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On the doubly connected domination number of a graph

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Abstract: For a given connected graph $G = (V, E)$, a set $D \subseteq V(G)$ is a *doubly connected dominating set* if it is dominating and both $\langle D \rangle$ and $\langle V(G) - D \rangle$ are connected. The cardinality of the minimum doubly connected dominating set in G is the *doubly connected domination number*. We investigate several properties of doubly connected dominating sets and give some bounds on the doubly connected domination number.

Keywords: Doubly connected domination number, connected domination number
MSC (2000): 05C69

1 Introduction

Let $G = (V, E)$ be a simple connected graph with $|V(G)| = n(G)$ and $|E(G)| = m(G)$. The *neighbourhood* $N_G(v)$ of a vertex v is the set of all vertices adjacent to v in G and the *closed neighbourhood* $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $d_G(v) = |N_G(v)|$ of a vertex v is the number of edges incident to v in G. The *minimum* and *maximum degrees* of vertices of $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex x such that $d_G(x) = \Delta(G) = n(G) - 1$ we call a *universal vertex*. Let $\Omega(G)$ be the set of all end-vertices of G, that is the set of vertices degree 1, and let $n_1(G)$ be the cardinality of $\Omega(G)$.

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A vertex that is a neighbour of an end-vertex is called a *support*. Let $S(G)$ be the set of supports in G.

The *corona* $G = H \circ K_1$ is the graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. For disjoint graphs G_1 and G_2, the *join* $G = G_1 + G_2$ is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \wedge v \in V(G_2)\}$. Let us denote by $G - v$ the graph obtained from G by removing the vertex $v \in V(G)$ and all edges incident to v.

For any connected graph G, a vertex $x \in V(G)$ is called *a cut-vertex* of G if $G-x$ is no longer connected. The *vertex-connectivity* or simply *connectivity* $\kappa(G)$ is the minimum number of vertices whose removal from G results in disconnected graph or a graph with only one vertex.

A set $D \subseteq V(G)$ is a *dominating set of* G if for every vertex $v \in V(G)-D$, there exists a vertex $u \in D$ such that v is adjacent to u. The minimum cardinality of a dominating set in G is the *domination number* $\gamma(G)$.

Sampathkumar and Walikar [7] defined a *connected dominating set* D to be a dominating set whose induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of a connected dominating set in G is the *connected domination number of* G and is denoted by $\gamma_c(G)$.

In this paper we introduce a new type of domination: a set $D \subseteq V(G)$ is a *doubly connected dominating set of* G if it is dominating and both $\langle D \rangle$ and $\langle V(G) - D \rangle$ are connected. The cardinality of a minimum doubly connected dominating set of G is the *doubly connected domination number of* G and is denoted by $\gamma_{cc}(G)$. We define that for each connected graph G the set of all vertices of G is a doubly connected dominating set of G.

For unexplained terms and symbols see [1, 5].

2 Preliminary results

We begin with some basic properties of doubly connected dominating sets.

Proposition 2.1. *Let D be a minimum doubly connected dominating set of a connected graph G on $n \geq 3$ vertices. Then*

(i) *every cut-vertex is in D;*
(ii) *every support is in D;*
(iii) *at least $n_1(G) - 1$ end-vertices are in D;*
(iv) *$\gamma_{cc}(G) \geq n_1(G)$, with equality if and only if G is a star $K_{1,n-1}$;*
(v) *$\gamma_{cc}(G) \geq n_1(G) + |S(G)| - 1$, with equality if and only if each vertex $v \in V(G)$ is either an end-vertex or a support.*

Proof. (i) Assume v is a cut-vertex of G that does not belong to a minimum doubly connected dominating set D. As $G - v$ is disconnected, it is not possible to choose a connected dominating set $D \subseteq V(G) - \{v\}$, a contradiction.

(ii) As every support is a cut-vertex, by (i) our claim follows.

(iii) If not, assume there are two end-vertices not belonging to D. As every support is in D it follows, that $\langle V(G) - D \rangle$ is not connected, a contradiction.

(iv) By (iii), at least $n_1(G) - 1$ end-vertices are in D. If $\Omega(G) \subseteq D$ our claim follows. Similarly, if there exists a vertex $x \in \Omega(G)$ such that $x \notin D$, then $\gamma_{cc}(G) = n(G) - 1$ and since $n \geq 3$ we have $n(G) - 1 \geq n_1(G)$, which completes the proof of the bound.

