\partial^2/\partial r^2 & 1 \end{pmatrix}.$$

Only the first two Taylor terms were retained for the problem with weak dispersion. Projectors do commute both with $\partial/\partial t$ and L. In order to obtain the rightwards progressive wave at any instant, for example, it is sufficient to apply P_1 to the total field: $P_1\psi = \psi_1$; ψ_1 means rightwards input, just the same for other inputs: $P_2\psi = \psi_2$; $P_3\psi = \psi_3$. The projectors satisfy general properties of orthogonal operators with an accuracy of order $\mathrm{O}(\varepsilon^2)$: $P_1 + P_2 + P_3 = \tilde{I},\ P_1P_1 = P_1, \ldots, P_1P_2 = P_1P_3 = \cdots = \tilde{0}$, where $\tilde{I},\ \tilde{0}$ are the unit and zero matrices, respectively.

3.2. *Coupled KdV equations*

Now, acting projectors on both sides of nonlinear system (2.7), one can write down the evolution equations for the certain type variables, where nonlinear part is presented by superposition of variables products of the second-order in agreement with (2.7). The type of mode (stationary or progressive) is defined strictly for the linear flow. In a problem with weak dispersion and nonlinearity we define modes as quasi-leftwards or rightwards progressive and quasi-stationary, assuming that the relations of perturbations for every mode take place. That is just a convenient choice of variables which yields further successful analytical solutions. For the disturbances of density of every mode the coupled equations are

$$\frac{\partial\rho'_n}{\partial t}+c_n\frac{\partial\rho'_n}{\partial r}+\frac{\varepsilon}{2}\sum_{i,m=1}^{3}\Phi^n_{i,m}\rho'_i\frac{\partial}{\partial r}\rho'_{\mathrm{m}}+\varepsilon\frac{M_n}{2}\frac{\partial^3}{\partial r^3}\rho'_n+\mathrm{O}(\varepsilon^2)=0,\quad n=1,2,3,\ \ c_{1,2}=\pm1,\ \ c_3=0$$

and the coefficients are determined in tables:

$\Phi^1_{i,m}$	m			$\Phi^2_{i,m}$	m			$\Phi^3_{i,m}$	m		
	1	2	3		1	2	3		1	2	3
i				i				i			
1	K	$-K$	1	1	K	$-(K-4)$	1	1	$2(2-K)$	$2(K-2)$	0
2	$K-4$	$-K$	-1	2	K	$-K$	-1	2	$2(2-K)$	$2(K-2)$	0
3	$T-2$	$-T$	0	3	T	$-(T-2)$	0	3	$2(2-T)$	$2(T-2)$	0

$$\begin{aligned}&M_1=-M_2=D,\quad M_3=0,\\&K=(1-\alpha_0)^3c_{\mathrm{m}}^4\left(\frac{\gamma_{\mathrm{ld}}+1}{c_{\mathrm{ld}}^4}+\frac{\alpha_0\rho_{\mathrm{ld0}}^2(\gamma_{\mathrm{g}}+1)}{(1-\alpha_0)(\gamma_{\mathrm{g}}p_{\mathrm{g0}})^2}\right),\\&T=(1-\alpha_0)c_{\mathrm{m}}^2(\gamma_{\mathrm{ld}}+1)/c_{\mathrm{ld}}^2.\end{aligned}\tag{3.1}$$

4. Nonlinear evolution of a dominant mode interacting with other modes. Solitons

Let us do some remarks concerning with general methods of approximate solution of nonlinear systems of equations with dispersion. The ordinary method of the perturbation theory, when solution of (3.1) is found in the form $\rho'_n=\sum_{i=0}^{\infty}\varepsilon^i\rho_n^{(i)}$ cannot be useful since a difference between the approximated and exact solutions increases over the time interval $\sim[0,\varepsilon^{-1}]$ continually. The so-called method of non-singular perturbations implies iterations inside operators of nonlinear system. We use a development of this approach similar to one from [6,14]. A solution of zero approximation of the system (3.1) $\rho_n^{(0)}(r,t)$ is taken to satisfy an equation

