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Stability Analysis of Two-Step Runge-Kutta Methods for Delay Differential Equations

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Abstract—We investigate stability properties of two-step Runge-Kutta methods with respect to the linear test equation

$$\begin{aligned} y'(t) &= ay(t) + by(t-\tau), \qquad && t \geq 0, \\ y(t) &= g(t), && t \in [-\tau, 0], \end{aligned}$$

where a and b are complex parameters. It is known that the solution $y(t)$ to this equation tends to zero as $t \to \infty$ if $|b| < -\operatorname{Re}(a)$. We will show that under some conditions this property is inherited by any A-stable two-step Runge-Kutta method applied on a constrained mesh to delay differential equations, i.e., that the corresponding method is P-stable. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Two-step Runge-Kutta methods, Delay differential equations, Absolute stability, P-stability.

1. INTRODUCTION

Consider the initial-value problem for delay differential equation (DDE)

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), \qquad && t \in [t_0, T], \\ y(t) &= g(t), && t \in [t_0 - \tau, t_0], \end{aligned} \tag{1.1}$$

$\tau > 0$, where g is a specified initial function and f satisfies conditions which guarantee the existence of the unique solution y to (1.1). Such conditions can be found in, for example, [1–4].

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Many numerical methods have been proposed for the numerical solution of problem (1.1); the recent surveys are given by Jackiewicz and Kwapisz [5] and Zennaro [6]. In this paper, we are concerned with two-step Runge-Kutta (TSRK) methods which can be defined by

$$\begin{aligned} K_{i+1}^j &= f\Bigg(t_i + c_j h, u_j y_h(t_{i-1}) + (1-u_j) y_h(t_i) \\ &\qquad + h \sum_{s=1}^{\nu} \left(a_{js} K_i^s + b_{js} K_{i+1}^s\right), y_h(t_i + c_j h - \tau)\Bigg), \\ y_h(t_i + \xi h) &= \eta(\xi) y_h(t_{i-1}) + (1 - \eta(\xi)) y_h(t_i) + h \sum_{s=1}^{\nu} \left(v_s(\xi) K_i^s + w_s(\xi) K_{i+1}^s\right), \end{aligned} \tag{1.2}$$

$i = 0, 1, \ldots, n-1$, $\xi \in (0, 1]$, $nh = T - t_0$, $t_i = t_0 + ih$. Here, ν is the number of stages, $y_h(t)$ is a continuous approximation to $y(t)$, K_{i+1}^j are approximations (possibly of low order) to $y'(t_i + c_j h)$, and $\eta(\xi)$, $v_i(\xi)$, and $w_i(\xi)$, $i = 1, 2, \ldots, \nu$, are polynomials such that $\eta(0) = 0$, $v_i(0) = 0$, and $w_i(0) = 0$. These methods form a subclass of general linear methods introduced by Butcher [7] and could be possibly also referred to as two-step hybrid methods. They generalize k-step collocation methods (with $k = 2$) for ordinary differential equations (ODEs) studied by Lie and Nørsett [8] and Lie [9], and TSRK methods for ODEs investigated by Byrne and Lambert [10], Renaut [11,12], Caira *et al.* [13], Jackiewicz *et al.* [14], Jackiewicz and Zennaro [15], and Jackiewicz *et al.* [16]. The discrete version of these methods (in somewhat different so-called Y-notation) was introduced by Jackiewicz and Tracogna [17] in the context of ODEs. They were further investigated by Jackiewicz and Tracogna [18], Butcher and Tracogna [19], Tracogna [20], Tracogna and Welfert [21], Jackiewicz and Vermiglio [22], Hairer and Wanner [23], and Bartoszewski and Jackiewicz [24]. The variable stepsize continuous TSRK methods for ODEs were investigated by Jackiewicz and Tracogna [25]; they result in the formulation (1.2) when applied with a constant stepsize h to the DDE (1.1). The A-stable TSRK methods have been constructed in [14,17,25].

Following [25], we will represent these methods by the following table of the coefficients:

$$\begin{array}{c|c|c} u & A & B \\ \hline \eta(\xi) & v^{\mathsf{T}}(\xi) & w^{\mathsf{T}}(\xi) \end{array} = \begin{array}{c|ccc|ccc} u_1 & a_{11} & \cdots & a_{1\nu} & b_{11} & \cdots & b_{1\nu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_\nu & a_{\nu 1} & \cdots & a_{\nu\nu} & b_{\nu 1} & \cdots & b_{\nu\nu} \\ \hline \eta(\xi) & v_1(\xi) & \cdots & v_\nu(\xi) & w_1(\xi) & \cdots & w_\nu(\xi) \end{array}.$$

We would like to stress that the methods (1.2) given by the above table of coefficients are more general than the methods studied in [8,9] (with $k = 2$) and in [10–16] since for all these methods the coefficient matrix A is identically equal to zero. As illustrated in [17], the presence of extra parameters a_{ij} in (1.2) makes it possible to construct high-order methods with relatively few stages, and we refer to [17,24,25] for specific examples.