It is easy to see that $\gamma_{cc}(K_{1,n-1}) = n_1(G)$. Conversely, assume that $\gamma_{cc}(G) = n_1(G)$. In this case, by (ii) and (iii), each support and at least all end-vertices except one are in a minimum doubly connected dominating set D. Thus $|S(G)| = 1$, $|V(G) - D| = 1$ and $|D| = n - 1 = n_1(G)$. We conclude G is a star $K_{1,n-1}$.

(v) By (ii) and (iii), the inequality is straightforward. If $\Omega(G) \cup S(G) = V(G)$ then obviously $n_1(G) + |S(G)| - 1 = \gamma_{cc}(G)$. Conversely, assume that $\gamma_{cc}(G) = n_1(G) + |S(G)| - 1$. In this case, the minimum doubly connected dominating set D consists of all vertices of the set $S(G)$ and all except one end-vertices. Thus $\gamma_{cc}(G) = n - 1$ and $V(G) = S(G) \cup \Omega(G)$.

□

As an immediate consequence of Proposition 2.1 we have

Corollary 2.2. *If $G = H \circ K_1$, then $\gamma_{cc}(G) = n(G) - 1$.*

Corollary 2.3. *For a tree T on $n \geq 3$ vertices, $\gamma_{cc}(T) = n - 1$.*

Proof. In a tree T each vertex is either a cut-vertex or an end-vertex. By Proposition 2.1, we conclude that $\gamma_{cc}(T) \geq n - 1$. On the other hand, if x is an end-vertex of a tree T, then $D = V(T) - \{x\}$ is a doubly connected dominating set. Thus, $\gamma_{cc}(T) = n - 1$. □

Since every doubly connected dominating set is a connected dominating and every connected dominating set is dominating, we have the following inequality chain for every connected graph G:

$$\gamma(G) \leq \gamma_c(G) \leq \gamma_{cc}(G).$$

We characterize now some graphs for which the numbers $\gamma_{cc}(G)$ and $\gamma_c(G)$ are the same.

Proposition 2.4. *Let G be a connected graph on $n \geq 3$ vertices.*

(i) *If G is a cycle, then $\gamma_{cc}(G) = \gamma_c(G) = n - 2$.*

(ii) *If $\gamma_{cc}(G) = \gamma_c(G)$, then $\gamma_{cc}(G) \leq n - 2$.*

(iii) *If $\gamma_{cc}(G) = \gamma_c(G)$, then $\delta(G) \geq 2$.*

(iv) *For any unicyclic graph G we have $\gamma_c(G) = \gamma_{cc}(G)$ if and only if G is a cycle.*

Proof. (i) It is obvious.

(ii) It is known [7] that for every connected graph G with $n \geq 3$ we have $\gamma_c(G) \leq n - 2$. Thus, for the equality $\gamma_{cc}(G) = \gamma_c(G)$ we conclude that $\gamma_{cc}(G) \leq n - 2$.

(iii) Suppose $\gamma_{cc}(G) = \gamma_c(G)$ and x is an end-vertex in G. By (ii), $\gamma_{cc}(G) \leq n - 2$. Let D

be a minimum doubly connected dominating set of G of cardinality $|D| \leq n-2$. If $x \in D$, then $D - \{x\}$ is also a connected dominating set of G, a contradiction with equality $\gamma_{cc}(G) = \gamma_c(G)$. If $x \notin D$, then x is the unique vertex in $V(G) - D$, because $\langle V(G) - D \rangle$ is connected. Thus $\gamma_{cc}(G) = n-1$, a contradiction.

(iv) If G is a cycle on n vertices, then by (i) $\gamma_{cc}(G) = \gamma_c(G) = n-2$. Suppose now G is unicyclic with $\gamma_c(G) = \gamma_{cc}(G)$ and G is not a cycle. By (ii), $\gamma_{cc}(G) \leq n-2$. Moreover, there exists a vertex $x \in V(G)$ such that $d_G(x) = 1$. Let D be a minimum doubly connected dominating set of G. If $x \notin D$, then $\gamma_{cc}(G) = n-1$, a contradiction. On the other hand, $x \in D$ implies, that $D - \{x\}$ is a connected dominating set of G, a contradiction, as $\gamma_c(G) = \gamma_{cc}(G)$.