$$\partial\rho_n^{(0)}/\partial t+c_n\partial\rho_n^{(0)}/\partial r+\frac{\varepsilon}{2}\Phi^n_{n,n}\rho_n^{(0)}\partial\rho_n^{(0)}/\partial r+\frac{\varepsilon}{2}M_n\partial^3\rho_n^{(0)}/\partial r^3=0\tag{4.1}$$

with an initial condition $\rho_n^{(0)}(r,0)=\phi_n^{(0)}(r)$.

Then, the approximate solution that accounts of the interaction effects of the first order is

$$\rho_n^{(1)}(r,t)=\rho_n^{(0)}(r,t)-\frac{\varepsilon}{2}\int_0^t\sum_{m,k\neq n}\Phi^n_{m,k}\rho_{\mathrm{m}}^{(0)}\frac{\partial}{\partial r}\rho_k^{(0)}\Big|_{r-c_n(t-\tau),\tau}\,\mathrm{d}\tau+\mathrm{O}(\varepsilon^2).\tag{4.2}$$

Suppose that only one mode which has number i is excited initially, $\phi_n^{(0)}(r)=0$, $n\neq i$. Therefore, the zero-order approximation solution for all other modes is zero: $\rho_n^{(0)}(r,t)=0$, $n\neq i$. The first-order approximation solution of these modes is described by (4.2)

$$\rho_n^{(1)}(r,t)=\frac{\varepsilon\Phi^n_{i,i}}{4(c_i-c_n)}\left[\left(\rho_i^{(0)}(r,t)\right)^2-\left(\varphi_i^{(0)}(r-c_nt)\right)^2\right]+\mathrm{O}(\varepsilon^2).\tag{4.3}$$

The main additive in (4.3) that does influence on the dominant mode number i is the first term since it possesses the same velocity. We would say that a perturbation referring to the second additive in (4.3) "goes away" without leaving any sufficient trace, but the influence of the first one is resonant and stored over time. The inverse influence is apparently of the order of ε^2. A new evolution equation for the dominant mode number i follows from (4.2) and (4.3) and is KdV–MKdV type

$$\partial\rho_i^{(1)}/\partial t + c_i\partial\rho_i^{(1)}/\partial r + \frac{\varepsilon}{2}\left(\Phi_{i,i}^i\rho_i^{(1)} + \frac{\varepsilon}{2}A_i\left(\rho_i^{(1)}\right)^2\right)\partial\rho_i^{(1)}/\partial r + \frac{\varepsilon M_i}{2}\partial^3\rho_i^{(1)}/\partial r^3 = 0, \tag{4.4}$$

where $A_i = \sum_{m\neq i}(\Phi_{mi}^i/2 + \Phi_{im}^i)\Phi_{ii}^m/(c_i - c_\mathrm{m})$. Eq. (4.4) is explicitly integrated in the sense of Lax-pair existence [15] and its solitonary solution looks

$$\begin{aligned}
&\rho_i^{(1)} = \rho_i^{(1)}(r - r_0 - \tilde{c}_i t),\\
&\rho_i^{(1)}(\xi) = \varepsilon^{-1}\left[\frac{\Phi_{i,i}^i}{12(\tilde{c}_i - c_i)} + \sqrt{\left[\frac{\Phi_{i,i}^i}{12(\tilde{c}_i - c_i)}\right]^2 + \frac{A_i}{24(\tilde{c}_i - c_i)}}\cosh\left(\sqrt{\frac{2(\tilde{c}_i - c_i)}{\varepsilon M_i}}\xi\right)\right]^{-1},
\end{aligned} \tag{4.5}$$

$\tilde{c}_i$ is the soliton velocity. Indeed, there is a family of stationary solutions of KdV–MKdV equation that can be reduced to the pure MKdV equation by linear shift of ξ. We are interested here in the solution satisfying the boundary conditions $\lim_{\xi\to\pm\infty}\rho_i^{(1)}(\xi) = 0$, that refers to the physical problem of impulse evolution in an undisturbed background medium. A solution (4.5) satisfies to these boundary conditions. A solution of the zero-order equation (4.1) is a limit of (4.5) when $A_i \to 0$.