Assuming that $h = \tau/m$ for some positive integer m we can approximate the delayed arguments $y_h(t_i + c_j h - \tau) = y_h(t_{i-m} + c_j h)$ by

$$y_h(t_{i-m} + c_j h) = \eta(c_j) y_h(t_{i-m-1}) + (1 - \eta(c_j)) y_h(t_{i-m}) + h \sum_{s=1}^{\nu} \left(v_s(c_j) K_{i-m}^s + w_s(c_j) K_{i-m+1}^s\right),$$

$j = 1, 2, \ldots, \nu$. Substituting this relation into (1.2) with $\xi = 1$, we obtain

$$\begin{aligned} K_{i+1}^j &= f\Bigg(t_i + c_j h, u_j y_{i-1} + (1-u_j) y_i + h \sum_{s=1}^{\nu} \left(a_{js} K_i^s + b_{js} K_{i+1}^s\right), \\ &\qquad \eta_j y_{i-m-1} + (1 - \eta_j) y_{i-m} + h \sum_{s=1}^{\nu} \left(\gamma_{js} K_{i-m}^s + \delta_{js} K_{i-m+1}^s\right)\Bigg), \\ y_{i+1} &= \eta y_{i-1} + (1-\eta) y_i + h \sum_{s=1}^{\nu} \left(v_s K_i^s + w_s K_{i+1}^s\right), \end{aligned} \tag{1.3}$$

$i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, \nu$, where $y_i = y_h(t_i)$, $\eta = \eta(1)$, $\eta_j = \eta(c_j)$, $\gamma_{js} = v_s(c_j)$, and $\delta_{js} = w_s(c_j)$.

It is the purpose of this paper to investigate stability properties of (1.3) with respect to the linear test equation

$$\begin{aligned} y'(t) &= ay(t) + by(t-\tau), \qquad && t \geq 0, \\ y(t) &= g(t), && t \in [-\tau, 0], \end{aligned} \tag{1.4}$$

where a and b are complex parameters. Stability properties of Runge-Kutta methods with respect to this test equation have been investigated by Koto [26,27], Zennaro [28], in 't Hout and Spijker [29], in 't Hout [30], Guglielmi [31], and Guglielmi and Hairer [32]. Stability properties of linear multistep methods for DDEs with respect to (1.4) have been investigated by Cryer [33], Bickart [34,35], Wiederholt [36], and Watanabe and Roth [37].

It was proved by Barwell [38] that the solution $y(t)$ to (1.4) tends to zero as $t \to \infty$ if

$$|b| < -\operatorname{Re}(a), \tag{1.5}$$

and it was proved by Zennaro [28] that if $y(t) \to 0$ as $t \to \infty$, then $|b| \leq -\operatorname{Re}(a)$. Hence, inequality (1.5) defines the interior of the region of asymptotic stability of equation (1.4). In this paper, we investigate under what conditions the asymptotic stability properties of this test equation are inherited by the numerical approximation to (1.4) by the TSRK method (1.3).

The approach of this paper is based mainly on the technique proposed by Zennaro [28] in the context of RK methods for DDEs. However, in spite of some similarities to the RK case there are also many important differences, and the extension of the approach of [28] to TSRK methods for DDEs is far from trivial.

The main result of the paper is Theorem 3.3 in Section 3, which states that under some technical conditions the TSRK method (1.2) for DDEs is P-stable if the underlying TSRK method for ODEs is A-stable.

2. PRELIMINARY RESULTS AND DEFINITIONS

Applying (1.3) to the test equation (1.4) with $h = \tau/m$, we obtain

$$\begin{aligned} K_{i+1}^j &= a\left(u_j y_{i-1} + (1-u_j)y_i + h\sum_{s=1}^{\nu}\left(a_{js}K_i^s + b_{js}K_{i+1}^j\right)\right) \\ &\quad + b\left(\eta_j y_{i-m-1} + (1-\eta_j)y_{i-m} + h\sum_{s=1}^{\nu}\left(\gamma_{js}K_{i-m}^s + \delta_{js}K_{i-m+1}^s\right)\right), \\ y_{i+1} &= \eta y_{i-1} + (1-\eta)y_i + h\sum_{s=1}^{\nu}\left(v_s K_i^s + w_s K_{i+1}^s\right), \end{aligned} \tag{2.1}$$

$i = 0, 1, \ldots$, $j = 1, 2, \ldots, \nu$. Putting $\alpha = ha$, $\beta = hb$, $\tilde{u} = [\eta_1, \ldots, \eta_\nu]^\top$, $\Gamma = [\gamma_{js}]_{j,s=1}^{\nu}$, $\Delta = [\delta_{js}]_{j,s=1}^{\nu}$, $K_i = [K_i^1, \ldots, K_i^\nu]^\top$, and $e = [1, \ldots, 1]^\top \in R^\nu$, equation (2.1) can be written as

$$\begin{aligned} hK_{i+1} &= \alpha(y_{i-1}u + y_i(e-u) + hAK_i + hBK_{i+1}) \\ &\quad + \beta\left(y_{i-m-1}\tilde{u} + y_{i-m}\left(e-\tilde{u}\right) + h\Gamma K_{i-m} + h\Delta K_{i-m-1}\right), \\ y_{i+1} &= \eta y_{i-1} + (1-\eta)y_i + v^\top hK_i + w^\top hK_{i+1}, \end{aligned} \tag{2.2}$$

$i = 0, 1, \ldots$. Denote by $\{y_i(m;\alpha,\beta)\}_{i=0}^{\infty}$ the solution to (2.2) with $h = \tau/m$. Following Zennaro [28], we introduce the following definition.