□

We have shown that there exist graphs G for which the equality $\gamma_c(G) = \gamma_{cc}(G)$ holds. However the difference between $\gamma_{cc}(G)$ and $\gamma_c(G)$ can be arbitrarily large.

Lemma 2.5. *The difference $\gamma_{cc} - \gamma_c$ can be arbitrarily large.*

Proof. Consider a star $K_{1,n-1}$ with $n-1$ end-vertices. Of course, $\gamma_c(K_{1,n-1}) = 1$. By Proposition 2.1, $\gamma_{cc}(K_{1,n-1}) = n-1$. Thus $\gamma_{cc}(K_{1,n-1}) - \gamma_c(K_{1,n-1}) = n-2$. □

Observation 2.6. Let $G = K_{m_1,m_2,\ldots,m_k}$ be the complete k partite graph, $k \geq 3$ with $m_1 \leq m_2 \leq \cdots \leq m_k$.

- If $m_1 = 1$, then $\gamma_{cc}(G) = 1$;
- If $m_1 \geq 2$, then $\gamma_{cc}(G) = 2$.

Observation 2.7. If G_1 and G_2 are disjoint connected graphs, then

$$\gamma_{cc}(G_1 + G_2) = \begin{cases} 1 & \text{if } \gamma_{cc}(G_1) = 1 \text{ or } \gamma_{cc}(G_2) = 1; \\ 2 & \text{otherwise.} \end{cases}$$

A connected subgraph B of G is called *a block* if B has no cut-vertex and every subgraph $B' \subseteq G$ with $B \subseteq B'$ and $B \neq B'$ has at least one cut-vertex. A connected graph G is called *a block graph* if every block in G is complete. A vertex v of a graph G is called a *simplicial vertex* if every two vertices of $N_G(v)$ are adjacent in G.

Theorem 2.8. *If G is a block graph, then $\gamma_{cc}(G) = n(G) - t$, where t is the maximal number of simplicial vertices in a block with a largest number of simplicial vertices.*

Proof. Let D be a minimum doubly connected dominating set of a block graph G. By Proposition 2.1, each cut-vertex belongs to D. Hence $\gamma_{cc}(G) \geq n(G) - t$, where t is maximal number of simplicial vertices in a block with a largest number of simplicial vertices.

Conversely, let B be a block with a largest number of simplicial vertices. Denote by F the set of all simplicial vertices belonging to B and let $|F| = t$. Then $V(G) - F$ is a doubly connected dominating set of G and we have $\gamma_{cc}(G) \le n(G) - t$. Thus $\gamma_{cc}(G) = n(G) - t$. $\square$

3 Bounds

Now we find some bounds on the doubly connected domination number. For this purpose, denote by $\mathcal{A}$ a family of graphs such that $K_2 \in \mathcal{A}$ and G belongs to $\mathcal{A}$ if and only if for each pair of adjacent non-cut-vertices $u, v \in V(G)$, $\langle V(G) - \{u, v\}\rangle$ is disconnected.

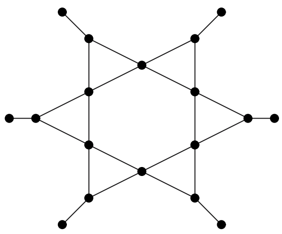

Fig. 1 A graph $G \in \mathcal{A}$.

Theorem 3.1. *For every connected graph G on $n \ge 2$ vertices,*

$$1 \le \gamma_{cc}(G) \le n - 1$$

with equality for the lower bound if and only if there exists a connected graph H such that $G = H + K_1$ and equality for the upper bound if and only if $G \in \mathcal{A}$.

Proof. The inequality $1 \le \gamma_{cc}(G)$ is obvious. If $G = H + K_1$ and H is connected, then obviously $\gamma_{cc}(G) = 1$. Assume now that $\gamma_{cc}(G) = 1$ and let $D = \{x\}$ be a minimum doubly connected dominating set of G. Since D is dominating, x must be a universal vertex. Moreover, $\langle V(G) - D\rangle = \langle V(G) - \{x\}\rangle$ is connected, so x is a non-cut-vertex. We conclude that $G = H + K_1$, where H is connected.