Returning to the dimensional (r,t)-variables, for rightwards progressive mode $Z_1 = p'_{\mathrm{ld1}}/p_{\mathrm{g0}}$, one obtains equations like (4.1) and (4.4) with coefficients

$$\begin{aligned}
&c_1 \to c_\mathrm{m}, \quad c_2 \to -c_\mathrm{m}, \quad \frac{\varepsilon M_1}{2} \to R_0^2 c_\mathrm{m}/(6\alpha_0(1-\alpha_0)); \quad \frac{\varepsilon\Phi_{11}^1}{2} \to (\gamma_\mathrm{g}+1)c_\mathrm{m}/(2\gamma_\mathrm{g});\\
&\frac{\varepsilon^2 A_1}{4} \to -((\gamma_\mathrm{g}+1)^2/16 + (\gamma_\mathrm{g}+1)/4)c_\mathrm{m}/\gamma_\mathrm{g}^2,
\end{aligned}$$

in agreement with formulae (3.1). Equation of zero order corresponds to the case of the first mode self-action and looks just the same that has been studied by van Wijngaarden [9].

5. Numerical calculations

For a system of equations (3.1) simulation a variant of the LW scheme [16] is used

$$\begin{aligned}
&\frac{\rho_{n,i}^{j+1} - \rho_{n,i}^{j}}{\tau} + c_n\frac{\rho_{n,i+1}^{j+1/2} - \rho_{n,i-1}^{j+1/2}}{2h} + \frac{\varepsilon}{2}\sum_{k,m=1}^{3}\Phi_{k,m}^n\rho_{k,i}^{j+1/2}\frac{\rho_{m,i+1}^{j+1/2} - \rho_{m,i-1}^{j+1/2}}{2h} + \frac{\varepsilon M_n}{2}\\
&\quad\times\frac{\rho_{n,i+2}^{j+1/2} - 2\rho_{n,i+1}^{j+1/2} + 2\rho_{n,i-1}^{j+1/2} - \rho_{n,i-2}^{j+1/2}}{(2h)^3} = 0.
\end{aligned} \tag{5.1}$$

Here n, k, m mark numbers of modes and j, i are discrete time and space co-ordinates.

One should complete Eq. (5.1) with values of intermediate layers $\rho_{n,i}^{j+1/2}$. To calculate these values, we use a scheme

$$\frac{\rho_{n,i}^{j+1/2}-\rho_{n,i}^{j}}{\tau/2}+c_n\frac{\rho_{n,i+1}^{j+1/2}-\rho_{n,i-1}^{j+1/2}}{2h}+\frac{\varepsilon}{2}\sum_{k,m=1}^{3}\Phi_{k,m}^{n}\rho_{k,i}^{j+1/2}\frac{\rho_{m,i+1}^{j+1/2}-\rho_{m,i-1}^{j+1/2}}{2h}+\frac{\varepsilon M_n}{2}$$
$$\times\frac{\rho_{n,i+2}^{j+1/2}-2\rho_{n,i+1}^{j+1/2}+2\rho_{n,i-1}^{j+1/2}-\rho_{n,i-2}^{j+1/2}}{(2h)^3}=0$$

and accept zero boundary conditions.

