

Local properties of the solution set of the operator equation in Banach spaces in a neighbourhood of a bifurcation point

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Abstract: In this work we study the problem of the existence of bifurcation in the solution set of the equation $F(x, \lambda) = 0$, where $F: X \times R^k \rightarrow Y$ is a C^2 -smooth operator, X and Y are Banach spaces such that $X \subset Y$. Moreover, there is given a scalar product $\langle \cdot, \cdot \rangle: Y \times Y \rightarrow R^1$ that is continuous with respect to the norms in X and Y . We show that under some conditions there is bifurcation at a point $(0, \lambda_0) \in X \times R^k$ and we describe the solution set of the studied equation in a small neighbourhood of this point.

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1 Introduction

Let X and Y be real Banach spaces and $F: X \times R^k \rightarrow Y$ be a continuous map. Suppose that the equation

$$F(x, \lambda) = 0, \tag{1}$$

where $x \in X$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in R^k$, possesses the trivial family of solutions

$$\Lambda = \{(0, \lambda) \in X \times R^k : \lambda \in R^k\}.$$

A point (x, λ) such that $F(x, \lambda) = 0$ and $x \neq 0$ is called a nontrivial solution of (1). Bifurcation theory is concerned in part with the existence of nontrivial solutions of (1)

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in a small neighbourhood of Λ . A point $(0, \lambda_0) \in \Lambda$ is called a bifurcation point of (1) if every neighbourhood of $(0, \lambda_0)$ contains a nontrivial solution of (1).

Methods of bifurcation theory are often applied in mathematical physics. Let us mention some applications to mechanics of elastic constructions and hydromechanics. In [3] the buckling of a thin elastic plate subject to arbitrary forces and stresses along its boundary is studied by the use of a perturbation theory and a variational method. In [6] to describe a deformation of the minimal interface of two fluids in a vertical tube in a gravitational field one applies a method based on the Crandall-Rabinowitz bifurcation theorem and representation theory. In [9] the buckling of a thin elastic rectangular plate simply supported on sides is studied numerically. In [14] the forms of equilibrium of a thin elastic circular plate lying on an elastic foundation and simply supported along its boundary are investigated via a finite-dimensional reduction and the Krasnosielski bifurcation theorem. Finally, in [16] the buckling of a homogeneous finite beam clamped at the edges to an elastic foundation is studied by a method of a key function due to Sapronov.

Assume that F is C^1 -smooth. For every $\lambda \in R^k$, let $F'_x(0, \lambda): X \rightarrow Y$ denote the Fréchet derivative of F with respect to x at $(0, \lambda)$. Let $N(\lambda) = \ker F'_x(0, \lambda)$ and $R(\lambda) = \text{im } F'_x(0, \lambda)$. It is easily seen that if $F'_x(0, \lambda_0): X \rightarrow Y$ is a Fredholm operator of index zero then a necessary condition for $(0, \lambda_0)$ to be a bifurcation point of (1) is

$$\dim N(\lambda_0) > 0.$$

In this paper we investigate bifurcation at $(0, \lambda_0)$ when X is a linear subspace of Y , there is given a scalar product $\langle \cdot, \cdot \rangle: Y \times Y \rightarrow R^1$ that is continuous with respect to the norms in X and Y , and F is a C^p -smooth map ($p \geq 2$) that satisfies the following conditions:

- (I₁) $F(0, \lambda) = 0$ for every $\lambda \in R^k$,
- (I₂) $\dim N(\lambda_0) = 1$,
- (I₃) $N(\lambda_0) \perp R(\lambda_0)$,
- (I₄) $F'_x(0, \lambda_0): X \rightarrow Y$ is a Fredholm operator of index 0.

Our aim is to prove a theorem on bifurcation at $(0, \lambda_0)$ and a local structure of a solution set of equation (1) in a neighbourhood of a bifurcation point. We apply a kind of finite-dimensional reduction of Liapunov-Schmidt type and the implicit function theorem. We are motivated by applications in mathematical physics [6], [14], [16] in which the problems under considerations (see above) are described by (1) with F that satisfies (I₁)–(I₄) and is a variational gradient. The main results of this work are Theorem 3.7 and its variational version: Conclusion 3.10. Theorem 3.7 is an analogue of the Crandall-Rabinowitz bifurcation theorem (see [17], [21]). However, our theorem is formulated in terms of a finite-dimensional reduction and in a variational case it seems to be easier to apply. Conclusion 3.10 is well adapted to a class of nonlinear problems of elasticity described by the von Kármán equations with one or a few parameters (see [4], [15], [16]) in the case when the linearization space is one-dimensional. An example is given in Section 4.

The paper is divided into four sections. In Section 2 we introduce some notions and we briefly sketch a scheme of finite-dimensional reduction. Section 3 is devoted to the study of bifurcation and local properties of the solution set of (1) near a bifurcation point. In Section 4 some applications of our results are indicated.

In practice it suffices to suppose that F is defined in a neighbourhood of $(0, \lambda_0)$ in $X \times R^k$, but we want to omit inessential details.

2 Finite-dimensional reduction

In this section we describe a kind of a finite-dimensional reduction of the Liapunov-Schmidt type. The scheme we present is adapted from [21] (see also [10], [11], [17], [20]).