Now we prove that $\gamma_{cc}(G) \le n-1$. The inequality and the equality are straightforward when $G = K_2$. Suppose $n \ge 3$. Then there exist in G at least two non-cut-vertices, for example two leaves of a spanning tree of G. Let x be a non-cut-vertex. Then $D = V(G) - \{x\}$ is a doubly connected dominating set of G.

If $G \in \mathcal{A}$, then $\gamma_{cc}(G) = n - 1$, because every support of G is in D and for each pair of adjacent non-cut-vertices $u, v \in V(G)$, the induced subgraph $\langle V(G) - \{u, v\}\rangle$ is disconnected. Now let $G \notin \mathcal{A}$. It suffices to show that $\gamma_{cc}(G) \le n - 2$. If $G \notin \mathcal{A}$, then there exist adjacent non-cut-vertices $v, u \in V(G)$ such that $\langle V(G) - \{u, v\}\rangle$ is connected. In this case $D = V(G) - \{u, v\}$ is a doubly connected dominating set of G, as $n \ge 3$, G is connected and neither of u, v is a support. $\square$

Proposition 3.2. *Let G be a connected graph on $n \ge 2$ vertices. Then $\gamma_{cc}(G) \le n - \kappa(G) + 1$.*

Proof. If $\kappa(G) \leq 2$, then by Theorem 3.1 our claim follows. Thus assume now $\kappa(G) \geq 3$. It is obvious that $\kappa(G) \leq \delta(G)$. Let A be a set of an arbitrary vertex $x \in V(G)$ and $\kappa(G) - 2$ of its neighbours. Obviously, $\langle V(G) - A\rangle$ is connected. Observe that $D = V(G) - A$ is dominating in G. Thus D is a doubly connected dominating set in G with $|D| = n - \kappa(G) + 1$. □

In [7] Sampathkumar and Walikar showed that for every connected graph G with $n \geq 3$ vertices and m edges we have inequalities $\frac{n}{\Delta(G)+1} \leq \gamma_c(G) \leq 2m - n$. Now we present similar inequalities for the number γ_{cc}.

Theorem 3.3. *For any connected graph G with $n \geq 2$ vertices and m edges,*

$$\frac{n}{\Delta(G)+1} \leq \gamma_{cc}(G) \leq 2m - n + 1$$

with equality for the lower bound if and only if $\gamma_{cc}(G) = 1$ and equality for the upper bound if and only if G is a tree.

Proof. Since $\frac{n}{\Delta(G)+1} \leq \gamma_c(G) \leq \gamma_{cc}(G)$ the lower bound follows. If $\gamma_{cc}(G) = 1$, then by Theorem 3.1 there exists a vertex $v \in V(G)$ such that $d_G(v) = n - 1$. Thus $\frac{n}{\Delta(G)+1} = 1 = \gamma_{cc}(G)$.

Conversely, let G be a graph such that $\gamma_{cc}(G) = \frac{n}{\Delta(G)+1}$ and $\gamma_{cc}(G) > 1$. Let D be a minimum doubly connected dominating set of G. Since $\langle D\rangle$ is connected, for each $v \in D$ we have $|N_G(v) \cap (V(G) - D)| \leq \Delta(G) - 1$. Hence $|V(G) - D| \leq (\Delta(G) - 1)|D|$ and $n - \gamma_{cc}(G) \leq (\Delta(G) - 1)\gamma_{cc}(G)$, which gives $\gamma_{cc}(G) \geq \frac{n}{\Delta(G)}$, a contradiction.

By Theorem 3.1, $\gamma_{cc}(G) \leq n-1 = 2(n-1)-n+1$ and since G is connected, $m \geq n-1$. Thus $\gamma_{cc}(G) \leq 2m - n + 1$.

We now show that $\gamma_{cc}(G) = 2m - n + 1$ if and only if G is a tree. If G is a tree, then $m = n - 1$ and $\gamma_{cc}(G) = n - 1 = 2m - n + 1$. Conversely, let $\gamma_{cc}(G) = 2m - n + 1$. By Theorem 3.1 we have $2m - n + 1 \leq n - 1$, which implies $m \leq n - 1$, so G must be a tree with $m = n - 1$. □

As an immediate consequence of the second paragraph of the proof of Theorem 3.3 we have what follows.