The problem with initial conditions is considered. Values $\alpha_0=0.01$, $R_0=0.0001$ m, $p_{g0}=10^5$ Pa, $\gamma_g=1.4$ are taken for calculations. To illustrate the method, we treated an incompressible liquid with bubbles, though the formulae are derived for more general case of a compressible one. Formally, the limit of incompressible liquid follows from (3.1) when $c_{ld}\to\infty$. The sound velocity in compressible liquid with bubbles is

$$c_m=\sqrt{\frac{\gamma_g p_{g0}}{\alpha_0(1-\alpha_0)\rho_{ld0}}}$$

(see (2.6)) and equals to 118 m/s for the data presented. During the calculations a space step $h=10^{-4}$ m was chosen, and the time step τ relating to h as well

$$\tau=\min\left(h/c_1,h^3/(R_0c_1^2/3\alpha_0(1-\alpha_0))\right)=0.25\times10^{-7}\ \text{s}.$$

The initial disturbance of the first mode is centered. We treat space interval from 0.0 to 0.2 m, so $r_0=0.1$ m. Theoretically, evolution of the second and third modes should be described by (4.3).

In general, the stability of soliton solutions of nonlinear equations is not well-studied. The coefficients of (3.1) are responsible for stability of soliton solutions. According to our numerical investigations, whether soliton is stable depends on the sign of A_i. For the concrete problem of a bubbly liquid flow we got the negative sign of A_1. In the numerical calculations we treated matrices leading to both negative and positive A_1 to be sure in the scheme and meaning that other media (and other physical problems) would lead to a positive multiplier.

In contrast to pure MKdV equation, KdV–MKdV one does not possesses a symmetry when solution changes its sign: if $u(r,t)$ is its solution, $-u(r,t)$ is not. The test desired is to take initial conditions for a soliton with the changed sign, so that stationary numerical evolution would not expected.

The following series of numerical calculations of the non-dimensional pressure fluctuation for the rightwards progressive dominant mode $Z_1=p'_{ld1}/p_{g0}$ were carried out:

- (I) with coefficients of (3.1), and initial conditions for the three modes:

$$\rho_1(r-r_0,0)=\varepsilon^{-1}\left[\frac{\Phi_{1,1}^{1}}{12(\tilde{c}_1-c_1)}+\sqrt{\left[\frac{\Phi_{1,1}^{1}}{12(\tilde{c}_1-c_1)}\right]^2+\frac{A_1}{24(\tilde{c}_1-c_1)}}\right.$$
$$\left.\times\cosh\left(\sqrt{\frac{2(\tilde{c}_1-c_1)}{\varepsilon M_1}}(r-r_0)\right)\right]^{-1},$$

$$\rho_2(r-r_0,0)=\rho_3(r-r_0,0)\equiv0;$$

- (II) with the changed matrix $\Phi^2_{i,m}$: we treat $\Phi\mathrm{new}^2_{i,m} = -\Phi^2_{i,m}$ for all i, m. That yields positive $A\mathrm{new}_1 = -A_1$, and initial conditions like that in series I;
- (III) with initial conditions:

$$\rho_1(r - r_0, 0) = -\varepsilon^{-1}\left[\frac{\Phi^1_{1,1}}{12(\tilde{c}_1 - c_1)} + \sqrt{\left[\frac{\Phi^1_{1,1}}{12(\tilde{c}_1 - c_1)}\right]^2 + \frac{A_1}{24(\tilde{c}_1 - c_1)}} \times \cosh\left(\sqrt{\frac{2(\tilde{c}_1 - c_1)}{\varepsilon M_1}}(r - r_0)\right)\right]^{-1},$$

$$\rho_2(r - r_0, 0) = \rho_3(r - r_0, 0) \equiv 0,$$

and with $\Phi^2_{i,m}$ like that in series II.

Since the initial condition for the first mode includes the function $\cosh(\xi)$, we just account it zero for arguments ξ with absolute value more than 8. Theoretically, evolution of leftwards and stationary modes is described by (4.3) and consists of two parts, one moving with the soliton velocity of the first mode and other moving with velocity of the corresponding mode.