From now on we assume that $X \subset Y$ are real Banach spaces with a scalar product $\langle \cdot, \cdot \rangle: Y \times Y \rightarrow R^1$ that is continuous with respect to the norms in X and Y . The norms in X and Y can be defined independently of the scalar product $\langle \cdot, \cdot \rangle$, and the norm in X does not have to be induced by the norm in Y . In particular, X and Y with $\langle \cdot, \cdot \rangle$ may be Hilbert spaces. Let $F: X \times R^k \rightarrow Y$ be a C^p -smooth map, where $p \geq 1$, satisfying conditions: (I_1) , (I_3) , (I_4) and (I'_2) $\dim N(\lambda_0) = n \neq 0$.

The aim is to show that under the above assumptions the problem of bifurcation for equation (1) at the point $(0, \lambda_0) \in X \times R^k$ is reducible to the problem of bifurcation for the equation $\varphi(\xi, \lambda) = 0$ with a certain map $\varphi: S \subset R^n \times R^k \rightarrow R^n$ at the point $(0, \lambda_0) \in R^n \times R^k$. The reader may find the proofs of the propositions given below in [13] and [15].

Proposition 2.1. For every $\lambda \in R^k$ the following equality holds:

$$Y = R(\lambda) \oplus N(\lambda). \tag{2}$$

Let $G: X \times R^n \times R^k \rightarrow Y$ be a map defined by

$$G(x, \xi, \lambda) = F(x, \lambda) + \sum_{i=1}^n (\xi_i - \langle x, e_i \rangle) e_i, \tag{3}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ and $\{e_1, e_2, \dots, e_n\}$ is a fixed orthonormal base of $N(\lambda_0)$.

Proposition 2.2. The operator $G'_x(0, 0, \lambda_0): X \rightarrow Y$ is an isomorphism.

It is easily seen that G is C^p -smooth. From the implicit function theorem it follows that there exist two open sets $U \subset X$ and $S \subset R^n \times R^k$ such that $0 \in U$, $(0, \lambda_0) \in S$ and the solution set of the equation

$$G(x, \xi, \lambda) = 0 \tag{4}$$

in $U \times S$ is a graph of a certain C^p -smooth function $x: S \rightarrow U$ such that $x(0, \lambda_0) = 0$. Moreover, it is obvious that $x(0, \lambda) = 0$ for all $(0, \lambda) \in S$, because $G(0, 0, \lambda) = 0$. Let

$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n): S \rightarrow R^n$ be defined by coordinates as follows:

$$\varphi_i(\xi, \lambda) = \xi_i - \langle x(\xi, \lambda), e_i \rangle, \quad i = 1, \dots, n. \quad (5)$$

Proposition 2.3. $(0, \lambda_0) \in \Lambda$ is a bifurcation point of equation (1) if and only if $(0, \lambda_0) \in S$ is a bifurcation point of equation

$$\varphi(\xi, \lambda) = 0. \quad (6)$$

3 Theorem on bifurcation

In this section our main results are stated and proved.

Let $F: X \times R^k \rightarrow Y$ be a C^p -smooth map, $p \geq 2$, satisfying conditions (I_1) – (I_4) (see p. 562). Fix $e \in N(\lambda_0)$ such that $\langle e, e \rangle = 1$ and denote $\lambda_0 = (\lambda_{01}, \lambda_{02}, \dots, \lambda_{0k})$. We will describe the solution set of (1) in terms of the finite-dimensional reduction. Notice that now in the formulas of maps G and φ there are $n = 1$ and $e_1 = e$. Differentiating the equality $G(x(\xi, \lambda), \xi, \lambda) = 0$ with respect to ξ at $(0, \lambda_0)$ we obtain

$$F'_x(0, \lambda_0)x'_\xi(0, \lambda_0) + (1 - \langle x'_\xi(0, \lambda_0), e \rangle)e = 0.$$

From this and (I_3) it follows that $x'_\xi(0, \lambda_0) = e$.

Theorem 3.1. There exist open sets $V_0 \subset X$ and $V \subset R^k$ such that $(0, \lambda_0) \in V_0 \times V$ and for every $(x, \lambda) \in V_0 \times V$ we have $F(x, \lambda) = 0$ if and only if $(\langle x, e \rangle, \lambda) \in S$ and $x = x(\langle x, e \rangle, \lambda)$.

Proof 3.2. Suppose contrary to our claim, that there are no open sets $V_0 \subset X$ and $V \subset R^k$ with the above properties. Then for every $n \in N$ there exists $(x_n, \lambda_n) \in X \times R^k$ such that $\|x_n\|_X \leq \frac{1}{n}$, $|\lambda_n - \lambda_0| \leq \frac{1}{n}$ and one of the following conditions is satisfied:

1. $F(x_n, \lambda_n) = 0$ and $(\langle x_n, e \rangle, \lambda_n) \notin S$,
2. $F(x_n, \lambda_n) = 0$, $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n \neq x(\langle x_n, e \rangle, \lambda_n)$,
3. $F(x_n, \lambda_n) \neq 0$, $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n = x(\langle x_n, e \rangle, \lambda_n)$.