Corollary 3.4. *For each connected graph G with $\gamma_{cc}(G) > 1$ is $\gamma_{cc}(G) \geq \frac{n}{\Delta(G)}$.*

Now we introduce the following notation: if T_1 and T_2 are vertex disjoint trees, then by $\mathcal{P}(T_1, T_2)$ we denote the set of all graphs G that can be obtained from T_1 and T_2 by adding $n(T_2)$ edges, one edge joining each vertex of T_2 to one arbitrarily chosen vertex of T_1. We say that a graph G belongs to the family $\mathcal{U}$ if there exist trees T_1 and T_2 such that $G \in \mathcal{P}(T_1, T_2)$.

Theorem 3.5. *For any connected graph G on $n \geq 2$ vertices and with m edges,*

$$2n - m - 2 \leq \gamma_{cc}(G)$$

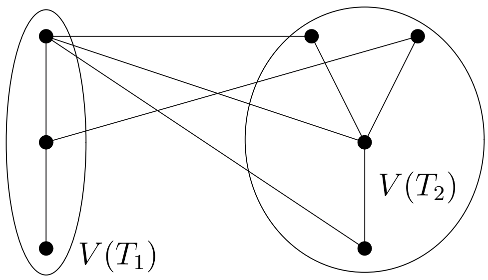

Fig. 2 A graph $G \in \mathcal{P}(T_1, T_2)$.

with equality for the bound if and only if G belongs to the family $\mathcal{U}$.

Proof. Let D be a minimum doubly connected dominating set in G. Since $\langle D \rangle$ and $\langle V(G) - D \rangle$ are connected and D is dominating, we have the following inequalities:

$$\begin{aligned} m(\langle D \rangle) &\geq \gamma_{cc}(G) - 1, \\ m(\langle V(G) - D \rangle) &\geq n - \gamma_{cc}(G) - 1, \\ m_{\gamma_{cc}} &\geq n - \gamma_{cc}(G), \end{aligned}$$

where $m_{\gamma_{cc}}$ is the number of the edges connecting vertices of $V(G) - D$ to vertices of D. By summing the inequalities we obtain

$$m = m(\langle D \rangle) + m(\langle V - D \rangle) + m_{\gamma_{cc}} \geq 2n - \gamma_{cc}(G) - 2$$

and thus $2n - m - 2 \leq \gamma_{cc}(G)$.

We now show that $\gamma_{cc}(G) = 2n - m - 2$ if and only if G belongs to the family $\mathcal{U}$. Let $G \in \mathcal{U}$. Then there exist trees T_1 and T_2 such that $G \in \mathcal{P}(T_1, T_2)$. In such a graph G the set $V(T_1)$ is a doubly connected dominating set. Thus $\gamma_{cc}(G) \leq n(T_1)$. Of course $n = n(T_1) + n(T_2)$ and

$$m = m(T_1) + m(T_2) + n(T_2) = n(T_1) - 1 + n(T_2) - 1 + n(T_2) = n(T_1) + 2n(T_2) - 2.$$

It follows that

$$2n - m - 2 = 2(n(T_1) + n(T_2)) - (n(T_1) + 2n(T_2) - 2) - 2 = n(T_1).$$

Consequently $\gamma_{cc}(G) \geq n(T_1)$, which together with $\gamma_{cc}(G) \leq n(T_1)$ gives $\gamma_{cc}(G) = n(T_1) = 2n - m - 2$.

Conversely, suppose $\gamma_{cc}(G) = 2n - m - 2$. This implies that

$$\begin{aligned} m(\langle D \rangle) &= \gamma_{cc}(G) - 1 = n(\langle D \rangle) - 1, \\ m(\langle V(G) - D \rangle) &= n - \gamma_{cc}(G) - 1 = n(\langle V(G) - D \rangle) - 1, \\ m_{\gamma_{cc}} &= n - \gamma_{cc}(G). \end{aligned}$$

It follows that $\langle D \rangle$ and $\langle V(G) - D \rangle$ are trees and each vertex of $V(G) - D$ has exactly one neighbour in D. Thus G is a graph obtained from two trees T_1 and T_2 by adding $n(T_2)$ edges, one edge joining each vertex of T_2 to one arbitrarily chosen vertex of T_1. $\square$

Duchet and Meyniel [3] have shown that for any connected graph G is $\gamma_c(G) \leq 2\beta_0(G) - 1$ and $\gamma_c(G) \leq 2\Gamma(G) - 1$, where $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G and β_0 is the maximum cardinality of an independent set of G. The next theorem shows that there is no similar result for the doubly connected domination number of a graph.