DEFINITION. *Method (1.3) is said to be stable for given* (α, β) *if the sequence* $\{y_i(m; \alpha, \beta)\}_{i=0}^{\infty}$ *tends to zero as* $t \to \infty$ *for any integer* $m \geq 1$. *The region* S_P *of stability of (1.3) is the set of all values* (α, β) *for which this method is stable. Method (1.3) is said to be P-stable if its region of stability* S_P *includes the set* $\{(\alpha, \beta) : |\beta| < -\operatorname{Re}(\alpha)\}$.

Observe that it follows from (1.5) that if the method (1.3) is P-stable, then the numerical solution $\{y_i(m; \alpha, \beta)\}_{i=0}^{\infty}$ defined by (2.2) tends to zero as $i \to \infty$ whenever the analytical solution $y(t)$ to (1.4) tends to zero as $t \to \infty$. Observe also that α and β depend on the stepsize h.

Assuming that the matrix $I - \alpha B$ is nonsingular we can compute hK_{i+1} from the first relation of (2.2), and substituting the resulting expression into the second relation of (2.2) leads to the following vector recurrence equation:

$$Y_{i+1} = LY_i + MY_{i-1} + NY_{i-m+1} + RY_{i-m} + SY_{i-m-1}, \tag{2.3}$$

$i = 0, 1, \ldots$, where $Y_i = [y_i, hK_i^{\top}]^{\top}$ and the matrices L, M, N, R, and S are defined by

$$\begin{aligned}
L &= \begin{bmatrix} 1 - \eta + \alpha w^{\top}(I - \alpha B)^{-1}(e - u) & v^{\top} + \alpha w^{\top}(I - \alpha B)^{-1}A \\ \alpha(I - \alpha B)^{-1}(e - u) & \alpha(I - \alpha B)^{-1}A \end{bmatrix}, \\
M &= \begin{bmatrix} \eta + \alpha w^{\top}(I - \alpha B)^{-1}u & 0 \\ \alpha(I - \alpha B)^{-1}u & 0 \end{bmatrix}, \\
N &= \begin{bmatrix} 0 & \beta w^{\top}(I - \alpha B)^{-1}\Delta \\ 0 & \beta(I - \alpha B)^{-1}\Delta \end{bmatrix}, \\
R &= \begin{bmatrix} \beta w^{\top}(I - \alpha B)^{-1}(e - \tilde{u}) & \beta w^{\top}(I - \alpha B)^{-1}\Gamma \\ \beta(I - \alpha B)^{-1}(e - \tilde{u}) & \beta(I - \alpha B)^{-1}\Gamma \end{bmatrix}, \\
S &= \begin{bmatrix} \beta w^{\top}(I - \alpha B)^{-1}\tilde{u} & 0 \\ \beta(I - \alpha B)^{-1}\tilde{u} & 0 \end{bmatrix}.
\end{aligned}$$

The characteristic equation of (2.3) is

$$\det\left(\lambda^{m+2}I - \lambda^{m+1}L - \lambda^m M - \lambda^2 N - \lambda R - S\right) = 0. \tag{2.4}$$

Observe that λ is a root of this equation if and only if there exists $x^* \in C^{\nu+1}$, $x^* \neq 0$, such that

$$\lambda^{m+2}x^* - \lambda^{m+1}Lx^* - \lambda^m Mx^* - \lambda^2 Nx^* - \lambda Rx^* - Sx^* = 0. \tag{2.5}$$

Put $x^* = [\rho, x^{\top}]^{\top}$, where $\rho \in C$ and $x \in C^{\nu}$, and assume that $\rho \neq 0$. This seems to happen for most TSRK methods. Then (2.5) is equivalent to

$$\begin{aligned}
&\lambda^{m+2} - \lambda^{m+1}\left(\left(1 - \eta + \alpha w^{\top}(I - \alpha B)^{-1}(e - u)\right)\right. \\
&\quad \left.+ \left(v^{\top} + \alpha w^{\top}(I - \alpha B)^{-1}A\right)x\right) - \lambda^m\left(\eta + \alpha w^{\top}(I - \alpha B)^{-1}u\right) \\
&\quad - \lambda^2\beta w^{\top}(I - \alpha B)^{-1}\Delta x - \lambda\left(\beta w^{\top}(I - \alpha B)^{-1}(e - \tilde{u})\right. \\
&\quad \left.+ \beta w^{\top}(I - \alpha B)^{-1}\Gamma x\right) - \beta w^{\top}(I - \alpha B)^{-1}\tilde{u} = 0,
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\lambda^{m+2}x - \lambda^{m+1}\left(\alpha(I - \alpha B)^{-1}(e - u) + \alpha(I - \alpha B)^{-1}Ax\right) \\
&\quad - \lambda^m\alpha(I - \alpha B)^{-1}u - \lambda^2\beta(I - \alpha B)^{-1}\Delta x \\
&\quad - \lambda\left(\beta(I - \alpha B)^{-1}(e - \tilde{u}) + \beta(I - \alpha B)^{-1}\Gamma x\right) \\
&\quad - \beta(I - \alpha B)^{-1}\tilde{u} = 0.
\end{aligned} \tag{2.7}$$