If $(\langle x_n, e \rangle, \lambda_n) \in S$ and $x_n = x(\langle x_n, e \rangle, \lambda_n)$ then $F(x_n, \lambda_n) = F(x(\langle x_n, e \rangle, \lambda_n), \lambda_n) + (\langle x_n, e \rangle - \langle x(\langle x_n, e \rangle, \lambda_n), e \rangle)e = G(x(\langle x_n, e \rangle, \lambda_n), \langle x_n, e \rangle, \lambda_n) = 0$.

Since $x_n \rightarrow 0$ in X , there exists $n_0 \in N$ such that $x_n \in U$ for every $n \geq n_0$. If for some $n \geq n_0$ we have $F(x_n, \lambda_n) = 0$ and $(\langle x_n, e \rangle, \lambda_n) \in S$ then $0 = F(x_n, \lambda_n) + (\langle x_n, e \rangle - \langle x_n, e \rangle)e = G(x_n, \langle x_n, e \rangle, \lambda_n)$, and so $x_n = x(\langle x_n, e \rangle, \lambda_n)$.

Since $(\langle x_n, e \rangle, \lambda_n) \rightarrow (0, \lambda_0) \in S$ there exists $n_1 \in N$ such that $(\langle x_n, e \rangle, \lambda_n) \in S$ for every $n \geq n_1$ — a contradiction.

The equality $\langle G(x(\xi, \lambda), \xi, \lambda), e \rangle = 0$ implies

$$\varphi(\xi, \lambda) = -\langle F(x(\xi, \lambda), \lambda), e \rangle. \quad (7)$$

From (7) we obtain

$$\varphi'_\xi(\xi, \lambda) = -\langle F'_x(x(\xi, \lambda), \lambda)x'_\xi(\xi, \lambda), e \rangle,$$

and hence $\varphi'_\xi(0, \lambda_0) = 0$. Moreover, since $\varphi(0, \lambda) = 0$ for every $(0, \lambda) \in S$ we have $\varphi_{\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m}}^{(m)}(0, \lambda_0) = 0$ for all $i_1, i_2, \dots, i_m \in \{1, 2, \dots, k\}$ and $m \in \mathbb{N}$. In order to get our main result we have to assume that there is $i \in \{1, 2, \dots, k\}$ such that $\varphi''_{\xi \lambda_i}(0, \lambda_0) \neq 0$. There is no loss of generality if we assume

$$(I_5) \quad \varphi''_{\xi \lambda_k}(0, \lambda_0) \neq 0.$$

From now on, if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k-1}, \lambda_k) \in R^k$, $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_{k-1}) \in R^{k-1}$ we will write $\lambda = (\lambda', \lambda_k)$.

Proposition 3.3. There exist open sets $\Omega_0 \subset R^1 \times R^{k-1}$ and $\Omega \subset R^1$ such that $(0, \lambda'_0) \in \Omega_0$, $\lambda_{0k} \in \Omega$ and there exists a C^p -smooth map $f: \Omega_0 \rightarrow \Omega$ that satisfies the following conditions:

- (1) $f(0, \lambda'_0) = \lambda_{0k}$,
- (2) for every $(\xi, \lambda') \in \Omega_0$ and $\lambda_k \in \Omega$ we have $\varphi(\xi, \lambda', \lambda_k) = 0$ if and only if $\xi = 0$ or $\lambda_k = f(\xi, \lambda')$.

Proof 3.4. Let $\psi: S \rightarrow R^1$ be a function defined by

$$\psi(\xi, \lambda) = \int_0^1 \varphi'_\xi(t\xi, \lambda) dt. \tag{8}$$

Observe that we have

$$\varphi(\xi, \lambda) = \xi \psi(\xi, \lambda). \tag{9}$$

Hence $\varphi(\xi, \lambda) = 0$ only if $\xi = 0$ or $\psi(\xi, \lambda) = 0$. From (8) and (I_5) it follows that $\psi(0, \lambda_0) = \varphi'_\xi(0, \lambda_0) = 0$ and $\psi'_{\lambda_k}(0, \lambda_0) = \varphi''_{\xi \lambda_k}(0, \lambda_0) \neq 0$. Applying the implicit function theorem we get the desired claim.

Let $B_r(\lambda'_0)$ denote a ball in R^{k-1} of radius r centered at λ'_0 , and $B_\delta(0)$ a ball in X of radius δ centered at 0.

Theorem 3.5. Let $f: \Omega_0 \rightarrow \Omega$ be a function of Proposition 3.3 and $r > 0$ be a number such that $(-r, r) \times B_r(\lambda'_0) \subset \Omega_0$. There exist open sets $\tilde{V}_0 \subset X$ and $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$ such that $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$ and for every $(x, \lambda) \in \tilde{V}_0 \times \tilde{V}$ we have $F(x, \lambda) = 0$ if and only if $x = 0$ or there exists $\xi \in (-r, r)$ such that $\lambda_k = f(\xi, \lambda')$ and $x = x(\xi, \lambda', f(\xi, \lambda'))$.

Proof 3.6. There exists $\delta \in (0, r)$ such that for every $x \in X$ if $\|x\|_X < \delta$ then $|\langle x, e \rangle| < r$. Let $\tilde{V}_0 = V_0 \cap B_\delta(0)$ and $\tilde{V} = V \cap (B_r(\lambda'_0) \times \Omega)$, where $V_0 \subset X$ and $V \subset R^k$ are open sets of Theorem 3.1. Take $(x, \lambda) \in \tilde{V}_0 \times \tilde{V}$.