6. Discussion

The soliton velocity defines an amplitude of soliton accordingly to (4.5) and its value is always greater than the sound velocity in pure liquid. A value $\tilde{c}_1 = 1.1c_1$ is treated in calculations of all series. All figures are plotted for non-dimensional fluctuations relating to the three modes $Z_1 = p'_{\mathrm{ld}1}/p_{\mathrm{g}0}$, $Z_2 = p'_{\mathrm{ld}2}/p_{\mathrm{g}0}$, $Z_3 = \rho'_3 c^2_{\mathrm{m}}/p_{\mathrm{g}0}$, we return also to the dimensional (r, t)-variables.

Fig. 1 presents evolution of the first mode Z_1, series I. The dispersion of this mode shows instability of the soliton solution. Nevertheless, the velocity of the numerically calculated disturbance corresponds to $\tilde{c}_1$. Time of evolution equals to 1.75×10^{-4} s. In spite of instability observed, both induced modes consist of two parts, one moving with the velocity of the dominant mode, and the second with its own velocity as follows from (4.3).

Figs. 2–4 represent the corresponding evolutions of the first, second and third modes, series II. Time of evolution is 2.25×10^{-4} s for every picture. For the series II, the soliton is stable, and an excellent correspondence of the first mode (Fig. 2) with the theoretical formula (4.5) is clear. The theoretical and calculated curves are undistinguishable. As for the second and third modes, they decay into two parts accordingly to (4.3), one having the soliton velocity of the first mode. Since these parts are proportional to $(\rho_1^{(0)}(\xi))^2$ (the soliton solution of zero approximation for the first mode) and have the same velocity as soliton of the first mode, there exists a complex soliton consisting of three components indeed. In another words, the soliton is formed by corresponding fluctuations of every mode that move with the same velocity without changing their shape.

Some other calculations with other coefficients of (3.1) were carried out showing that stability of soliton depends on sign of A_1 that is a combination of elements of all three matrices Φ. Figures of the series III are not presented: as expected, in this case stationary motion is not observed.

Systems of KdV type equations describe a wide variety of physical phenomena: dynamics of the atmosphere and the ocean, waves in plasma, refer to astrophysical problems and nonlinear

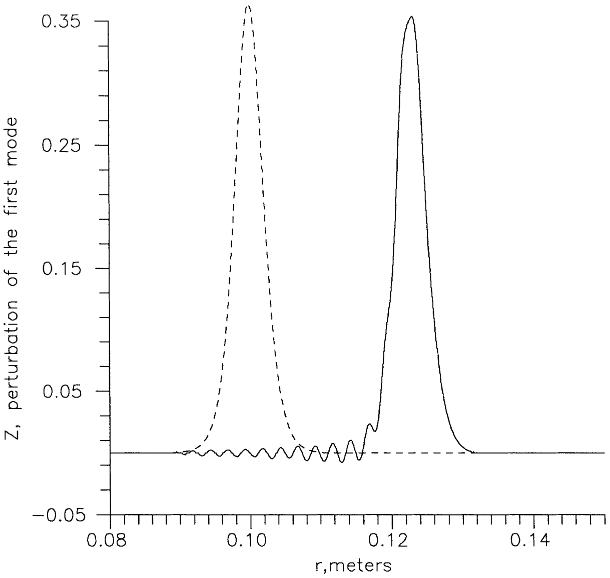

Fig. 1. Evolution of the first mode (solid), an initial disturbance is marked by dashed line. Series I.

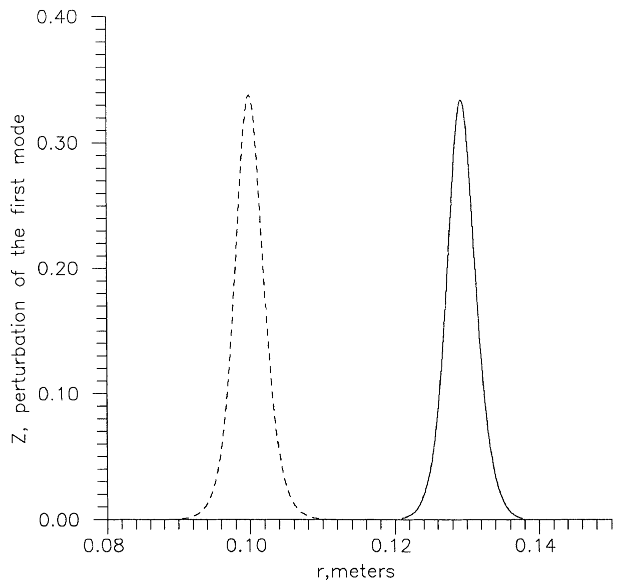

Fig. 2. Evolution of the first mode (solid), an initial disturbance is marked by dashed line. Series II.