Consider also the following relation:

$$\lambda^2 w^{\top}x - \lambda^2 + \lambda\left(1 - \eta + v^{\top}x\right) + \eta = 0, \tag{2.8}$$

and the quadratic function

$$\lambda^2 - \lambda R_1\left(\alpha, \frac{\beta}{\lambda^m}\right) - R_2\left(\alpha, \frac{\beta}{\lambda^m}\right) = 0, \tag{2.9}$$

where the rational functions

$$R_1(\alpha, z) = 1 - \eta + v^\top x + w^\top (I - \alpha B - z\Delta)^{-1} \left(\alpha(e-u) + z\left(e - \tilde{u}\right) + \alpha A x + z\Gamma x\right)$$

and

$$R_2(\alpha, z) = \eta + w^\top (I - \alpha B - z\Delta)^{-1} \left(\alpha u + z\tilde{u}\right)$$

are well defined if the matrix $I - \alpha B - z\Delta$ is nonsingular. We have the following theorem.

THEOREM 2.1. *Assume that the matrix $I-\alpha B-z\Delta$ is singular if and only if z is a pole of $R_1(\alpha, z)$ or $R_2(\alpha, z)$. Then $\lambda \neq 0$ such that β/λ^m is not a pole of $R_1(\alpha, z)$, and $R_2(\alpha, z)$ satisfies (2.6) and (2.7) if and only if λ is a root of (2.8) and (2.9).*

PROOF OF NECESSITY. Assume that $\lambda \neq 0$ such that β/λ^m is not a pole of $R_1(\alpha, z)$, and $R_2(\alpha, z)$ satisfies (2.6) and (2.7). Then it follows from the assumptions of the theorem that the matrix $I - \alpha B - (\beta/\lambda^m)\Delta$ is nonsingular. Consider the relation

$$\lambda^{m+2}(I - \alpha B)x - \lambda^{m+1}(\alpha(e-u) + \alpha A x) - \lambda^m \alpha u - \lambda^2 \beta \Delta x - \lambda\left(\beta\left(e - \tilde{u}\right) + \beta\Gamma x\right) - \beta\tilde{u} = 0,$$

which is equivalent to (2.7). Dividing this relation by λ^m and then multiplying it by $w^\top(I - \alpha B - (\beta/\lambda^m)\Delta)^{-1}$ we obtain

$$\begin{aligned}\lambda^2 w^\top x - \lambda w^\top \left(I - \alpha B - \frac{\beta}{\lambda^m}\Delta\right)^{-1} \times \left(\alpha(e-u) + \frac{\beta}{\lambda^m}\left(e - \tilde{u}\right) + \alpha A x + \frac{\beta}{\lambda^m}\Gamma x\right) \\ - w^\top \left(I - \alpha B - \frac{\beta}{\lambda^m}\Delta\right)^{-1} \left(\alpha u + \frac{\beta}{\lambda^m}\tilde{u}\right) = 0.\end{aligned} \tag{2.10}$$

To compute $\lambda^2 w^\top x$, we multiply (2.7) by $w^\top$ and then compare the resulting relation with equation (2.6). This leads to

$$\lambda^2 w^\top x = \lambda^2 - \lambda\left(1 - \eta + v^\top x\right) - \eta,$$

which is equivalent to (2.8). Substituting this equation into (2.10), we obtain (2.9). This completes the proof of necessity. ∎

PROOF OF SUFFICIENCY. From (2.9), it follows that

$$\begin{aligned}&\lambda^2 - \lambda\left(1 - \eta + v^\top x\right) - \eta \\ &= \lambda w^\top \left(I - \alpha B - \frac{\beta}{\lambda^m}\Delta\right)^{-1} \left(\alpha(e-u) + \frac{\beta}{\lambda^m}\left(e - \tilde{u}\right) + \alpha A x + \frac{\beta}{\lambda^m}\Gamma x\right) \\ &\quad - w^\top \left(I - \alpha B - \frac{\beta}{\lambda^m}\Delta\right)^{-1} \left(\alpha u + \frac{\beta}{\lambda^m}\tilde{u}\right).\end{aligned}$$

Using relation (2.8) it can be verified that (2.7) is satisfied. Multiplying (2.8) by λ^m and (2.7) by $w^\top$ and comparing the resulting relations, we then obtain (2.6). This completes the proof of sufficiency. ∎

REMARK. Assumptions of the Theorem 2.1 are not very restrictive and seem to be satisfied for most TSRK methods of practical interest. For example, they are satisfied for the TSRK methods constructed in Section 4.