(\Rightarrow) By Theorem 3.1, if $F(x, \lambda) = 0$ then $(\langle x, e \rangle, \lambda) \in S$ and $x = x(\langle x, e \rangle, \lambda)$, which gives $\varphi(\langle x, e \rangle, \lambda) = 0$. From Proposition 3.3 it follows that $\langle x, e \rangle = 0$ or $\lambda_k = f(\langle x, e \rangle, \lambda')$. If $\langle x, e \rangle = 0$ then $x = x(0, \lambda) = 0$. If $\lambda_k = f(\langle x, e \rangle, \lambda')$ then $x = x(\langle x, e \rangle, \lambda', f(\langle x, e \rangle, \lambda'))$.

(\Leftarrow) Assume now that $x = 0$ or there exists $\xi \in (-r, r)$ such that $\lambda_k = f(\xi, \lambda')$ and $x = x(\xi, \lambda', f(\xi, \lambda'))$. In the first case, $F(x, \lambda) = F(0, \lambda) = 0$. In the second case, by Proposition 3.3, we have $\varphi(\xi, \lambda) = 0$, and hence $F(x, \lambda) = F(x, \lambda) + \varphi(\xi, \lambda)e = F(x, \lambda) + (\xi - \langle x(\xi, \lambda'), e \rangle)e = F(x, \lambda) + (\xi - \langle x, e \rangle)e = G(x, \xi, \lambda) = 0$.

We are now in a position to prove our main result.

Theorem 3.7. Under assumptions (I_1) - (I_5) , the solution set of equation (1) in a certain neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two sets: Λ and Ξ . The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi, \lambda'), \lambda', f(\xi, \lambda')) : |\xi| < r, |\lambda' - \lambda'_0| < r\},$$

where \hat{x} and f are C^p -smooth functions such that $\hat{x}(0, \lambda'_0) = 0, f(0, \lambda'_0) = \lambda_{0k}, \hat{x}'_\xi(0, \lambda'_0) = e, f'_\xi(0, \lambda'_0) = -\frac{1}{2} \frac{\varphi''_{\xi\xi}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}, \hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$ and $f'_{\lambda_s}(0, \lambda'_0) = -\frac{\varphi''_{\xi\lambda_s}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$ for every $s \in \{1, 2, \dots, k - 1\}$.

Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0, \lambda_0)$ can be parametrized as follows

$$I_{\Lambda, \Xi} = \{(0, \lambda', f(\hat{\xi}(\lambda'), \lambda')) : |\lambda' - \lambda'_0| < \varrho\}$$

where $0 < \varrho \leq r$ and $\hat{\xi}$ is a C^p -smooth function such that $\hat{\xi}(\lambda'_0) = 0$ and $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$ for every $s \in \{1, 2, \dots, k - 1\}$, which gives that $(0, \lambda_0)$ is a bifurcation point of (1).

Proof 3.8. Let $f : \Omega_0 \rightarrow \Omega$ be a function of Proposition 3.3. Fix $r > 0$ such that $(-r, r) \times B_r(\lambda'_0) \subset \Omega_0$. Let $\hat{x} : (-r, r) \times B_r(\lambda'_0) \rightarrow X$ be given by $\hat{x}(\xi, \lambda') = x(\xi, \lambda', f(\xi, \lambda'))$. Then $f(0, \lambda'_0) = \lambda_{0k}$ and $\hat{x}(0, \lambda'_0) = x(0, \lambda_0) = 0$. Differentiating \hat{x} we get $\hat{x}'_\xi(0, \lambda'_0) = e$ and $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$ for every $s \in \{1, 2, \dots, k - 1\}$. Moreover, differentiating the equality $\psi(\xi, \lambda', f(\xi, \lambda')) = 0$ we obtain $f'_\xi(0, \lambda'_0) = -\frac{\psi'_\xi(0, \lambda_0)}{\psi'_{\lambda_k}(0, \lambda_0)} = -\frac{1}{2} \frac{\varphi''_{\xi\xi}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$ and $f'_{\lambda_s}(0, \lambda'_0) = -\frac{\psi'_{\lambda_s}(0, \lambda_0)}{\psi'_{\lambda_k}(0, \lambda_0)} = -\frac{\varphi''_{\xi\lambda_s}(0, \lambda_0)}{\varphi''_{\xi\lambda_k}(0, \lambda_0)}$ for every $s \in \{1, 2, \dots, k - 1\}$. From Theorem 3.5 it follows that there exist open sets $\tilde{V}_0 \subset X$ and $\tilde{V} \subset B_r(\lambda'_0) \times \Omega$ such that $(0, \lambda_0) \in \tilde{V}_0 \times \tilde{V}$ and $\{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : F(x, \lambda) = 0\} = \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : x = 0\} \cup \{(x, \lambda) \in \tilde{V}_0 \times \tilde{V} : \exists \xi \in (-r, r) x = x(\xi, \lambda', f(\xi, \lambda')) \wedge \lambda_k = f(\xi, \lambda')\} = (\Lambda \cup \Xi) \cap \tilde{V}_0 \times \tilde{V}$. A point $(x, \lambda) \in \Lambda \cap \Xi$ only if it satisfies the following system

$$\begin{cases} x = \hat{x}(\xi, \lambda'), \\ \lambda_k = f(\xi, \lambda'), \quad \xi \in (-r, r), \lambda' \in B_r(\lambda'_0), \\ x = 0. \end{cases}$$