Theorem 3.6. *Each of the differences $\gamma_{cc} - \beta_0$ and $\gamma_{cc} - \Gamma$ can be arbitrarily large.*

Proof. We show a graph G for which $\gamma_{cc}(G) - \beta_0(G) = \gamma_{cc}(G) - \Gamma(G) = k$ for any positive integer k. Let G be a corona $K_{k+1} \circ K_1$. It is easy to observe that the set of end-vertices $\Omega(G)$ is the maximum independent set of G and thus $\beta_0(G) = k + 1$. The set $\Omega(G)$ is also the maximum minimal dominating set of G, so $\Gamma(G) = k + 1$. Since G is a corona, from Corollary 2.2 we have $\gamma_{cc}(G) = |V(G)| - 1 = 2(k + 1) - 1 = 2k + 1$. It follows that $\gamma_{cc}(G) - \beta_0(G) = \gamma_{cc}(G) - \Gamma(G) = k$. □

4 Edge subdivision and vertex removing

Now we examine the effects on $\gamma_{cc}(G)$ when G is modified by an edge subdivision.

An *edge subdivision* in a nonempty graph G is an operation of removal of an edge $e = uv$ and the addition of a new vertex w and edges uw and vw. A graph obtained from G by subdividing the edge $e = uv$ is denoted by $G \oplus w_{uv}$.

Theorem 4.1. *For every connected graph G we have $\gamma_{cc}(G) \leq \gamma_{cc}(G \oplus w_{uv})$.*

Proof. Let $e = uv$ be the subdivided edge and let D_0 be a minimum doubly connected dominating set of $G \oplus w_{uv}$. We consider two cases:

a) $w \in D_0$. Then, since $\langle D_0 \rangle$ is connected, u or v belong to D_0. If both of these vertices belong to D_0, then $D_0 - \{w\}$ is a doubly connected dominating set of G and thus $\gamma_{cc}(G) < |D_0| = \gamma_{cc}(G \oplus w_{uv})$. If $u \in D_0$ and $v \notin D_0$, then $D_0 - \{w\}$ is a doubly connected dominating set of G and we have the required inequality.

b) $w \notin D_0$. Then, since D_0 is dominating, u or v belong D_0. Then, similarly as in case *a*), we have $\gamma_{cc}(G) \leq |D_0| = \gamma_{cc}(G \oplus w_{uv})$.

□

Theorem 4.2. *The difference $\gamma_{cc}(G \oplus w_{uv}) - \gamma_{cc}(G)$ can be arbitrarily large.*

Proof. We construct graphs G and $G \oplus w_{uv}$ for which $\gamma_{cc}(G \oplus w_{uv}) - \gamma_{cc}(G) = k$ for a non-negative integer $k \geq 2$.

We begin with two stars $K_{1,k-1}$, $k \geq 2$ and denote their centers by u and v. Next we add a vertex x and edges joining x with all vertices of the stars. Finally, to obtain a graph G, we add an edge $e = uv$ and a pendant edge xx' (see Fig. 3). It is easy to observe that the set $D = \{x, x'\}$ is a minimum doubly connected dominating set of G and thus $\gamma_{cc}(G) = 2$.

For the graph $G \oplus w_{uv}$ notice that the set $D_u = N[v] \cup \{x'\} - \{w\}$ is a minimum doubly connected dominating set and the size of this set is $k+2$. Thus $\gamma_{cc}(G \oplus w_{uv}) - \gamma_{cc}(G) = k+2-2 = k$.