3. P-STABILITY PROPERTIES OF TSRK METHODS

Introducing the notation by $z = \beta/\lambda^m$, relation (2.9) can be rewritten in the form

$$\lambda^2 - \lambda R_1(\alpha, z) - R_2(\alpha, z) = 0.$$

Now define the set

$$\Gamma_\alpha = \{z : \text{one of the roots of } \lambda^2 - \lambda R_1(\alpha, z) - R_2(\alpha, z) = 0 \text{ is on the unit circle and the other is inside or on the unit circle}\}$$

and the quantity

$$\sigma_\alpha = \min_{z \in \Gamma_\alpha} |z|.$$

We have the following theorem.

THEOREM 3.1. *Assume that $z = 0$ is not a pole of $R_1(\alpha, z)$ and $R_2(\alpha, z)$. Assume also that both roots of*

$$\lambda^2 - \lambda R_1(\alpha, 0) - R_2(\alpha, 0) = 0 \tag{3.1}$$

are inside of the unit circle and that $|\beta| < \sigma_\alpha$. Then all roots of

$$\lambda^2 - \lambda R_1\left(\alpha, \frac{\beta}{\lambda^m}\right) - R_2\left(\alpha, \frac{\beta}{\lambda^m}\right) = 0 \tag{3.2}$$

are inside of the unit circle for all integers $m \geq 1$.

PROOF. Since the functions $R_1(\alpha, 0)$ and $R_2(\alpha, 0)$ are well defined and the roots of (3.1) are inside of the unit circle, it follows that the roots of

$$\lambda^2 - \lambda R_1(\alpha, z) - R_1(\alpha, z) = 0 \tag{3.3}$$

are also inside of the unit circle for all $|z| < \sigma_\alpha$ (see Figure 1 for a geometrical explanation).

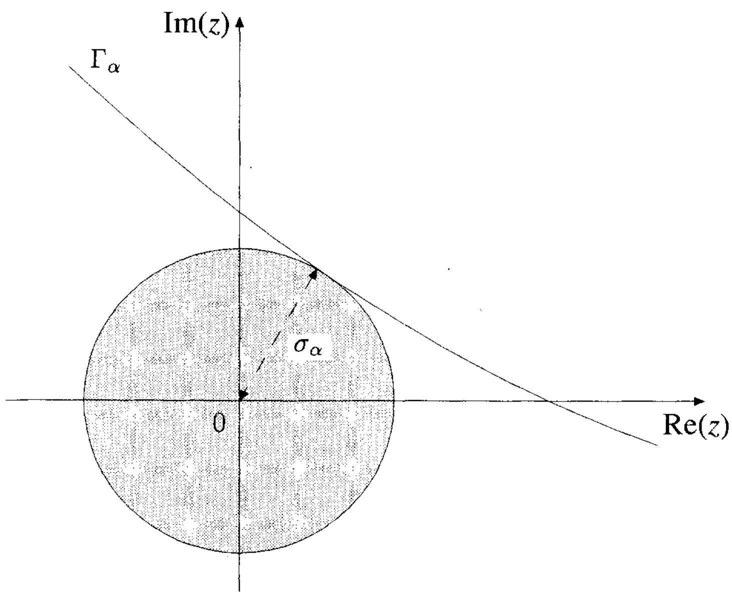

Figure 1. Geometrical interpretation of σ_α.

Assume now to the contrary that equation (3.2) has a root λ such that $|\lambda| \geq 1$. Then

$$\left|\frac{\beta}{\lambda^m}\right| \leq |\beta| < \sigma_\alpha,$$

for all integers $m \geq 1$. This means that for $z = \beta/\lambda^m$ we have $|z| < \sigma_\alpha$ and one root of equation (3.3) has modulus greater than or equal to one. This contradiction concludes the proof of the theorem. ∎

The next theorem gives a characterization of σ_α for some TSRK methods.

THEOREM 3.2. *Assume that $\tilde{u} = u$, $\Gamma = A$, and $\Delta = B$. Then*

$$\sigma_\alpha = \operatorname{dist}(\alpha, \partial S_A),$$

for every $\alpha \in S_A$, where S_A is the stability region of the TSRK method for ODEs.

PROOF. We have

$$R_1(\alpha, z) = 1 - \eta + v^\top x + w^\top (I - (\alpha + z)B)^{-1} \times ((\alpha + z)(e - u) + (\alpha + z)Ax) = R_1(\alpha + z, 0),$$

and

$$R_2(\alpha, z) = \eta + (\alpha + z)w^\top (I - (\alpha + z)B)^{-1} u = R_2(\alpha + z, 0).$$

By the definition of S_A, it follows that one of the roots of

$$\lambda^2 - \lambda R_1(\alpha, z) - R_2(\alpha, z) = \lambda^2 - \lambda R_1(\alpha + z, 0) - R_2(\alpha + z, 0) = 0$$

has modulus equal to one and the second has modulus less than or equal to one if and only if $\alpha + z \in \partial S_A$. It follows from the definition of Γ_α that $z \in \Gamma_\alpha$ if and only if $\alpha + z \in \partial S_A$ (see Figure 2).