Since $\hat{x}(0, \lambda'_0) = 0$ and $\hat{x}'_\xi(0, \lambda'_0) = e \neq 0$, there exist: $0 < \varrho \leq r$, an open set $B \subset (-r, r)$ such that $0 \in B$ and a C^p -smooth function $\hat{\xi} : B_\varrho(\lambda'_0) \rightarrow B$ such that $\hat{\xi}(\lambda'_0) = 0$ and for all $(\xi, \lambda') \in B \times B_\varrho(\lambda'_0)$ we have $\hat{x}(\xi, \lambda') = 0$ only if $\xi = \hat{\xi}(\lambda')$. Differentiating the equality $\hat{x}(\hat{\xi}(\lambda'), \lambda') = 0$ we receive $\hat{x}'_\xi(\hat{\xi}(\lambda'), \lambda') \hat{\xi}'_{\lambda_s}(\lambda') + \hat{x}'_{\lambda_s}(\hat{\xi}(\lambda'), \lambda') = 0$ for every $s \in \{1, 2, \dots, k - 1\}$, and hence $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$. Summarizing $I_{\Lambda, \Xi} \subset \Lambda \cap \Xi$ and in a sufficiently small neighbourhood of $(0, \lambda_0)$ the intersection $\Lambda \cap \Xi$ is equal to $I_{\Lambda, \Xi}$.

Conclusion 3.9. Assume that (I_1) - (I_5) hold and $k = 2$. Then the solution set of (1) in a small neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two surfaces: Λ and Ξ . The surface Ξ can be parametrized as follows

$$\Xi = \{(\hat{x}(\xi, \lambda_1), \lambda_1, f(\xi, \lambda_1)) : (\xi, \lambda_1) \in (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r)\},$$

where $\hat{x}: (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r) \rightarrow X$ and $f: (-r, r) \times (\lambda_{01} - r, \lambda_{01} + r) \rightarrow R^1$ are C^p -smooth functions such that $\hat{x}(0, \lambda_{01}) = 0$, $f(0, \lambda_{01}) = \lambda_{02}$, $\hat{x}'_{\xi}(0, \lambda_{01}) = e$, $\hat{x}'_{\lambda_1}(0, \lambda_{01}) = 0$, $f'_{\xi}(0, \lambda_{01}) = -\frac{1}{2} \frac{\varphi''_{\xi\xi}(0, \lambda_{01})}{\varphi''_{\xi\lambda_2}(0, \lambda_{01})}$ and $f'_{\lambda_1}(0, \lambda_{01}) = -\frac{\varphi''_{\xi\lambda_1}(0, \lambda_{01})}{\varphi''_{\xi\lambda_2}(0, \lambda_{01})}$. In a sufficiently small neighbourhood of $(0, \lambda_0)$ the surfaces Λ and Ξ intersect only along the curve

$$I_{\Lambda, \Xi} = \{(0, \lambda_1, f(\hat{\xi}(\lambda_1), \lambda_1)) : \lambda_1 \in (\lambda_{01} - \varrho, \lambda_{01} + \varrho)\},$$

where $0 < \varrho \leq r$ and $\hat{\xi}: (\lambda_{01} - \varrho, \lambda_{01} + \varrho) \rightarrow (-r, r)$ is a C^p -smooth function such that $\hat{\xi}(\lambda_{01}) = \hat{\xi}'(\lambda_{01}) = 0$, and hence $(0, \lambda_0)$ is a bifurcation point of (1).

Let us consider the following condition:

(I'_3) $F: X \times R^k \rightarrow Y$ is a variational gradient of a certain functional $E: X \times R^k \rightarrow R^1$ with respect to the scalar product $\langle \cdot, \cdot \rangle$, i.e. for all $x, y \in X$ and $\lambda \in R^k$

$$E'_x(x, \lambda)y = \langle F(x, \lambda), y \rangle.$$

It is evident that (I'_3) implies (I_3). Furthermore, by formula (7) we obtain

$$\varphi''_{\xi\lambda_s}(0, \lambda_0) = -E^{(3)}_{xx\lambda_s}(0, \lambda_0)(e, e, 1) \tag{10}$$

for $s \in \{1, 2, \dots, k\}$. From this it follows that if F satisfies (I'_3) then (I_5) can be replaced by the equivalent condition:

(I'_5) $E^{(3)}_{xx\lambda_k}(0, \lambda_0)(e, e, 1) \neq 0$.

By (7) we also obtain

$$\varphi''_{\xi\xi}(0, \lambda_0) = -E^{(3)}_{xxx}(0, \lambda_0)(e, e, e). \tag{11}$$

Summarizing, in a variational case we have the following result.