transmission lines [6]. The KdV–MKdV equation applies to the long internal waves in two-layer liquid with density discontinuity at an interface. For the dynamics of bubbly liquid, soliton solutions (4.5) include coefficients depending on mass gas concentration and compressibility of both

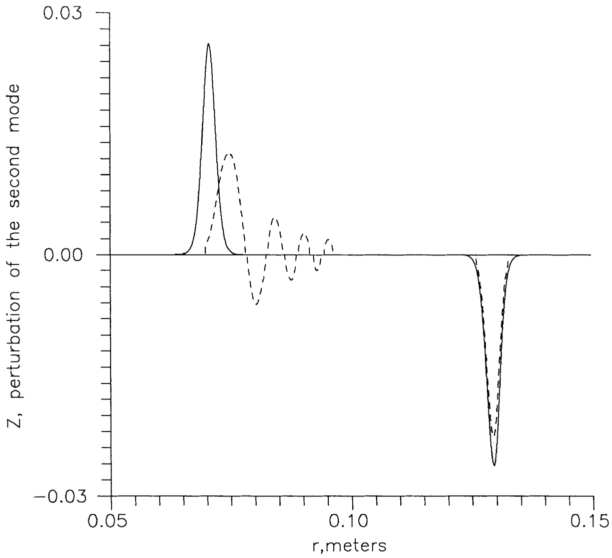

Fig. 3. Evolution of the second mode: theoretical (solid) and calculated (dashed). Series II.

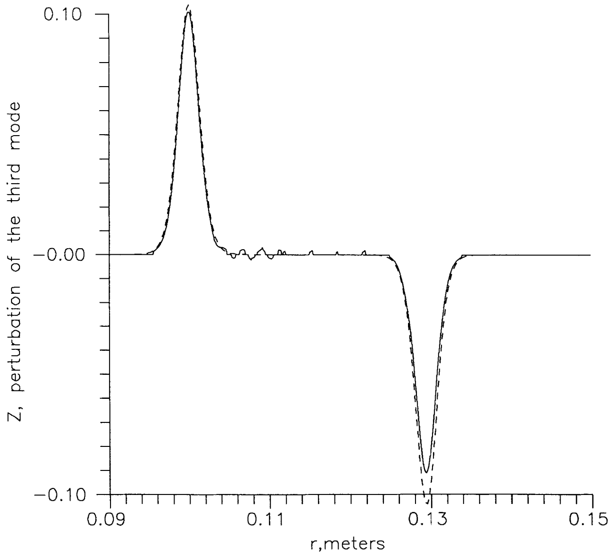

Fig. 4. Evolution of the third mode: theoretical (dashed) and calculated (solid). Series II.

liquid and gas, so there is a possibility to determine parameters of a mixture due to stationary solutions. Though the example of calculations relates to the limit of incompressible liquid ($c_{ld} \rightarrow \infty$), the basic formulae on modes interaction (3.1) are derived for the compressible liquid involving bubbles, as well as formula (4.5) describing an evolution stationary dominant mode. The results may be useful for mixtures, where drops of a non-soluble liquid are involved into a bulk liquid like gas bubbles [17].

Note that the very method of modes separating and further deriving of evolution equations may be applied to other problems of fluid dynamics. One-dimensional flow relates to the three modes, three-dimensional one relates to the five ones, with two vorticity modes involved. The different models of modes interaction including turbulence and viscous flow may be considered in the framework of the method.

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