Hence,

$$\sigma_\alpha = \inf_{z \in \Gamma_\alpha} |z| = \inf_{\alpha + z \in \partial S_A} |(\alpha + z) - \alpha| = \operatorname{dist}(\alpha, \partial S_A),$$

which completes the proof of the theorem. ∎

We are now ready to formulate and prove the main result of this paper.

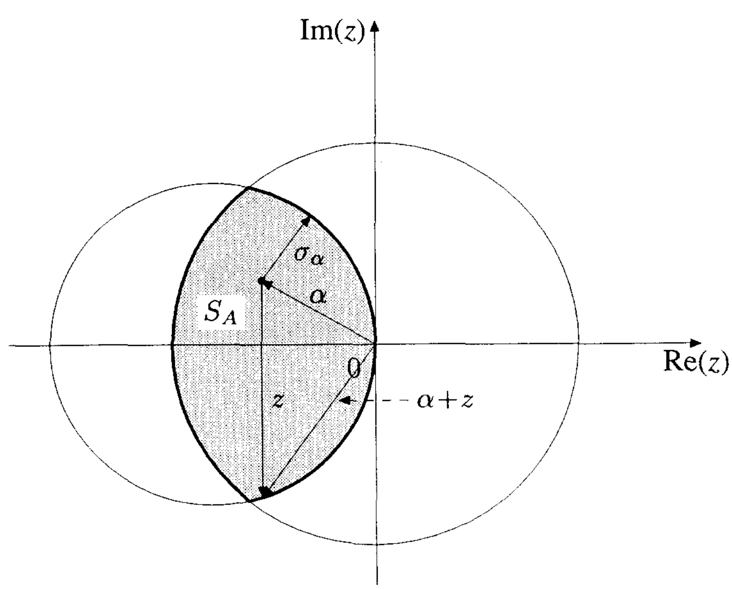

Figure 2. Region S_A and ∂S_A.

THEOREM 3.3. *Assume that the TSRK method for ODEs such that $\tilde{u} = u$, $\Gamma = A$, and $\Delta = B$ is A-stable. Then the corresponding TSRK method (1.2) for DDEs is P-stable.*

PROOF. We have

$$S_P = \left\{(\alpha, \beta) : \text{all roots of } \lambda^2 - \lambda R_1\left(\alpha, \frac{\beta}{\lambda^m}\right) - R_2\left(\alpha, \frac{\beta}{\lambda^m}\right) = 0 \right.$$
$$\left. \text{are inside of the unit circle for } m \geq 1\right\}.$$

It follows from Theorem 3.1 that S'_P given by

$$S'_P = \{(\alpha, \beta) : \text{both roots of } \lambda^2 - \lambda R_1(\alpha, 0) - R_2(\alpha, 0) = 0$$
$$\text{are inside of the unit circle and } |\beta| < \sigma_\alpha\}$$

satisfies

$$S'_P \subset S_P. \tag{3.4}$$

We have to show that

$$\{(\alpha, \beta) : \mathrm{Re}(\alpha) < 0 \text{ and } |\beta| < -\mathrm{Re}(\alpha)\} \subset S_P$$

(this is the definition of P-stability). Take (α, β) such that $\mathrm{Re}(\alpha) < 0$ and $|\beta| < -\mathrm{Re}(\alpha)$. Since the TSRK method for ODEs is A-stable we have

$$\{\alpha : \mathrm{Re}(\alpha) < 0\} \subset S_A.$$

This means that both roots of equation (3.1) are inside of the unit circle. It follows from Theorem 3.2 and A-stability that

$$\sigma_\alpha = \mathrm{dist}\,(\alpha, \partial S_A) \geq -\mathrm{Re}(\alpha).$$

Hence,

$$|\beta| < -\mathrm{Re}(\alpha) \leq \sigma_\alpha.$$

This means that $(\alpha, \beta) \in S'_P$, and as a consequence of (3.4) it follows that $(\alpha, \beta) \in S_P$. This completes the proof. ∎

4. EXAMPLES OF P-STABLE TSRK METHODS FOR DDES

In this section, we will illustrate by two examples how to construct TSRK methods for DDEs which are P-stable. We start with the TSRK method for ODEs given by

$$\begin{array}{c|c|c} u & A & B \\ \hline \eta & v^{\top} & w^{\top} \end{array} = \begin{array}{c|cc|cc} 0 & 0.164905 & -0.198522 & 0.75 & 0 \\ 0 & -0.210337 & -1.07121 & 2.70983 & 0.75 \\ \hline 0 & 0.128015 & -0.284316 & 1.12692 & 0.0293846 \end{array},$$

which, as demonstrated in [17], is A-stable and has order $p = 4$ and stage order $q = 4$. We compute next the continuous weights

$$v_i(\xi) = \xi\,\left(v_{i,1} + v_{i,2}\,\xi + v_{i,3}\,\xi^2\right), \qquad i = 1, 2,$$