Conclusion 3.10. Under assumptions: (I_1), (I_2), (I'_3), (I_4) and (I'_5), the solution set of equation (1) in a certain neighbourhood of $(0, \lambda_0) \in \Lambda$ is the union of two sets: Λ and Ξ . The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi, \lambda'), \lambda', f(\xi, \lambda')) : |\xi| < r, |\lambda' - \lambda'_0| < r\},$$

where \hat{x} and f are C^p -smooth functions such that $\hat{x}(0, \lambda'_0) = 0$, $f(0, \lambda'_0) = \lambda_{0k}$, $\hat{x}'_{\xi}(0, \lambda'_0) = e$, $f'_{\xi}(0, \lambda'_0) = -\frac{1}{2} \frac{E^{(3)}_{xxx}(0, \lambda_0)(e, e, e)}{E^{(3)}_{xx\lambda_k}(0, \lambda_0)(e, e, 1)}$, $\hat{x}'_{\lambda_s}(0, \lambda'_0) = 0$ and $f'_{\lambda_s}(0, \lambda'_0) = -\frac{E^{(3)}_{xx\lambda_s}(0, \lambda_0)(e, e, 1)}{E^{(3)}_{xx\lambda_k}(0, \lambda_0)(e, e, 1)}$ for every $s \in \{1, 2, \dots, k - 1\}$.

Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0, \lambda_0)$ can be parametrized as follows

$$I_{\Lambda, \Xi} = \{(0, \lambda', f(\hat{\xi}(\lambda'), \lambda')) : |\lambda' - \lambda'_0| < \varrho\}$$

where $0 < \varrho \leq r$ and $\hat{\xi}$ is a C^p -smooth function such that $\hat{\xi}(\lambda'_0) = 0$ and $\hat{\xi}'_{\lambda_s}(\lambda'_0) = 0$ for every $s \in \{1, 2, \dots, k - 1\}$, which gives that $(0, \lambda_0)$ is a bifurcation point of (1).

4 Applications

It is obvious that if we assume that F is a map from a small neighbourhood of the point $(0, \lambda_0)$ in $X \times R^k$ to Y , our results remain true. After this remark we are ready to give an example of application of Conclusion 3.10 to mathematical physics. All the results of Section 4 were proved either in [12] or [15]. However, to make this exposition self-sufficient we give the main ideas of the proofs.

For every $m \in N$ and $\mu \in (0, 1)$, let $C^{m,\mu}(\bar{D})$ denote the real Hölder space of functions defined on $D = \{(u, v) \in R^2: u^2 + v^2 < 1\}$ with the standard norm

$$\|x; C^{m,\mu}(\bar{D})\| = \max_{|\alpha| \leq m} \sup \{|D^\alpha x(u, v)|: (u, v) \in D\} + \max_{|\alpha| \leq m} \sup \left\{ \frac{|D^\alpha x(u, v) - D^\alpha x(\bar{u}, \bar{v})|}{|(u - \bar{u}, v - \bar{v})|^\mu} : (u, v), (\bar{u}, \bar{v}) \in D, (u, v) \neq (\bar{u}, \bar{v}) \right\},$$

where $D^\alpha x = \frac{\partial^{|\alpha|} x}{\partial^{\alpha_1} u \partial^{\alpha_2} v}$, $\alpha = (\alpha_1, \alpha_2) \in N_0 \times N_0$, $N_0 = N \cup \{0\}$ and $|\alpha| = \alpha_1 + \alpha_2$. It is well-known that $C^{m,\mu}(\bar{D})$ is a Banach space (see [1]). Let

- $C_{0,0}^{4,\mu}(\bar{D}) = \{f \in C^{4,\mu}(\bar{D}): \Delta f|_{\partial D} = f|_{\partial D} = 0\}$,
- $C_0^{2,\mu}(\bar{D}) = \{f \in C^{2,\mu}(\bar{D}): f|_{\partial D} = 0\}$,
- $X = C_{0,0}^{4,\mu}(\bar{D}) \times C_{0,0}^{4,\mu}(\bar{D})$,
- $Y = C^{0,\mu}(\bar{D}) \times C^{0,\mu}(\bar{D})$.

The norms in X and Y are defined by coordinates. That is as the maximum (or the sum) of norms of both coordinates of a given element. The function given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \frac{1}{\pi} \iint_D (x_1 y_1 + x_2 y_2) dudv$$

is a scalar product in Y , which is continuous with respect to the norms in X and Y . We define $F: X \times R_+^2 \rightarrow Y$ as follows

$$F(x, \lambda) = (\Delta^2 x_1 - [x_1, x_2] + 2\lambda_1 \Delta x_1 + \lambda_2 x_1 - \gamma x_1^3, -\Delta^2 x_2 - \frac{1}{2}[x_1, x_1]), \quad (12)$$

where $R_+ = (0, +\infty)$, $x = (x_1, x_2)$, $\lambda = (\lambda_1, \lambda_2)$, γ is a positive constant and $[\cdot, \cdot]: X \rightarrow Y$ is given by

$$[x_1, x_2] = \frac{\partial^2 x_1}{\partial u^2} \frac{\partial^2 x_2}{\partial v^2} - 2 \frac{\partial^2 x_1}{\partial u \partial v} \frac{\partial^2 x_2}{\partial u \partial v} + \frac{\partial^2 x_1}{\partial v^2} \frac{\partial^2 x_2}{\partial u^2}.$$