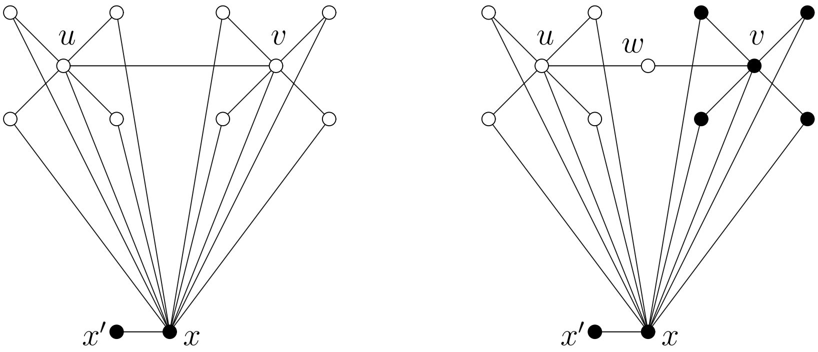

Fig. 3 Graphs G and $G \oplus w_{uv}$ for $k = 5$.

□

Theorem 4.3. *The difference $\gamma_{cc}(G) - \gamma_{cc}(G-x)$ can be arbitrarily large.*

Proof. Let H be the join $K_{1,k} + K_1$, $k \geq 2$, and let G be the graph that results if we add two pendant edges and two end-vertices x and y to the vertices of degree $k+1$ of the graph H (see Fig. 4). It is easy to observe that $V(G) - \{x\}$ is a minimum doubly connected dominating set of G. Thus, $\gamma_{cc}(G) = k+3$.

The set $N_G[y]$ is a minimum doubly connected dominating set of $G-x$. Thus $\gamma_{cc}(G-x) = 2$ and finally we have $\gamma_{cc}(G) - \gamma_{cc}(G-x) = k+1$. □

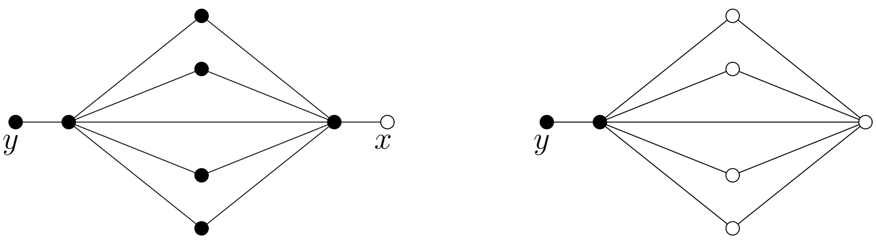

Fig. 4 Graphs G and $G-x$ for $k = 4$.

Theorem 4.4. *The difference $\gamma_{cc}(G-x) - \gamma_{cc}(G)$ can be arbitrarily large.*

Proof. Let G be the join of a path P on n vertices and K_1. Let x be the vertex of K_1. Clearly we have $\gamma_{cc}(G) = 1$.

As $G-x$ is a tree, by Corollary 2.3 we have $\gamma_{cc}(G-x) = n-1$. Thus $\gamma_{cc}(G-x) - \gamma_{cc}(G) = n-2$. □

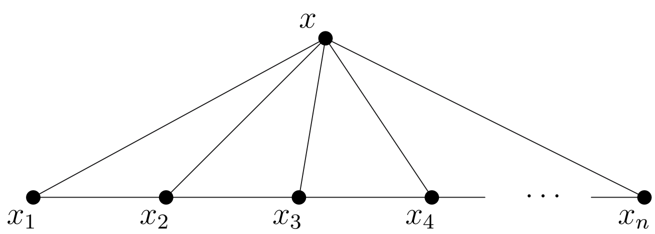

Fig. 5 Graph G.

5 Complexity issues for γ_{cc}

In this section we consider the decision problem of DOUBLY CONNECTED DOMINATING SET as follows

DOUBLY CONNECTED DOMINATING SET (DCDS)
INSTANCE: A connected graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a doubly connected dominating set of size at most k?

We show that the decision problem DCDS is NP–complete, even when restricted to connected bipartite graphs. We will use a well-known NP–completeness result, called DOMINATING SET, which is defined as follows.

DOMINATING SET (DS)
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a dominating set of size at most k?

Garey and Johnson in [4] proved that DS is NP–complete.

Theorem 5.1. *DCDS for bipartite graphs is NP–complete.*

Proof. We know that DCDS problem for bipartite graphs is in class NP of decision problems as it is easy to verify in polynomial time whether D is a doubly connected dominating set.