such that $v_i(1) = v_i$ and $\Gamma = A$, and the continuous weights

$$w_i(\xi) = \xi\,\left(w_{i,1} + w_{i,2}\,\xi + w_{i,3}\,\xi^2\right), \qquad i = 1, 2,$$

such that $w_i(1) = w_i$ and $\Delta = B$, where Γ and Δ are the matrices defined in Section 1. This leads to the linear systems of equations

$$v_i(c_j) = a_{ji}, \qquad v_i(1) = v_i,$$

and

$$w_i(c_j) = b_{ji}, \qquad w_i(1) = w_i,$$

$i, j = 1, 2$, for the coefficients v_{kl} and w_{kl}, $k = 1, 2$, $l = 1, 2, 3$. The solutions to the above systems correspond to the continuous weights $v_i(\xi)$ and $w_i(\xi)$ given by

$$\begin{aligned} v_1(\xi) &= \xi\,\left(0.57142 - 0.559464\,\xi + 0.116058\,\xi^2\right), \\ v_2(\xi) &= \xi\,\left(-0.33277 + 0.151525\,\xi - 0.10307\,\xi^2\right), \end{aligned}$$

and

$$\begin{aligned} w_1(\xi) &= \xi\,\left(0.755371 + 0.49649\,\xi - 0.124943\,\xi^2\right), \\ w_2(\xi) &= \xi\,\left(0.00598078 - 0.0885513\,\xi + 0.111955\,\xi^2\right). \end{aligned}$$

It can be verified using Theorem 3 in [25] that the resulting TSRK method for DDEs is convergent with uniform order $p = 4$; i.e., there is no superconvergence at the gridpoints. We think this is a very desirable property of the method since we can generate dense output without any additional cost. The method constructed above is also P-stable as can be easily verified using Theorem 3.3.

Theorem 3.3 can also be used to construct TSRK methods of the type considered in [8,9,14], i.e., with $A \equiv 0$. For example, starting with the A-stable method of order $p = 4$ for ODEs

$$\begin{array}{c|c|c} u & A & B \\ \hline \eta & v^{\mathsf T} & w^{\mathsf T} \end{array} = \begin{array}{c|cc|cc} 0 & 0 & 0 & 0.527766 & 1.06598 \\ 0 & 0 & 0 & -0.0679367 & 0.47028 \\ \hline 0.462626 & 0.592719 & 0.457494 & 0.0203561 & 0.392057 \end{array}$$

constructed in [14] and proceeding similarly as in the previous example, we obtain the P-stable method of uniform order $p = 4$ for DDEs. The coefficients $\eta(\xi)$, $v_i(\xi)$, and $w_i(\xi)$ of this method are given by

$$\begin{aligned} \eta(\xi) &= \xi\,\left(-0.835974 + 2.60229\,\xi - 1.30369\,\xi^2\right), \\ v_1(\xi) &= \xi\,\left(-1.07105 + 3.33407\,\xi - 1.6703\,\xi^2\right), \\ v_2(\xi) &= \xi\,\left(-0.8267 + 2.57343\,\xi - 1.28923\,\xi^2\right), \end{aligned}$$

and

$$\begin{aligned} w_1(\xi) &= \xi\,\left(-0.226373 + 0.073107\,\xi + 0.173622\,\xi^2\right), \\ w_2(\xi) &= \xi\,\left(2.28815 - 3.37831\,\xi + 1.48221\,\xi^2\right). \end{aligned}$$

It can be verified by direct computations that both methods constructed in this section satisfy assumptions of Theorem 2.1. For example, the matrix $I - (\alpha + z)B$ corresponding to the latter method is singular at $z = 1.55644 - \alpha \pm 0.834543\,i$ which are also the poles of the function $R_1(\alpha, z)$.

A systematic approach to the construction and implementation of highly stable TSRK methods for DDEs will be the subject of future work.