The equation

$$F(x, \lambda) = 0 \quad (13)$$

with F given by (12) is called the von Kármán equation for a thin circular elastic plate which lies on an elastic base and is uniformly radially compressed along its boundary. In mechanics x_1 is a deflection function, x_2 is a stress function, λ_1 is a value of a compressing force, λ_2 and γ are parameters of an elastic foundation. The solutions of (13) lying in a sufficiently small neighbourhood of the set of trivial solutions of (13) are called the forms

of equilibrium of a plate. The map F is C^∞ -smooth and an easy computation shows that for all $y = (y_1, y_2) \in X$

$$F'_x(x, \lambda)y = (\Delta^2 y_1 - [y_1, x_2] - [x_1, y_2] + 2\lambda_1 \Delta y_1 + \lambda_2 y_1 - 3\gamma x_1^2 y_1, -\Delta^2 y_2 - [x_1, y_1]). \tag{14}$$

Let $E: X \times R_+^2 \rightarrow R^1$ be given by

$$E(x, \lambda) = \frac{1}{2\pi} \iint_D ((\Delta x_1)^2 - (\Delta x_2)^2 - [x_1, x_1]x_2) dudv + \frac{1}{2\pi} \iint_D \left(-2\lambda_1 \left(\left(\frac{\partial x_1}{\partial u} \right)^2 + \left(\frac{\partial x_1}{\partial v} \right)^2 \right) + \lambda_2 x_1^2 - \frac{1}{2} \gamma x_1^4 \right) dudv. \tag{15}$$

E is easily seen to be C^∞ -smooth.

Theorem 4.1 (see Th. 2.4 of [12]). The map F is a variational gradient of the functional E with respect to the scalar product $\langle \cdot, \cdot \rangle$.

Sketch of the proof 4.2. For all $x, y \in X$ and $\lambda \in R_+^2$, we have

$$\begin{aligned} E'_x(x, \lambda)y &= \frac{d}{dt} E(x + ty, \lambda)|_{t=0} = \frac{1}{\pi} \iint_D \Delta x_1 \Delta y_1 dudv - \frac{1}{\pi} \iint_D \Delta x_2 \Delta y_2 dudv \\ &\quad - \frac{1}{\pi} \iint_D [x_1, y_1]x_2 dudv - \frac{1}{2\pi} \iint_D [x_1, x_1]y_2 dudv \\ &\quad - \frac{1}{\pi} \iint_D 2\lambda_1 \left(\frac{\partial x_1}{\partial u} \frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v} \frac{\partial y_1}{\partial v} \right) dudv \\ &\quad + \frac{1}{\pi} \iint_D (\lambda_2 x_1 y_1 - \gamma x_1^3 y_1) dudv. \end{aligned}$$

Integrating by part we receive

$$\begin{aligned} \iint_D \Delta x_1 \Delta y_1 dudv &= \iint_D (\Delta^2 x_1) y_1 dudv, \\ \iint_D \Delta x_2 \Delta y_2 dudv &= \iint_D (\Delta^2 x_2) y_2 dudv, \\ \iint_D [x_1, y_1]x_2 dudv &= \iint_D [x_1, x_2]y_1 dudv \end{aligned}$$

and

$$\iint_D \left(\frac{\partial x_1}{\partial u} \frac{\partial y_1}{\partial u} + \frac{\partial x_1}{\partial v} \frac{\partial y_1}{\partial v} \right) dudv = - \iint_D (\Delta x_1) y_1 dudv.$$

Hence $E'_x(x, \lambda)y = \langle F(x, \lambda), y \rangle$, which completes the proof.

Theorem 4.3 (see Th. 2.2 of [12]). For every $\lambda \in R_+^2$, $F'_x(0, \lambda): X \rightarrow Y$ is a Fredholm map of index 0.

Sketch of the proof 4.4. Fix $\lambda \in R_+^2$. By (14) we get

$$F'_x(0, \lambda)y = (\Delta^2 y_1 + 2\lambda_1 \Delta y_1 + \lambda_2 y_1, -\Delta^2 y_2). \tag{16}$$

We can write (16) as

$$F'_x(0, \lambda)y = A(y) + B(y),$$

where $A, B: X \rightarrow Y$ are given as follows:

$$A(y) = (\Delta^2 y_1, -\Delta^2 y_2), \quad B(y) = (2\lambda_1 \Delta y_1 + \lambda_2 y_1, 0).$$

It is known that $\Delta: C^{2,\mu}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$ is an isomorphism. Moreover, it is a simple matter to check that B is compact, which finishes the proof.

Let $J_k: R \rightarrow R, k \in N_0$, denote the k -th Bessel function. It is well-known (see [8], [18]) that $\alpha \in R$ is an eigenvalue of $\Delta: C^{2,\mu}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$ if and only if $\alpha < 0$ and there is $k \in N_0$ such that $J_k(\sqrt{-\alpha}) = 0$. Furthermore, if $J_0(\sqrt{-\alpha}) = 0$ then the eigenspace corresponding to α is one-dimensional. If $J_k(\sqrt{-\alpha}) = 0$ for a certain $k \in N$ then the corresponding eigenspace is two-dimensional.