For any given instance for DS, which is a graph $G = (V, E)$ and an integer k, we construct a graph H and an integer q as follows:

$$
\begin{aligned}
V(H) &= V(G) \times \{1,2,3\} \cup \{x,y\},\\
E(H) &= \{(v,1)(v,2)\colon v \in V(G)\}\\
&\quad \cup \{(v,2)(v,3)\colon v \in V(G)\}\\
&\quad \cup \{(v,1)x\colon v \in V(G)\}\\
&\quad \cup \{(v,1)y\colon v \in V(G)\}\\
&\quad \cup \{(v,3)x\colon v \in V(G)\}\\
&\quad \cup \{(v,3)y\colon v \in V(G)\}\\
&\quad \cup \{(v,1)(w,2)\colon vw \in E(G)\},\\
q &= k+1.
\end{aligned}
$$

The graph H is connected and bipartite, as every cycle in H has even length. (See Figure 6).

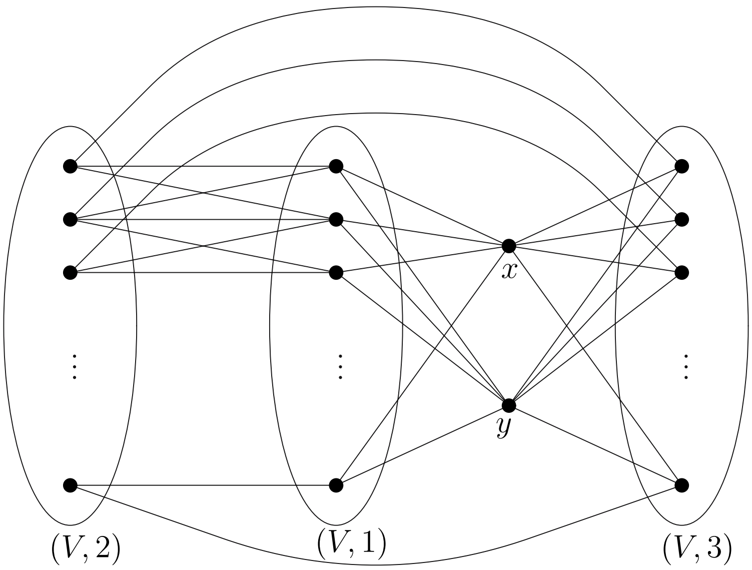

Fig. 6 Reduction from DS to DCDS for bipartite graphs.

Assume first that G has a dominating set $D = \{v_1, v_2, \ldots, v_{k'}\}$, $k' \le k$, of size at most k. Let $F = \{(v_1,1),(v_2,1),\ldots,(v_{k'},1),x\}$. Since x dominates all vertices in $(V,1)\cup(V,3)$ and D is a dominating set in G, the set F is dominating in H. Moreover, from the construction of H we see that induced subgraphs $\langle F\rangle$ and $\langle V(H)-F\rangle$ are connected. Thus F is a doubly connected dominating set of H of size at most $q = k+1$.

Conversely, assume that F is a doubly connected dominating set of cardinality at most q in H. We shall show that G contains a dominating set D of size at most $k = q-1$. It is easy to see that if $q > n(G)$, answers for problems DCDS and DS are "yes". So assume $q \le n(G)$. We claim that either vertex x or y is in every doubly connected dominating set of size $q \le n(G)$, because a connected dominating set of size at most $n(G)$ that dominates all vertices of $(V,3)$ and does not contain x nor y does not exist. (Observe that in $\langle V \times \{1,2,3\}\rangle$ the subset $(V,3)$ is a set of vertices of degree 1.) Thus assume $x \in F$. Moreover, every doubly connected dominating set F' of size $q_1 \leq n(G)$ can be transformed into a doubly connected dominating set $F \subseteq (V,1) \cup \{x\}$ of size $q \leq q_1$ as follows

- $x \in F$;
- if $(v_i,1) \in F'$, then $(v_i,1) \in F$;
- if $(v_i,3) \in F'$, then $(v_i,1) \in F$;
- if $(v_i,2) \in F'$, then $(v_i,1) \in F$.

Now, if $F = \{(v_1,1),(v_2,1),\ldots,(v_{q-1},1),x\}$ is a doubly connected dominating set of size q, then $D = \{v_1,v_2,\ldots,v_{q-1}\}$ is a dominating set in G of size $k = q-1$.

It is obvious that the transformation used is polynomial, as H has $3n(G)+2$ vertices and $4n(G)+2m(G)$ edges. $\square$

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