REFERENCES

1. J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, (1977).
2. V.B. Kolmanovskii and V.R. Nosov, *Stability of Functional Differential Equations*, Academic Press, London, (1986).
3. Y. Kuang, *Delay Differential Equations with Applications in Population Biology*, Academic Press, Boston, (1993).
4. Z. Kamont and M. Kwapisz, On the Cauchy problem for differential-delay equations in a Banach space, *Math. Nachr.* **74**, 173–190, (1976).
5. Z. Jackiewicz and M. Kwapisz, The numerical solution of functional differential equations, a survey, *Mat. Stos.* **33**, 57–78, (1991).
6. M. Zennaro, Delay differential equations: Theory and numerics, In *Theory and Numerics of Ordinary and Partial Differential Equations*, Advances in Numerical Analysis, Volume 4, (Edited by M. Ainsworth *et al.*), pp. 291–333, Clarendon Press, Oxford, (1995).
7. J.C. Butcher, *The Numerical Analysis of Ordinary Differential Equations*, John Wiley & Sons, Chichester, New York, (1987).
8. I. Lie and S.P. Nørsett, Superconvergence of multistep collocation, *Math. Comput.* **52**, 65–79, (1989).
9. I. Lie, Local error estimation for multistep collocation methods, *BIT* **30**, 126–144, (1990).
10. G.D. Byrne and R.J. Lambert, Pseudo-Runge-Kutta methods involving two points, *J. Assoc. Comput. Mach.* **13**, 114–123, (1966).
11. R.A. Renaut-Williamson, Numerical solution of hyperbolic partial differential equations, Ph.D. Thesis, Cambridge University, Cambridge, U.K., (1985).
12. R. Renaut, Two-step Runge-Kutta methods and hyperbolic partial differential equations, *Math. Comput.* **90**, 563–579, (1990).
13. R. Caira, C. Costabile and F. Costabile, A class of pseudo Runge-Kutta methods, *BIT* **30**, 642–649, (1990).
14. Z. Jackiewicz, R. Renaut and A. Feldstein, Two-step Runge-Kutta methods, *SIAM J. Numer. Anal.* **28**, 1165–1182, (1991).
15. Z. Jackiewicz and M. Zennaro, Variable-stepsize explicit two-step Runge-Kutta methods, *Math. Comput.* **59**, 421–438, (1992).
16. Z. Jackiewicz, R. Renaut and M. Zennaro, Explicit two-step Runge-Kutta methods, *Appl. Math.* **40**, 433–456, (1995).
17. Z. Jackiewicz and S. Tracogna, A general class of two-step Runge-Kutta methods for ordinary differential equations, *SIAM J. Numer. Anal.* **32**, 1390–1427, (1995).
18. Z. Jackiewicz and S. Tracogna, A representation formula for two-step Runge-Kutta methods, In *Proc. of the Second Hellenic European Conference on Mathematics and Informatics*, (Edited by E.A. Lipitakis), pp. 111–120, Hellenic Mathematical Society, Athens, (1994).
19. J.C. Butcher and S. Tracogna, Order conditions for two-step Runge-Kutta methods, *Appl. Numer. Math.* **24**, 351–364, (1997).
20. S. Tracogna, Implementation of two-step Runge-Kutta methods for ordinary differential equations, *J. Comp. Appl. Math.* **76**, 113–136, (1997).
21. S. Tracogna and B. Welfert, Two-step Runge-Kutta: Theory and practice, *BIT* **40**, 775–799, (2000).
22. Z. Jackiewicz and R. Vermiglio, General linear methods with external stages of different orders, *BIT* **36**, 688–712, (1996).
23. E. Hairer and G. Wanner, Order conditions for general two-step Runge-Kutta methods, *SIAM J. Numer. Anal.* **34**, 2087–2089, (1997).
24. Z. Bartoszewski and Z. Jackiewicz, Construction of two-step Runge-Kutta methods of high order for ordinary differential equations, *Numerical Algorithms* **18**, 51–70, (1998).
25. Z. Jackiewicz and S. Tracogna, Variable stepsize continuous two-step Runge-Kutta methods for ordinary differential equations, *Numerical Algorithms* **12**, 347–368, (1996).
26. T. Koto, A stability property of A-stable natural Runge-Kutta methods for systems of delay-differential equations, *BIT* **34**, 262–267, (1994).
27. T. Koto, A criterion for P-stability properties of Runge-Kutta methods, *BIT* **38**, 737–750, (1998).
28. M. Zennaro, P-stability properties of Runge-Kutta methods for delay differential equations, *Numer. Math.* **49**, 305–318, (1986).
29. K.J. in 't Hout and M.N. Spijker, Stability analysis of numerical methods for delay differential equations, *Numer. Math.* **59**, 807–814, (1991).
30. K.J. in 't Hout, The stability of a class of Runge-Kutta methods for delay differential equations, *Appl. Numer. Math.* **9**, 347–355, (1992).
31. N. Guglielmi, On the asymptotic stability properties of Runge-Kutta methods for delay differential equations, *Numer. Math.* **77**, 467–485, (1997).
32. N. Guglielmi and E. Hairer, Order stars and stability for delay differential equations, *Numer. Math.* **83**, 371–383, (1999).
33. C.W. Cryer, Highly stable multistep methods for retarded differential equations, *SIAM J. Numer. Anal.* **11**, 788–797, (1974).
34. T.A. Bickart, P-stable and $P[\alpha,\beta]$-stable integration/interpolation methods in the solution of retarded differential-difference equations, *BIT* **22**, 464–476, (1982).

35. T.A. Bickart, F-stable and $F[\alpha, \beta]$-stable integration/interpolation methods in the solution of retarded differential-difference equations, *J. Comp. Appl. Math.* **8**, 279–284, (1982).
36. L.F. Wiederholt, Stability of multistep formulas for delay differential equations, *Math. Comp.* **30**, 283–290, (1976).
37. D.S. Watanabe and M.G. Roth, The stability of difference formulas for delay differential equations, *SIAM J. Numer. Anal.* **22**, 132–145, (1985).
38. V.K. Barwell, On the asymptotic behavior of the solution of a differential difference equation, *Utilitas Math.* **6**, 189–194, (1974).