For $\lambda = (\lambda_1, \lambda_2) \in R^2_+$, let $\delta = (\lambda_1)^2 - \lambda_2, a = -\lambda_1 - \sqrt{\delta}$ and $b = -\lambda_1 + \sqrt{\delta}$. Of course, a and b are determined on condition $\delta \geq 0$. Let $\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I: C^{4,\mu}_{0,0}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$ and $\Delta - aI, \Delta - bI: C^{2,\mu}(\overline{D}) \rightarrow C^{0,\mu}(\overline{D})$, where $I(h) = h$ are natural embeddings of the appropriate Hölder spaces.

Lemma 4.5 (see Lemmas 4.1-4.3 of [12]). Under the above assumptions:

- (i) If $\delta < 0$ then $\ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) = \{0\}$.
- (ii) If $\delta = 0$ then $\ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) = \ker(\Delta + \lambda_1 I)$.
- (iii) If $\delta > 0$ then $\ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) = \ker(\Delta - aI) \oplus \ker(\Delta - bI)$.

By (16), $N(\lambda) = \ker(\Delta^2 + 2\lambda_1 \Delta + \lambda_2 I) \times \{0\}$. From this and Lemma 4.5 we obtain what follows.

Theorem 4.6. $\dim N(\lambda) = 1$ if and only if one of the below conditions is satisfied:

- (I) $\delta = 0$ and $J_0(\sqrt{\lambda_1}) = 0$,
- (II) $\delta > 0, J_0(\sqrt{-a}) = 0$ and $J_k(\sqrt{-b}) \neq 0$ for every $k \in N_0$,
- (III) $\delta > 0, J_0(\sqrt{-b}) = 0$ and $J_k(\sqrt{-a}) \neq 0$ for every $k \in N_0$.

Suppose that $\lambda_0 = (\lambda_{01}, \lambda_{02})$ and $\dim N(\lambda_0) = 1$. Fix $e = (e_1, 0) \in N(\lambda_0)$ such that $\langle e, e \rangle = 1$. Set

$$c_0 = \begin{cases} a_0 & \text{if (I) or (II),} \\ b_0 & \text{if (III),} \end{cases}$$

where $a_0 = -\lambda_{01} - \sqrt{\delta_0}, b_0 = -\lambda_{01} + \sqrt{\delta_0}$ and $\delta_0 = (\lambda_{01})^2 - \lambda_{02}$. A trivial verification combining Theorem 4.1 with (14) shows that

$$E'''_{xx\lambda_1}(x, \lambda)(y, z, 1) = \frac{2}{\pi} \iint_D (\Delta y_1) z_1 dudv,$$

$$E'''_{xx\lambda_2}(x, \lambda)(y, z, 1) = \frac{1}{\pi} \iint_D y_1 z_1 dudv,$$

and

$$E'''_{xxx}(x, \lambda)(y, z, w) = -\frac{1}{\pi} \iint_D ([y_1, z_2] + [y_2, z_1] + 6\gamma x_1 y_1 z_1) w_1 dudv - \frac{1}{\pi} \iint_D [y_1, z_1] w_2 dudv,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, $w = (w_1, w_2)$. From this and Lemma 4.5 we receive

$$E'''_{xx\lambda_1}(0, \lambda_0)(e, e, 1) = \frac{2}{\pi} \iint_D (\Delta e_1) e_1 dudv = 2c_0 \langle e, e \rangle = 2c_0,$$

$$E'''_{xx\lambda_2}(0, \lambda_0)(e, e, 1) = \frac{1}{\pi} \iint_D e_1^2 dudv = \langle e, e \rangle = 1,$$

$$E'''_{xxx}(0, \lambda_0)(e, e, e) = 0.$$

Applying Conclusion 3.10 we get the following theorem.

Theorem 4.7. Let $\lambda_0 \in R^2_+$ satisfy the above assumptions. Then the solution set of equation (13) in a certain neighbourhood of $(0, \lambda_0) \in X \times R^2_+$ is the union of two sets: Λ and Ξ . The set Ξ is given by

$$\Xi = \{(\hat{x}(\xi, \lambda_1), \lambda_1, f(\xi, \lambda_1)) : |\xi| < r, |\lambda_1 - \lambda_{01}| < r\},$$

where \hat{x} and f are C^∞ -smooth functions such that $\hat{x}(0, \lambda_{01}) = 0$, $f(0, \lambda_{01}) = \lambda_{02}$, $\hat{x}'_\xi(0, \lambda_{01}) = e$, $f'_\xi(0, \lambda_{01}) = 0$, $\hat{x}'_{\lambda_1}(0, \lambda_{01}) = 0$ and $f'_{\lambda_1}(0, \lambda_{01}) = -2c_0$.

Moreover, the intersection of Λ and Ξ in a sufficiently small neighbourhood of $(0, \lambda_0)$ can be parametrized as follows

$$I_{\Lambda, \Xi} = \{(0, \lambda_1, f(\hat{\xi}(\lambda_1), \lambda_1)) : |\lambda_1 - \lambda_{01}| < \varrho\},$$

where $0 < \varrho \leq r$ and $\hat{\xi}$ is a C^∞ -smooth function such that $\hat{\xi}(\lambda_{01}) = 0$ and $\hat{\xi}'(\lambda_{01}) = 0$, which gives that $(0, \lambda_0)$ is a bifurcation point of (13).

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